Chapter 13

Coupled oscillators

Some oscillations are fairly simple, like the small-amplitude swinging of a pendulum, and can be modeled by a single mass on the end of a Hooke's-law spring. Others are more complex, but can still be modeled by two or more masses and two or more springs. Examples include compound mechanical systems, oscillating electrical circuits with several branches, multi-atom molecules, and elastic solids. Here we display the techniques used to understand oscillations like these.

13.1   Linear systems of masses and springs

We are given two blocks, each of mass m, sitting on a frictionless horizontal surface. The blocks are attached to three springs, and the outer springs are also attached to stationary walls, as shown in Figure 13.1.   The two outer springs each have force constant k, and the inner spring has force constant k'. When the blocks are at rest the springs are unstretched.

Let the displacements of block 1 and block 2 from their equilibrium positions be x_1 and x_2, respectively, each positive to the right. Our goal is to find the differential equations of motion of each block, and then solve them to find $x_1(t)$ and $x_2(t)$. The Lagrangian of the system is

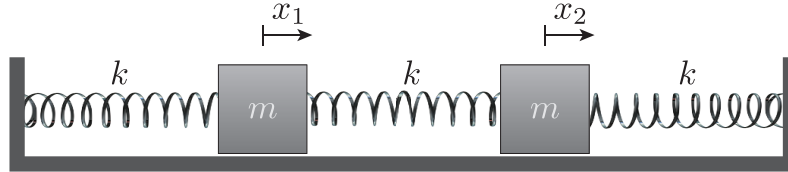$$L = T(\dot{x}_1, \dot{x}_2) - U(x_1, x_2)$$

FIGURE 13.1

$$= \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}kx_1^2 - \frac{1}{2}k'(x_2 - x_1)^2 - \frac{1}{2}kx_2^2, \tag{13.1}$$

taking into account the potential energy stored in each of the three springs (note that the stretch in the middle spring is $x_2 - x_1$.) Lagrange's equations give

$$
\begin{aligned}
m\ddot{x}_1 &= -kx_1 + k'(x_2 - x_1) \\
m\ddot{x}_2 &= -kx_2 - k'(x_2 - x_1),
\end{aligned}
\tag{13.2}
$$

which we could have written down directly using $F = ma$ for each block.

We want to solve these coupled equations to find $x_1(t)$ and $x_2(t)$, given the initial conditions. The problem is that each equation involves both x_1 and x_2, so we have to begin by decoupling them.

First method

Looking closely at the two equations, note that if we *sum* them we eliminate the terms with the difference $x_2 - x_1$, giving

$$m(\ddot{x}_1 + \ddot{x}_2) = -k(x_1 + x_2) \tag{13.3}$$

in terms of the single variable x_1+x_2. If instead we *subtract* the first equation from the second, every resulting term contains *only* the difference $x_2 - x_1$:

$$\begin{aligned} m(\ddot{x}_2 - \ddot{x}_1) &= -k(x_2 - x_1) - 2k'(x_2 - x_1) \\ &= -(k + 2k')(x_2 - x_1). \end{aligned} \tag{13.4}$$

That is, in terms of new, composite coordinates $\xi_1 \equiv x_2+x_1$ and $\xi_2 \equiv x_2-x_1$, the equations become

$$m\ddot{\xi}_1 + k\xi_1 = 0 \qquad \text{and} \qquad m\ddot{\xi}_2 + (k + 2k')\xi_2 = 0, \tag{13.5}$$

which are two *decoupled* simple harmonic oscillator equations. Each has sinusoidal solutions, the first with frequency $\omega_1 = \sqrt{k/m}$ and the second with the higher frequency $\omega_2 = \sqrt{(k + 2k')/m}$. A mathematically convenient solution is

$$\begin{aligned} \xi_1 &= A_1 e^{i\omega_1 t} \quad (\omega_1 = \sqrt{k/m}) \\ \xi_2 &= A_2 e^{i\omega_2 t} \quad (\omega_2 = \sqrt{(k + 2k')/m}) \end{aligned} \tag{13.6}$$

where A_1 and A_2 are arbitrary constants.[1]

Substituting the trial solutions into the differential equations of motion, we find that

$$(k - m\omega^2)A_1 = 0 \quad \text{and} \quad ((k + 2k') - m\omega^2)A_2 = 0 \tag{13.9}$$

which (for later use) we can write in matrix form

$$\begin{pmatrix} k - m\omega^2 & 0 \\ 0 & (k + 2k') - m\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0. \tag{13.10}$$

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[1]If we let A_1 and A_2 be complex, so that each has both a real and an imaginary part, we get the requisite four arbitrary constants for solutions of two second-order differential equations. The final solutions must be real, in terms of the real initial conditions, so letting $A_1 \equiv a_1 - ib_1$ and $A_2 \equiv a_2 - ib_2$ (using minus signs for later convenience) and recalling Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$, the physical solutions are

$$\xi_1^{phys} = \mathcal{Re}(\xi_1) = a_1 \cos\omega_1 t + b_1 \sin\omega_1 t \tag{13.7}$$

$$\xi_2^{phys} = \mathcal{Re}(\xi_2) = a_2 \cos\omega_2 t + b_2 \sin\omega_2 t. \tag{13.8}$$

with four arbitrary (real) constants, which can be determined by the initial positions and velocities.
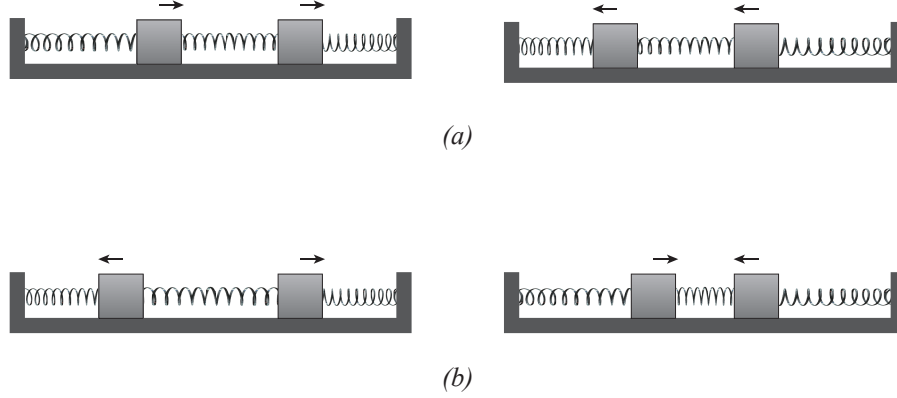
(a)

(b)

FIGURE 13.2

Note that the square matrix is diagonal, which is consistent with equations that are decoupled (if the matrix were not diagonal, each equation would involve both A_1 and A_2.)

Now consider two special cases.

