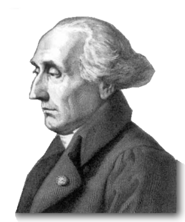
Lagrange Multipliers

Wednesday, 28 September 2011

Physics 111



Sometimes it is convenient to use redundant coordinates, and to effect the variation of the action consistent with the constraints via the method of Lagrange undetermined multipliers. As a bonus, we obtain the generalized forces of constraint.

Consider a mechanical system with $N - P$ degrees of freedom, which we describe with N generalized coordinates and P equations of constraint. We will assume that dissipation may be neglected so that the system may be described by a Lagrangian. Let us suppose that the equations of constraint may be written in the form

$$G_j(q_i, t) = 0 \qquad \Longrightarrow \qquad dG_j = \sum_i \frac{\partial G_j}{\partial q_i}\, dq_i + \frac{\partial G_j}{\partial t}\, dt = 0$$

for $j = 1, 2, \ldots, P$. For notational convenience[1], define

$$\frac{\partial G_j}{\partial q_i} = a_{ji} \qquad \text{and} \qquad \frac{\partial G_j}{\partial t} = a_{jt}$$

Then the jth constraint equation may be written

$$a_{ji}\, dq_i + a_{jt}\, dt = 0 \qquad \text{or} \qquad a_{ji}\dot{q}_i + a_{jt} = 0 \qquad\qquad (P \text{ equations})$$

Note that the coefficients a_{ji} and a_{jt} may be functions of the generalized coordinates q_i and the time t, but not the generalized velocities $\dot{q}_i$.

Hamilton's principle says that of all possible paths, the one the system follows is that which minimizes the action, which is the time integral of the Lagrangian:

$$\delta S = \delta \int_{t_a}^{t_b} L(q_i, \dot{q}_i, t)\, dt = 0 \tag{1}$$

Expanding the variation in the Lagrangian and integrating by parts, we obtain

$$\delta S = \int_{t_a}^{t_b} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q_i\, dt$$

which must vanish on the minimum path. Remember that we are using the summation convention; we sum over i. When the coordinates q_i formed a minimal complete set, we argued that the virtual displacements δq_i were arbitrary and that the only way to ensure that the action be minimum is for each term in the square brackets to vanish.

[1] Translation: just to confuse you.

Now, however, an arbitrary variation in the coordinates will send us off the constraint surface, leading to an impossible solution. One way to "trick" Prof. Hamilton into finding the right solution (the solution consistent with the constraint equations) would be to sneak something inside the brackets that would make the variation in the action zero for displacements that take us off the surface of constraint. That's hardly cheating; it just takes away the incentive to cheat! If we managed to find such terms, then it wouldn't matter which way we varied the path; we'd get zero change in the action, to first order in the variation. We could then treat all the variations δq_i as independent.

Before running that little operation, consider what it might mean for those terms to represent the generalized forces of constraint. Since the allowed virtual displacements of the system are all orthogonal to the constraint forces, those forces do no work and they don't change the value of the action integral. In fact, they are just the necessary forces to ensure that the motion follows the constrained path. So, if we can figure out what terms we need to add to the Lagrangian to make the illegal variations vanish, we will have also found the forces of constraint.

Each of the constraint equations is of the form $G_j(q_i, t) = 0$, so if we were to add a multiple of each constraint equation to the Lagrangian, it would leave the action unchanged. So, we form the augmented Lagrangian:

$$L' = L + \lambda_j G_j$$

where I'm using the summation convention and the λ_j are Lagrange's undetermined multiplier, one per equation of constraint; all may be functions of the time. Then the (augmented) action is

$$S = \int_{t_1}^{t_2} L' \, dt$$

(but since we added zero, it's the same as the un-augmented action). By Hamilton's principle, the action is minimized along the true path followed by the system. We may now effect the variation of the action and force it to vanish:

$$\delta S = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \lambda_j \frac{\partial G_j}{\partial q_i} \delta q_i \right] dt = 0$$

Integrate the middle term by parts (and remember we're using the summation convention):

$$\delta S = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) + \lambda_j a_{ji} \right] \delta q_i \, dt = 0$$

We now have N variations δq_i, only $N - P$ of which are independent (since the system has only $N - P$ degrees of freedom). Using the P independent Lagrange multipliers, we may ensure that all N terms inside the square bracket vanish, so that no matter what the variations δq_i the value of the integral doesn't vary. Thus we have

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = \lambda_j a_{ji} \qquad (N \text{ equations})$$

$$a_{ji} \, dq_i + a_{jt} \, dt = 0 \qquad \text{or} \qquad a_{ji} \dot{q}_i + a_{jt} = 0 \qquad (P \text{ equations})$$

Since there are N generalized coordinates and P Lagrange multipliers, we now have a closed algebraic system.