(1) Suppose that $A_2 = 0$, so $\xi_2 = x_2 - x_1 = 0$. In that case $x_1 = x_2 = (A_1/2)e^{i\omega_1 t}$, so each block oscillates with the same frequency ω_1, and with the same amplitude $A_1/2$. They oscillate *in phase* with one another, sliding back and forth on the table together, so that the middle spring is never stretched or compressed, as illustrated in Figure 13.2(a). That is why the oscillation frequency $\omega_1 = \sqrt{k/m}$ is independent of k', and why it is the same as the frequency each block would have if it were simply oscillating on the end of a single spring k. This motion is called a *normal mode of oscillation*, in which (by definition) both blocks oscillate with the same frequency. In fact, it is the *first* normal mode, in which the blocks also have the same amplitude and the same phase.

(2) Suppose instead that $A_1 = 0$, so $\xi_1 = x_1 + x_2 = 0$. In that case $x_2 = -x_1 = -(A_2/2)e^{i\omega_2 t}$, so each block oscillates with the same frequency $\omega_2 = \sqrt{(k + 2k')/m}$ and with equal but *opposite* amplitudes. That is, they move alternately apart and together, with the center of the middle spring remaining fixed, as illustrated in Figure 13.2(b). In effect each block oscillates on the end of an outer spring plus *half* of the middle spring. The force constant of a

half-spring is *twice* that of a full spring (because a half-spring is twice as stiff as the corresponding full spring, since it stretches only half as much for a given applied force). That is why the frequency in this case is $\omega_2 = \sqrt{(k + 2k')/m}$. This motion is the *second normal mode of oscillation.* The blocks move with the same frequency in opposition to one another, with equal but opposite amplitudes (i.e., they are $180°$ out of phase with one another.)

This method of solving the differential equations of motion, to find the normal mode frequencies and relative amplitudes, involved the slightly clever guess that adding or subtracting the original $F = ma$ equations decouples them. Now we will discuss an alternative approach that is more straightforward.

Second method

In a normal mode, by definition each block oscillates with the same frequency, so we can simply try the solutions

$$x_1 = A_1 e^{i\omega t} \qquad x_2 = A_2 e^{i\omega t} \tag{13.11}$$

with the same frequency for each. Substituting these into the original $F = ma$ equations equation (13.1)

$$(-m\omega^2 + (k + k'))A_1 - k'A_2 \;=\; 0 \quad \text{and}$$

$$-k'A_1 + (-m\omega^2 + (k + k'))A_2 \;=\; 0, \tag{13.12}$$

which in matrix form becomes

$$\begin{pmatrix} (k + k') - m\omega^2 & -k' \\ -k' & (k + k') - m\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0, \tag{13.13}$$

forming a pair of homogeneous equations in the unknown amplitudes. For arbitrary frequencies the only solution of the equation is the "trivial" solution $A_1 = A_2 = 0$, where both blocks remain at rest at their equilibrium positions. But from linear algebra we know that with just the *right* choice(s) of ω there are also *non*-trivial solution(s) if and only if the determinant of the coefficients vanishes, i.e., if and only if

$$\begin{vmatrix} (k + k') - m\omega^2 & -k' \\ -k' & (k + k') - m\omega^2 \end{vmatrix} = (-m\omega^2 + (k + k'))^2 - k'^2$$

$$= 0. \tag{13.14}$$

From this so-called *secular equation* it follows that

$$-m\omega^2 + (k + k') = \pm k', \tag{13.15}$$

with the two solutions

$$\omega_1 = \sqrt{\frac{k}{m}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{k + 2k'}{m}}, \tag{13.16}$$

the same results we found for the frequencies using the "clever" technique. We have once again found the normal mode frequencies, which are also called the *characteristic* frequencies, *eigenfrequencies*, or *frequency eigenvalues*. Now we can substitute them one at a time into the original differential equations of motion to find the relative amplitudes of the two blocks.

First, let $\omega = \omega_1 = \sqrt{k/m}$. Then equation (13.12) becomes

$$k'A_1 - k'A_2 = 0 \quad \text{and} \quad -k'A_1 + k'A_2 = 0, \tag{13.17}$$

with solutions $A_2/A_1 = 1$. The two blocks slide back in forth in phase with equal amplitudes, just as we found using the "clever" technique. If instead we substitute $\omega = \omega_2 = \sqrt{(k + 2k')/m}$, then tequation (13.12) becomes

$$-k'A_1 - k'A_2 = 0 \quad \text{and} \quad -k'A_1 - k'A_2 = 0, \tag{13.18}$$

so that $A_2/A_1 = -1$, where the two blocks slide in and out with equal but opposite amplitudes. Note that we can find only the *ratios* A_2/A_1 for each normal-mode frequency. That makes sense, because we can have any amplitude A_1 we like, as provided by the initial conditions.

We can write normalized "eigenvectors" corresponding to each of the eigenfrequencies in the form

$$E(1) \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{13.19}$$

for the first normal mode, and

$$E(2) \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{13.20}$$

for the second normal mode. Note that each has the correct relative amplitudes of the two blocks. If each eigenvector is multiplied by the same constant, as determined by the initial conditions, we get both A_1 and A_2. The eigenvectors are *normalized*, because the inner product of each with itself is unity:

$$\frac{1}{\sqrt{2}}(1 \quad 1)\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \tag{13.21}$$

and

$$\frac{1}{\sqrt{2}}(-1 \quad 1)\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1 \tag{13.22}$$

by matrix multiplication, where in the product we multiply the column matrix by the corresponding row matrix. The two eigenvectors are also *normal*, i.e., "perpendicular" to one another, because their inner products are

$$\frac{1}{\sqrt{2}}(1 \quad 1)\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \tag{13.23}$$

and

$$\frac{1}{\sqrt{2}}(-1 \quad 1)\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0. \tag{13.24}$$

This is the reason the two solutions are said to be *normal* modes.

The differential equations are linear, so any linear combination of the two normal mode solutions is also a solution. In fact, the most general solution of the equations is an arbitrary linear combination of the two normal modes. So for example if both blocks begin at their respective origins, but only block 2 is given an initial velocity, the subsequent motion of the two blocks will involve *both* normal-mode frequencies, so will appear to be rather chaotic.

If again both blocks begin at their respective origins, we can use equation (13.7) and equation (13.8) in the case of arbitrary initial velocities. Differentiating these equations, we get

$$\dot{\xi}_1^{phys}(t) \equiv \dot{x}_2(t) + \dot{x}_1(t) = \omega_1 b_1 \cos \omega_1 t \tag{13.25}$$

$$\dot{\xi}_2^{phys}(t) \equiv \dot{x}_2(t) - \dot{x}_1(t) = \omega_2 b_2 \cos \omega_2 t. \tag{13.26}$$

so that the constants b_1 and b_2 can be found in terms of the initial velocities $(v_1)_0$ and $(v_2)_0$. In fact,

$$b_1 = \frac{(v_2)_0 + (v_1)_0}{\omega_1} \quad \text{and} \quad b_2 = \frac{(v_2)_0 - (v_1)_0}{\omega_2}. \tag{13.27}$$

Then

$$x_2(t) + x_1(t) = \left(\frac{(v_2)_0 + (v_1)_0}{\omega_1} \right) \sin \omega_1 t \quad \text{and}$$

$$\tag{13.28}$$

$$x_2(t) - x_1(t) = \left(\frac{(v_2)_0 - (v_1)_0}{\omega_2} \right) \sin \omega_2 t, \tag{13.29}$$

so

$$x_1(t) = \frac{1}{2} \left[\left(\frac{(v_2)_0 + (v_1)_0}{\omega_1} \right) \sin \omega_1 t - \left(\frac{(v_2)_0 - (v_1)_0}{\omega_2} \right) \sin \omega_2 t \right] \tag{13.30}$$

and

$$x_2(t) = \frac{1}{2} \left[\left(\frac{(v_2)_0 + (v_1)_0}{\omega_1} \right) \sin \omega_1 t + \left(\frac{(v_2)_0 - (v_1)_0}{\omega_2} \right) \sin \omega_2 t \right]. \tag{13.31}$$

Now in particular if we give only block 1 an initial velocity,

$$x_1(t) = \left(\frac{(v_1)_0}{2\omega_1} \right) \left[\sin \omega_1 t + \left(\frac{\omega_1}{\omega_2} \right) \sin \omega_2 t \right] \tag{13.32}$$

and

$$x_2(t) = \left(\frac{(v_1)_0}{2\omega_1} \right) \left[\sin \omega_1 t - \left(\frac{\omega_1}{\omega_2} \right) \sin \omega_2 t \right] \tag{13.33}$$

so each block oscillates with a iinear combination of the two normal-mode frequencies.

EXAMPLE 13-1: Weak coupling and strong coupling

Sometimes the middle spring is much *weaker* than the others, with $k' << k$, so that the coupling between the two blocks is said to be *weak*. In that case we can approximate ω_2 using the binomial expansion,

$$\omega_2 = \left(\frac{k + 2k'}{m} \right)^{1/2} = \sqrt{\frac{k}{m}} \left(1 + \frac{2k'}{k} \right)^{1/2} \cong \omega_1 \left(1 + \frac{k'}{k} \right). \tag{13.34}$$

Then

$$\sin \omega_2 t = \sin \omega_1 (1 + k'/k)t \cong \cos[(k'/k)\omega_1 t] \sin \omega_1 t, \tag{13.35}$$

using $\sin(a + b) = \sin a \cos b + \cos a \sin b$. Therefore

$$\begin{aligned}
x_1(t) &\cong \left(\frac{(v_1)_0}{2\omega_1} \right) \sin \omega_1 t \left[1 + \cos[(k'/k)\omega_1 t] \right] \\
&\cong \left(\frac{(v_1)_0}{\omega_1} \right) \cos^2 \left[\left(\frac{k'}{2k} \right) \omega_1 t \right] \sin \omega_1 t
\end{aligned} \tag{13.36}$$

and

$$\begin{aligned}
x_2(t) &\cong \left(\frac{(v_1)_0}{2\omega_1} \right) \sin \omega_1 t \left[1 - \cos[(k'/k)\omega_1 t] \right] \\
&\cong \left(\frac{(v_1)_0}{\omega_1} \right) \sin^2 \left[\left(\frac{k'}{2k} \right) \omega_1 t \right] \sin \omega_1 t
\end{aligned} \tag{13.37}$$

so that $x_1(t)$ and $x_2(t)$ are each products of a rapidly-oscillating function $\sin \omega_1 t$ and an "envelope" oscillating with the slower frequency $(k'/2k)\omega_1$. This behavior is illustrated in Figure 13.3 for the case $k'/2k = 1/20$. Note that both x_1 and x_2 start at zero, in accordance with our initial conditions, and that the swings of x_1 are initially large, since we gave ball 1 alone an initial velocity. However, gradually the swing amplitudes of ball 1 decrease while those of ball 2 increase. That is, the energy originally put into ball 1 is gradually taken over by ball 1, and then returned to ball 1, and so on.

Now suppose the center spring is much *stronger* than the others, with $k' >> k$, corresponding to *strong coupling* between the two blocks. Then $\omega_1/\omega_2 \cong \sqrt{k/2k'} << 1$, so Eqs. show that to a good approximation the two blocks slide back and forth in phase with the first normal mode frequency ω_1, but superimposed on this back and forth motion is a small-amplitude high-frequency oscillation in which the blocks oscillate in opposition, a small-amplitude jittery motion superimposed on the much larger amplitude motion in which the blocks slide back and forth together. ————
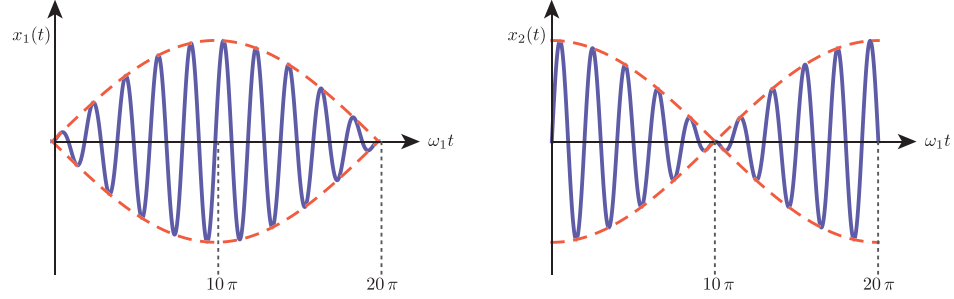
FIGURE 13.3

EXAMPLE 13-2: Coupled pendulums

Two balls, each of mass m, are attached to two strings of equal length ℓ to form side-by-side pendulums of equal period. Now a weak spring k' is attached to the two balls, as shown in Figure 13.4.

We want to find the motion of each ball in the small-amplitude limit, in which the angles θ_1 and θ_2 are both very small. In that case the spring stretch is very nearly $\ell(\theta_2 - \theta_1)$, and the gravitational potential energy of the first ball is $mgh = mg\ell(1 - \cos\theta_1) \cong mg\ell[1 - (1 - \theta_1^2/2)] = mg\ell\theta_1^2/2$, with a similar expression for the second ball. The Lagrangian of the small-amplitude system is therefore

$$L = \frac{1}{2}m\ell^2\dot{\theta}_1^2 + \frac{1}{2}m\ell^2\dot{\theta}_1^2 - \frac{1}{2}mg\ell\theta_1^2 - \frac{1}{2}mg\ell\theta_2^2 - \frac{1}{2}k'\ell^2(\theta_2 - \theta_1)^2. \tag{13.38}$$