When we derived Lagrangian mechanics starting from Newton's laws, we showed that

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = \mathscr{F}_i^{\text{tot}} = \mathbf{F}^{\text{tot}} \cdot \frac{\partial \mathbf{r}}{\partial q_i}$$

where $\mathbf{F}$ is the *total* force on the particle and $\mathscr{F}_i^{\text{tot}}$ is the generalized force corresponding to the ith generalized coordinate. If we separate the forces into those expressible in terms of a scalar potential depending only on positions (not velocities), the forces of constraint, and anything left over, then this becomes

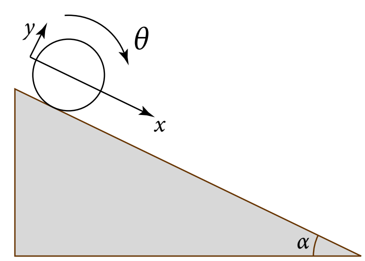
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \mathscr{F}_i^{\text{constraint}} + \mathscr{F}_i^{\text{noncons}} \tag{2}$$

Comparing with the N Lagrange equations above, we see that when all the forces are conservative,

$$\mathscr{F}_i^{\text{constraint}} = \mathbf{F}^{\text{constraint}} \cdot \frac{\partial \mathbf{r}}{\partial q_i} = \lambda_j a_{ji} = \sum_j \lambda_j \frac{\partial G_j}{\partial q_i}$$

where I have written the sum explicitly in the last expression, just to remind you. In other words, the sum $\lambda_j a_{ji}$ is the generalized constraint force.

Example 1



A hoop of mass m and radius R rolls without slipping down a plane inclined at angle α with respect to the horizontal. Solve for the motion, as well as the generalized constraint forces.

Using the indicated coordinate system, we have no motion in y, but coordinated motion between x and θ, which are linked by the constraint condition

$$R\,d\theta = dx \qquad \text{or} \qquad R\,d\theta - dx = 0$$

Therefore,

$$a_{1\theta} = R, \qquad a_{1x} = -1, \qquad a_{1t} = 0$$

The kinetic energy is $T = \frac{m}{2}\dot{x}^2 + \frac{mR^2}{2}\dot{\theta}^2$ and the potential energy is $V = -mgx\sin\alpha$, so the Lagrangian is

$$L = T - V = \frac{m}{2}\dot{x}^2 + \frac{mR^2}{2}\dot{\theta}^2 + mgx\sin\alpha$$

We will first solve by using the constraint equation to eliminate θ: $R\dot{\theta} = \dot{x}$, so

$$L = \frac{m}{2}\dot{x}^2 + \frac{m}{2}\dot{x}^2 + mgx\sin\alpha = m\dot{x}^2 + mgx\sin\alpha$$

This Lagrangian has a single generalized coordinate, x, and thus we obtain the equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \qquad \Longrightarrow \qquad 2m\ddot{x} - mg\sin\alpha = 0 \qquad \Longrightarrow \qquad \ddot{x} = \frac{g}{2}\sin\alpha$$

which is half as fast as it would accelerate if it slid without friction.

If we delay the gratification of inserting the constraint and instead use the Lagrangian with two generalized coordinates, we get

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= \lambda_1 a_{1x} & \Longrightarrow \qquad && m\ddot{x} - mg\sin\alpha = \lambda_1(-1) \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= \lambda_1 a_{1\theta} & \Longrightarrow \qquad && mR^2\ddot{\theta} - 0 = \lambda_1 R
\end{aligned}$$

From the second equation, we obtain $\lambda_1 = mR\ddot{\theta} = m\ddot{x}$, where I have used the constraint equation $R\ddot{\theta} = \ddot{x}$ in the last step. Substituting into the first equation, we again obtain $\ddot{x} = g/2 \sin\alpha$.

What about the constraint forces? The generalized constraint force in x is $\mathscr{F}_x = \lambda_1 a_{1x} = -m\ddot{x} = -\frac{mg}{2}\sin\alpha$. This is the force heading up the slope produced by friction; it is responsible for the slowed motion of the center of mass. The generalized constraint force in θ is $\mathscr{F}_\theta = \lambda_1 R = mR\ddot{x} = \frac{mgR}{2}\sin\alpha$. This is the torque about the center of mass of the hoop caused by the frictional force.

Summary

When you wish to use redundant coordinates, or when you wish to determine forces of constraint using the Lagrangian approach, here's the recipe:

1. Write the equations of constraint, $G_j(q_i, t) = 0$, in the form

$$a_{ji}\,dq_i + a_{jt}\,dt = 0$$

   where $a_{ji} = \frac{\partial G_j}{\partial q_i}$.

2. Write down the N Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \lambda_j a_{ji} \qquad \text{(summation convention)}$$

   where the $\lambda_j(t)$ are the Lagrange undetermined multipliers and $\mathscr{F}_i = \lambda_j a_{ji}$ is the generalized force of constraint in the q_i direction.

3. Solve, using the N Lagrange equations and the P constraint equations.
4. Compute the generalized constraint forces, $\mathscr{F}_i$, if desired.

Problem 1 Use the method of Lagrange undetermined multipliers to calculate the generalized constraint forces on our venerable bead, which is forced to move without friction on a hoop of radius R whose normal is horizontal and forced to rotate at angular velocity ω about a vertical axis through its center. Interpret these generalized forces. What do they correspond to physically?