Note that this Lagrangian is identical to the Lagrangian of the two-mass, three-spring problem we just discussed, if we replace $\ell\theta_1$ and $\ell\theta_2$ by x_1 and x_2, and replace the constant quantity mg/ℓ by k. That is, for small displacements from equilibrium the gravitational force on each mass acts like a spring of force-constant mg/ℓ. Therefore the behavior of the small-amplitude coupled pendulum is just like that of the two-mass, three-spring problem. In particular, if the coupling spring k' is very weak, we can start the first pendulum mass swinging back and forth while the second pendulum is initially at rest; then after a while the motion (and energy) is gradually transferred from the first pendulum to the second, so that the second pendulum eventually swings back and forth while the first pendulum comes instantaneously to rest. This alternating behavior would continue indefinitely were it not for friction, which eventually robs the system of its energy, and both pendulums come to rest.
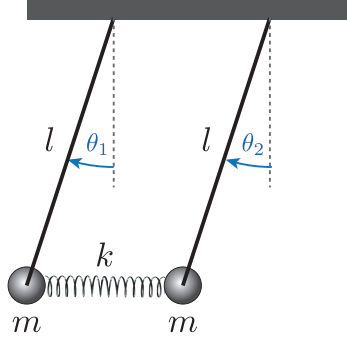
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FIGURE 13.4

In contrast, if $k >> mg/\ell$ the two pendulums are strongly coupled: they swing back and forth together, upon which is superimposed a small-amplitude high-frequency oscillation between the two balls.

## EXAMPLE 13-3: Three blocks and four springs

Now suppose there are *three* blocks attached to *four* springs: Again, the springs are un-stretched in the equilibrium position, and the blocks are free to move in the horizontal direction only. The far end of each outer spring is attached to a stationary wall. The displacements of the three blocks from equilibrium are x_1, x_2, x_3, positive to the right. For simplicity, suppose the blocks have equal mass m and all four springs have the same force constant k, as shown in Figure 13.5. Now the Lagrangian is

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - \frac{1}{2}kx_1^2 - \frac{1}{2}k(x_2 - x_1)^2 - \frac{1}{2}k(x_3 - x_2)^2 - \frac{1}{2}kx_3^2, \quad (13.39)$$

taking into account the potential energy stored in each of the four springs. Lagrange's equations then give

$$
\begin{aligned}
m\ddot{x}_1 &= -kx_1 + k(x_2 - x_1) \\
m\ddot{x}_2 &= -k(x_2 - x_1) + k(x_3 - x_2) \\
m\ddot{x}_3 &= -kx_3 - k(x_3 - x_2),
\end{aligned}
\quad (13.40)
$$

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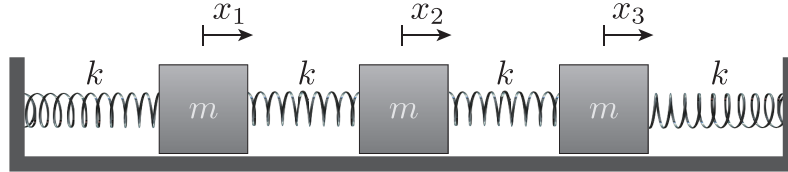
FIGURE 13.5

which again could instead be written down directly using $F = ma$ for each block.

As in the two-block case, these can be solved by writing $x_1 = A_1 e^{i\omega t}$, $x_2 = A_2 e^{i\omega t}$, $x_3 = A_3 e^{i\omega t}$, resulting in the matrix equation

$$\begin{pmatrix} (-m\omega^2 + 2k) & -k & 0 \\ -k & (-m\omega^2 + 2k) & -k \\ 0 & -k & (-m\omega^2 + 2k) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0 \tag{13.41}$$

which have a nontrivial solution only if the determinant of the coefficient matrix is zero. That is, the secular equation is

$$\begin{vmatrix} -m\omega^2 + 2k & -k & 0 \\ -k & -m\omega^2 + 2k & -k \\ 0 & -k & -m\omega^2 + 2k \end{vmatrix} = 0. \tag{13.42}$$

Expanding about the top row,

$$(-m\omega^2 + 2k)[(-m\omega^2 + 2k)^2 - k^2] + k(-k(-m\omega^2 + 2k)) = 0. \tag{13.43}$$

Factoring,

$$(-m\omega^2 + 2k)[(-m\omega^2 + 2k)^2 - 2k^2] = 0, \tag{13.44}$$

the product of a linear and a quadratic equation in ω^2, with altogether three solutions. The first factor is zero if

$$\omega = \omega_1 = \sqrt{\frac{2k}{m}}. \tag{13.45}$$

The *second* factor is zero if $(-m\omega^2 + 2k)^2 = 2k^2$, i.e., $-m\omega^2 + 2k = \pm\sqrt{2}k$, which gives the other two eigenfrequencies

$$\omega_2 = \sqrt{\frac{(2-\sqrt{2})k}{m}} \quad \text{and} \quad \omega_3 = \sqrt{\frac{(2+\sqrt{2})k}{m}}. \tag{13.46}$$

As with the two-block problem, we can find the three normal mode motions either intuitively or mathematically. Intuitively, it is clear that ω_1 corresponds to the frequency when the center block remains at rest and the outer blocks oscillate oppositely to one another, both moving outwards and then both moving inwards, etc. Each is connected to two springs whose opposite ends stay at rest, so the frequency should be $\omega = \sqrt{2k/m}$. This motion can be verified by substituting ω_1 into the algebraic equations, confirming that $A_3 = -A_1$ and $A_2 = 0$. The eigenfrequency ω_2 corresponds to the two outer blocks moving together in phase, with the middle block moving in the same direction with a different amplitude; and the eigenfrequency ω_3 corresponds to the two outer blocks moving together in phase, with the middle block moving always in the opposite direction, and with a different amplitude.

Let us find the exact motion for the second eigenfrequency ω_2. The algebraic equations then become

$$\begin{aligned}
(-(2-\sqrt{2})k + 2k)A_1 - kA_2 + 0 \cdot A_3 &= 0 \\
-kA_1 + (-(2-\sqrt{2}) + 2k)A_2 - kA_3 &= 0 \\
0 \cdot A_1 - kA_2 + (-(2-\sqrt{2} + 2k)A_3 &= 0,
\end{aligned} \tag{13.47}$$

so

$$A_2 = \sqrt{2}A_1, \quad -kA_1 + \sqrt{2}A_2 - kA_3 = 0, \quad A_2 = \sqrt{2}A_3 \tag{13.48}$$

which reduce to $A_2 = \sqrt{2}A_1 = \sqrt{2}A_3$. So in this mode the two outer blocks oscillate in phase with equal amplitudes $A_3 = A_1$, while the middle block oscillates with the *same* phase, but with an amplitude that is larger by the factor $\sqrt{2}$.

In a similar way we find for the third eigenfrequency $\omega_3 = \sqrt{(2+\sqrt{2})k/m}$ that the two outer blocks oscillate in phase with $A_3 = A_1$, while the middle block oscillates with the *opposite* phase, with an amplitude $A_2 = -\sqrt{2}A_1$.

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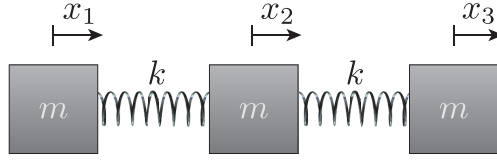**EXAMPLE 13-4: Tear down the walls**

FIGURE 13.6

Now we return to the case of three equal-mass blocks, but now we remove the walls and outer springs, so the three blocks are connected together linearly with only *two* springs, as shown in Figure 13.6.

The Lagrangian for this system is

$$L = T - U = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - \frac{1}{2}k(x_2 - x_1)^2 - \frac{1}{2}k(x_3 - x_2)^2, \tag{13.49}$$

from which Lagrange's equations give

$$\begin{aligned}
m\ddot{x}_1 &= k(x_2 - x_1) \\
m\ddot{x}_2 &= -k(x_2 - x_1) + k(x_3 - x_2) = k(x_1 + x_3) - 2kx_2 \\
m\ddot{x}_3 &= -k(x_3 - x_2).
\end{aligned} \tag{13.50}$$

Substituting in the expressions

$$x_1 = A_1 e^{i\omega t} \qquad x_2 = A_2 e^{i\omega t} \qquad x_3 = A_3 e^{i\omega t} \tag{13.51}$$

gives the set of algebraic equations

$$\begin{aligned}
(-m\omega^2 + k)A_1 - kA_2 + 0 \cdot A_3 &= 0 \\
-kA_1 + (-m\omega^2 + 2k)A_2 - kA_3 &= 0 \\
0 \cdot A_1 - kA_2 + (-m\omega^2 + k)A_3 &= 0,
\end{aligned} \tag{13.52}$$

so the secular determinant is

$$\begin{vmatrix}
-m\omega^2 + k & -k & 0 \\
-k & -m\omega^2 + 2k & -k \\
0 & -k & -m\omega^2 + k
\end{vmatrix} = 0. \tag{13.53}$$

Expanding about the top row,

$$(-m\omega^2 + k)[(-m\omega^2 + 2k)(-m\omega^2 + k) - k^2] + k[-k(-m\omega^2 + k)] = 0. \qquad (13.54)$$

This can be factored to give

$$(-m\omega^2)(-m\omega^2 + k)(-m\omega^2 + 3k) = 0, \qquad (13.55)$$

with the three solutions

$$(1) \ \ \omega_1 = 0 \qquad (2) \ \ \omega_2 = \sqrt{\frac{k}{m}} \qquad (3) \ \ \omega_3 = \sqrt{\frac{3k}{m}}. \qquad (13.56)$$

Substituting $\omega_1 = 0$ into the original algebraic equations, we find that $A_1 = A_2 = A_3$, which means that all three blocks have the same amplitude at all times: that is, there is no oscillation, and all blocks either are at rest or else are moving together at the same velocity. Substituting in ω_2 instead gives $A_3 = -A_1$ and $A_2 = 0$, so that the middle block remains at rest while the outer blocks move with equal amplitudes in opposition to one another. The frequency $\omega_2 = \sqrt{k/m}$ makes sense in this case, because in effect each outer block oscillates at the end of a single spring. Substituting in ω_3, we find that $A_2 = -2A_1 = -2A_3$; that is, the outer blocks move in phase with one another, with the same amplitude, while the center block moves in the opposite direction with twice the amplitude. Note that this motion keeps the center of mass always at rest. The frequency $\omega_3 > \omega_1$, because for a given amplitude of an outer block, the spring is stretched the most if the central block moves in opposition.

13.2 The carbon dioxide molecule

So far we have dealt only with masses attached to Hooke's-law springs, which of course are highly idealized systems. More realistically, for macroscopic one-dimensional mechanical motions there is often some sort of potential energy $U(x)$ between any two masses. If there is a minimum in $U(x)$ at x_0, as illustrated in Figure 13.7, then in equilibrium the two masses are separated by the distance x_0. If they are slightly disturbed, they will oscillate back and forth about the equilibrium point. Most often $U(x)$ rises quadratically from the minimum, so that a parabola can be fit to the bottom of the potential well. In that case the small oscillations will be sinusoidal, and it is as if a Hooke's-law spring were attached to the two masses.

The Taylor-series expansion of $U(x)$ about x_0 is

$$U(x) = U(x_0) + \frac{dU(x)}{dx}\Big|_{x_0}(x - x_0) + \frac{1}{2}\frac{d^2U(x)}{dx^2}\Big|_{x_0}(x - x_0)^2 + ... \quad (13.57)$$
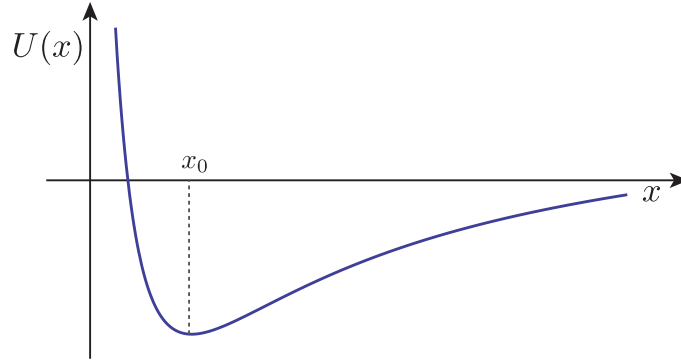
FIGURE 13.7

However, since $U(x)$ has a local minimum at x_0, the second term vanishes, and so $U(x) - U(x_0)$ has the form

$$U(x) - U(x_0) = \frac{1}{2}k(x - x_0)^2 + ...$$ (13.58)

of a Hooke's-law spring potential. The effective force constant is the second derivative

$$k = \frac{d^2U(x)}{dx^2}\Big|_{x_0}$$ (13.59)

evaluated at the potential energy minimum. If the second derivative happens to be zero at the minimum in $U(x)$, then we cannot model the system by Hooke's-law springs, and the motion is not sinusoidal.

Carbon dioxide is a linear molecule, with the carbon atom (of mass m and coordinate x_2) in the middle and the oxygen atoms (of mass M at coordinates x_1 and x_3) at the two ends, as shown in Figure 13.8. One can find a potential energy between the carbon atom and an oxygen atom, and neglect any interaction between the two oxygen atoms, which are relatively far apart. The potential energies are quadratic near the minimum, so there is an effective force-constant k, and the Lagrangian is

$$L = T - U = \frac{1}{2}M(\dot{x}_1^2 + \dot{x}_3^2) + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}k(x_2 - x_1)^2 - \frac{1}{2}k(x_3 - x_2)^2.$$ (13.60)
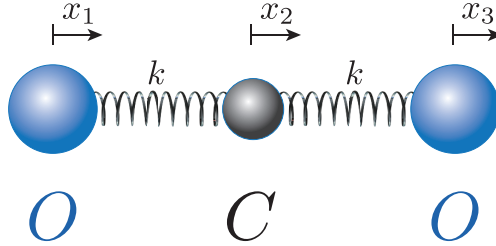
FIGURE 13.8

At the scale of molecules, the motion of the atoms is of course governed by quantum mechanics, and so classical mechanics is only a rough approximation to the actual behavior of the CO_2 molecule. However, the classical approximation is still interesting here, so we will continue to pursue it.

The Lagrange equations are

$$
\begin{aligned}
M\ddot{x}_1 &= k(x_2 - x_1) \\
m\ddot{x}_2 &= -k(x_2 - x_1) + k(x_3 - x_2) \\
M\ddot{x}_3 &= -k(x_3 - x_2).
\end{aligned}
\tag{13.61}
$$

The molecule is not subject to external forces, so if its center of mass begins at rest it will remain at rest.[2] We can find the normal modes by substituting $x_i = A_i e^{i\omega t}$ $(i = 1, 2, 3)$, giving

$$
\begin{aligned}
(-M\omega^2 + k)A_1 - kA_2 + 0 \cdot A_3 &= 0 \\
-kA_1 + (-m\omega^2 + 2k)A_2 - kA_3 &= 0 \\
0 \cdot A_1 - kA_2 + (-M\omega^2 + k)A_3 &= 0.
\end{aligned}
\tag{13.62}
$$

---

[2]If its center of mass is initially moving, the center of mass velocity will remain constant. Any oscillation frequency is independent of such an overall drift.

The secular determinant is

$$\begin{vmatrix} -M\omega^2 + k & -k & 0 \\ -k & -m\omega^2 + 2k & -k \\ 0 & -k & -M\omega^2 + k \end{vmatrix} = 0 \qquad (13.63)$$

for nontrivial solutions. Expanding about the top row,

$$(-M\omega^2+k)[(-m\omega^2+2k)(-M\omega^2+k)-k^2]+k[-k(-M\omega^2+k)] = 0, \quad (13.64)$$

which can be factored to give

$$(-m\omega^2)(-M\omega^2 + k)(-M\omega^2 + k(1 + 2M/m)) = 0, \qquad (13.65)$$

with the three solutions

$$(1)\ \ \omega_1 = 0 \qquad (2)\ \ \omega_2 = \sqrt{\frac{k}{M}} \qquad (3)\ \ \omega_3 = \sqrt{\frac{k(1 + 2M/m)}{M}}. \quad (13.66)$$

We can see that the first of these, ω_1, corresponds to *no oscillation at all*; all the atoms remain at rest. The second frequency, $\omega_2 = \sqrt{k/M}$, is a normal mode of oscillation in which the carbon atom remains at rest at the center while the oxygen atoms move in and out, in opposing directions, so the center of mass of the system remains at rest. This explains why the carbon mass m does not appear in the frequency; its mass is irrelevant. The third frequency, $\omega_3 = \sqrt{k(1 + 2M/m)/M}$, corresponds to the two oxygen atoms moving back and forth together, i.e., in phase, while the carbon atom moves in the opposite direction by the distance required to keep the center of mass of the system at rest.

Now let us derive these results analytically. Substitution of $\omega = \omega_1 = 0$ into the A_1, A_2, A_3 algebraic equations yields

$$\begin{aligned} kA_1 - kA_2 &= 0 \\ -kA_1 + 2kA_2 - kA_3 &= 0 \\ -kA_2 + kA_3 &= 0, \end{aligned} \qquad (13.67)$$

whose only solution is $A_1 = A_2 = A_3$, so that for this solution $x_1 = x_2 = x_3$ at all times, corresponding to the molecule remaining at rest or moving with uniform translation.

Substitution of $\omega = \omega_2 = \sqrt{k/M}$ into the algebraic equations yields the solution $A_3 = -A_1$ and $A_2 = 0$, so the oxygen atoms move with equal but opposite amplitudes while the carbon atom remains at rest, as expected.

Finally, substitution of $\omega = \omega_3 = \sqrt{k(1 + 2M/m)/M}$ into the algebraic equation yields

$$
\begin{aligned}
(-k(1 + 2M/m) + k)A_1 - kA_2 &= 0 \\
-kA_1 + (-(m/M)(1 + 2M/m)k) + 2k)A_2 - kA_3 &= 0 \\
-kA_2 + (-k(1 + 2M/m) + k)A_3 &= 0, \qquad (13.68)
\end{aligned}
$$

Eliminating A_2 between the first and third equations shows that $A_1 = A_3$. The second equation then shows that

$$
\frac{A_2}{A_1} = \frac{A_2}{A_3} = \frac{2}{2 - (m/M + 2)} = -\frac{2M}{m} \qquad (13.69)
$$

so that if the oxygen atoms move a certain distance to the right, the carbon atom in the middle moves the larger distance $2M/m$ to the *left*, by just the right amount to keep the center of mass of the system at rest, since in this mode $Mx_1 + mx_2 + Mx_3 = Mx_1 + m(-2M/m)x_1 + Mx_1 = 0$.

13.3   Degrees of freedom

How many normal modes of oscillation are there for a given system? Why does carbon dioxide have *two* normal modes of oscillation for linear motion? And why are there *three* normal modes for three masses connected in a straight line by four springs, the outer two connected to stationary walls?

Suppose there are N particles, free to move only in one dimension. Then we say there are N degrees of freedom, $x_1, x_2, ...x_N$, the positions measured along the single dimension of each particle. If instead the N particles are allowed to move in two dimensions, there are then $2N$ degrees of freedom, $x_1, y_1; x_2, y_2, ...x_N, y_N$, and so on. So for a carbon dioxide molecule free to move only along one dimension, there are three degrees of freedom. Rather than count by the x coordinate of each atom, we could alternatively say that the one-dimensional motion of the *center of mass* of the molecule counts for one degree of freedom, leaving two more corresponding to possible relative motions of the atoms, in particular the normal mode oscillations. Thus for

one-dimensional motion of three particles,

Three degrees of freedom = (1 *translational* + 2 *vibrational*) degrees of freedom.

Any arbitrary motion of the molecule can be written as some linear combination of these three motions, just as any arbitrary motion can be specified by $x_1(t), x_2(t), x_3(t)$.

Now suppose instead that the CO_2 molecule is free to move in a *plane*, i.e., in two dimensions. Including now two coordinates for each atom, there are six degrees of freedom. There are two translational degrees of freedom of the center of mass, in the x and y directions, plus now a single *rotational* degree of freedom, corresponding to a rigid rotation of the molecule in the x, y plane (i. e., about a z axis through the center of mass), plus vibrational modes. So for two-dimensional motion there must be *three* vibrational modes:

Six degrees of freedom = (2 *translational* + 1 *rotational* + 3 *vibrational*) degrees of freedom

We have now learned that there must be a *third* vibrational mode, which is a vibration in which the molecule bends, with (say) the carbon atom moving in one direction while the two oxygen atoms move in the opposite direction, while keeping the center of mass at rest and while keeping the angular momentum equal to zero. That is, any motion of the center of mass enters exclusively into the translational degrees of freedom, and any net rotation (any rotation with a non-zero angular momentum) enters into the rotational degree of freedom. The vibrational degrees of freedom are what is left over.

If, finally, the same CO_2 model is free to move in all three dimensions, there must be *nine* degrees of freedom. Three of these are translational, in the x, y, and z directions. We might think that there are now also three rotational degrees of freedom, corresponding to rotation about the x, y, or z axis. However, there are really only two: Any rotation about an axis passing through all three atoms is unobservable, so does not count. That leaves *four* vibrational degrees of freedom: the same two we found for one-dimensional motion, plus two bending oscillations. If the molecule is strung out along the x axis, for example, in one bending oscillation the carbon atom moves in the positive y direction while the two oxygen atoms move in the negative y direction (keeping the CM at rest and keeping angular momentum

equal to zero), and then vice versa, oscillating back and forth. In the other bending oscillation the carbon atom moves in the positive z direction while the oxygens move in the negative z direction, and then vice versa, oscillating back and forth. Any other bending oscillation is a linear combination of these two. So for the CO_2 molecule in three dimensions,

Nine degrees of freedom $= (3$ *translational* $+ 2$ *rotational* $+ 4$ *vibrational*$)$ degrees of freedom.

Now return to the special case of three masses with four springs, with the outer springs attached to rigid walls. How many vibrational modes do we expect? We take the motion to be one-dimensional, not considering any bending motion of the springs. There are then three degrees of freedom, $x_1(t), x_2(t), x_3(t)$. There are obviously no rotational degrees of freedom; *there is also no translational degree of freedom* in this case, because the center of mass of the system is not free to move as it likes, due to the constraint of the walls. So in this case

Three degrees of freedom $= 3$ *vibrational* degrees of freedom.

That is, any motion $x_1(t), x_2(t), x_3(t)$ permitted by the constraints in this case can be written as a linear combination of the three normal modes of oscillation alone.


EXAMPLE 13-5: A planar system: masses in an equilateral triangle

So far every explicit case we have worked out has been one-dimensional. All particles and springs have been aligned along a straight line, and have been allowed to oscillate only along that same straight line. However, the techniques we used can be extended to two- and three-dimensional systems as well. We will illustrate by working out the normal-mode oscillations of a system of three equal masses m and three equal springs k in the configuration of an equilateral triangle, as shown in Figure 13.9. We will suppose the masses are free to move only in the plane of the triangle, so there are $3 \times 2 = 6$ degrees of freedom for this system. Two of these are translational and one is rotational (about an axis perpendicular to the plane of the triangle), leaving three vibrational degrees of freedom. What are these three vibrational normal modes?

We begin as usual with the Lagrangian. Let us label the three masses 1, 2, and 3, beginning with No. 1 for the uppermost mass in the figure, then proceeding clockwise for Nos. 2 and 3.
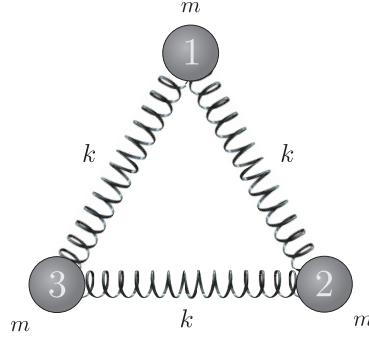
FIGURE 13.9

We describe displacements from the equilibrium positions by $x_1, y_1, x_2, y_2, x_3, y_3$. The kinetic energy of the system is then

$$T = \frac{1}{2}m[(\dot{x}_1^2 + \dot{y}_1^2) + (\dot{x}_2^2 + \dot{y}_2^2) + (\dot{x}_3^2 + \dot{y}_3^2)]. \tag{13.70}$$

The potential energy stored in the spring attached between Nos. 1 and 2 is

$$U_{12} = \frac{1}{2}k[(x_1 - x_2)^2 + (y_1 - y_2)^2] \tag{13.71}$$

with similar expressions for the potential energies in the remaining two springs. The total Lagrangian of the system is therefore $L = T - U$, where

$$T = \frac{1}{2}m[(\dot{x}_1^2 + \dot{y}_1^2) + (\dot{x}_2^2 + \dot{y}_2^2) + (\dot{x}_3^2 + \dot{y}_3^2)] \tag{13.72}$$

and

$$\begin{aligned} U &= \frac{1}{2}k[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (x_2 - x_3)^2 + (y_2 - y_3)^2 \\ &+ (x_3 - x_1)^2 + (y_3 - y_1)^2]. \end{aligned} \tag{13.73}$$

The corresponding Lagrange equations are

$$\begin{aligned} m\ddot{x}_1 + k[(x_1 - x_2) - (x_3 - x_1)] &= 0 \\ m\ddot{x}_2 + k[(x_2 - x_3) - (x_1 - x_2)] &= 0 \\ m\ddot{x}_3 + k[(x_3 - x_1) - (x_3 - x_3)] &= 0 \end{aligned} \tag{13.74}$$

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together with an exactly similar set in $y_1, y_2,$ and y_3. To find the normal modes, we substitute

$$
\begin{aligned}
x_1 &= A_{1x}e^{i\omega t}, \quad x_2 = A_{2x}e^{i\omega t}, \quad x_3 = A_{3x}e^{i\omega t}, \\
y_1 &= A_{1y}e^{i\omega t}, \quad y_2 = A_{2y}e^{i\omega t}, \quad y_3 = A_{3y}e^{i\omega t},
\end{aligned}
\tag{13.75}
$$

into the $F = ma$ differential equations, resulting in the algebraic equations

$$
\begin{aligned}
(-m\omega^2 + 2k)A_{1x} - kA_{2x} - kA_{3x} &= 0 \\
-kA_{1x} + (-m\omega^2 + 2k)A_{2x} - kA_{3x} &= 0 \\
-kA_{1x} - kA_{2x} + (-m\omega^2 + 2k)A_{3x} &= 0
\end{aligned}
\tag{13.76}
$$

along with an exactly similar set in A_{1y}, A_{2y} and A_{3y}. The secular determinant for each set is

$$
\begin{vmatrix}
-m\omega^2 + 2k & -k & -k \\
-k & -m\omega^2 + 2k & -k \\
-k & -k & -m\omega^2 + 2k
\end{vmatrix} = 0.
\tag{13.77}
$$

Expanding the determinant, it is straightforward to show that there is a solution $\omega^2 = 0$ and a *double*-solution $\omega = \sqrt{3k/m}$. The first of these is the translational degree of freedom, one in the x direction and one in the y direction. The second, when substituted into the algebraic equations, gives

$$
-kA_{1x} - kA_{2x} - kA_{3x} = 0
\tag{13.78}
$$

three times! And there is of course a similar equation in the y direction. Therefore the only guidance we have in finding normal-mode solutions are the equations

$$
A_{1x} + A_{2x} + A_{3x} = 0 \quad \text{and} \quad A_{1y} + A_{2y} + A_{3y} = 0,
\tag{13.79}
$$

together with the knowledge that the oscillation frequency $\omega = \sqrt{3k/m}$ for all three independent normal modes. That is, the three normal modes are said to be *degenerate*; all three have the same eigenfrequency. Note that the above equations are simply equivalent to the requirement that for vibrational modes, the center of mass of the three masses remains at rest, both in the x direction and in the y direction.

So what are the normal modes? Since they are degenerate, there are actually an infinite number of choices we might make. Having chosen three that are linearly independent (as normal modes are), any other mode must be a linear combination of these three.

One type of oscillation that obviously repeats itself is one in which all three masses move in and out relative to the center of mass with equal amplitudes and in phase with one another, as shown in Figure 13.10(a). This is often called the "breathing" mode: It is as though the system were breathing in and out. This type of oscillation clearly satisfies Eqs. XXX; the CM of the system, which is 2/3 of the way from each vertex on a line from that vertex to the midpoint of the opposite side, remains at rest, so there is never a net linear momentum. There is also no *angular* momentum of the system about the CM; this is important, because
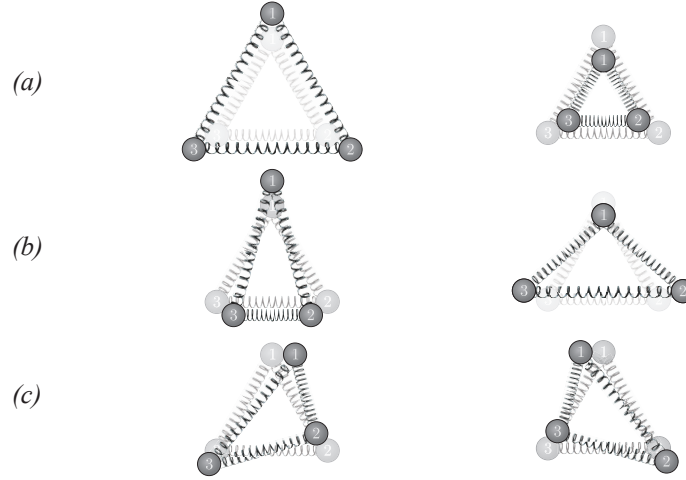
FIGURE 13.10

the only degree of freedom allowed to have a net angular momentum is the rotational degree of freedom.

Again, let the mass at the upper vertex in the diagram be designated No. 1, and the masses at the lower right and the lower left be No. 2 and No. 3, respectively. Then as the system breathes in and out, A_{1x} is always zero, and also $A_{3x} = -A_{2x}$, satisfying the first of Eqs. XXX; also $A_{2y} + A_{3y} = 2A_{2y} = -A_{1y}$, satisfying the second of Eqs. XXX. The first normal mode can therefore be assigned the matrix

$$E(1) = \begin{pmatrix} 0 & , & A_{1y} \\ A_{2x} & , & A_{2y} \\ A_{3x} & , & A_{3y} \end{pmatrix} = \begin{pmatrix} 0 & , & 1 \\ \sqrt{3}/2 & , & -1/2 \\ -\sqrt{3}/2 & , & -1/2 \end{pmatrix} A_{1y} \tag{13.80}$$

in terms of A_{1y} alone. For convenience we can drop the A_{1y} at the end, keeping only the relative numerical amplitudes.

Now let us try to find a *second* normal mode in which mass No. 1 also moves purely in the y direction. If one exists, then it must still be true that $A_{3x} = -A_{2x}$ and $A_{2y} + A_{3y} = 2A_{2y} = -A_{1y}$. Therefore the second normal mode should take the form

$$E(2) = \begin{pmatrix} 0 & , & 1 \\ a & , & -1/2 \\ -a & , & -1/2 \end{pmatrix} \tag{13.81}$$

times the actual amplitude A_{1y} in this mode. Here a could be any number. However, we also require that the second normal mode be *normal* to the first normal mode, so the matrix product

$$\begin{pmatrix} 0 , & 1 \\ a , & -1/2 \\ -a , & -1/2 \end{pmatrix} \left(0,\ 1\ ;\ \ \sqrt{3}/2,\ -1/2\ ;\ \ -\sqrt{3}/2,\ -1/2 \right) = 0 \tag{13.82}$$

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Multiplying through, component by component, we find

$$1 + (\sqrt{3}/2)a + 1/4 + (\sqrt{3}/2)a + 1/4 = 0, \tag{13.83}$$

whose solution is $a = -\sqrt{3}/2$. The second normal mode is therefore

$$E(2) = \begin{pmatrix} 0\,, & 1 \\ -\sqrt{3}/2\,, & -1/2 \\ \sqrt{3}/2\,, & -1/2 \end{pmatrix} \tag{13.84}$$

which is sketched in Figure 13.10(b). It has the same frequency $\sqrt{3k/m}$, and it is normal to the breathing mode.

Finally, we need a third normal mode $E(3)$, with the same frequency but normal to the other two modes. Let us write this mode in the totally general form

$$E(3) = \begin{pmatrix} a\,, & b \\ c\,, & d \\ e\,, & f \end{pmatrix} \tag{13.85}$$

for arbitrary values of $a, b, ..f$. We need to find five independent equations to determine all ratios of these six numbers; we cannot determine the values themselves, because these must be free to account for particular initial conditions.

First, we know that for vibrational modes the CM must stay at rest in both the x and y directions. This gives

(1) $a + c + e = 0$ $\tag{13.86}$

(2) $b + d + f = 0.$ $\tag{13.87}$

Also, the new mode must be normal to each of the first two modes. Using matrix multiplication $E(3) \times E(1) = 0$ and $E(3) \times E(2) = 0$ give

(3) $b + \sqrt{3}/2(c - e) - (1/2)(d + f) = 0$ $\tag{13.88}$

(4) $b - \sqrt{3}/2(c - e) - (1/2)(d + f) = 0$ $\tag{13.89}$

Subtracting and adding these last two equations gives $e = c$ and $2b = d + f$. So now using (1) we learn that $a = -2c$, and combining (2) with $2b = d + f$ we learn that $b = 0$ and $f = -d$. We are free to choose initial conditions such that $a = 1$, which normalizes to unity the amplitude of mass No. 1. So now only d (and $f = -d$) remains to be determined.

As the fifth equation we can require that the angular momentum of the system be zero. That means that at all times the sum over all three masses of the product of their displacements and their perpendicular lever arms remain zero, where the lever arms are measured from the CM of the system, 2/3 of the way from each vertex of the triangle to the center of the opposite side.

Let ℓ be the length of each side of the triangle in equilibrium; then the distance from a vertex to the middle of the opposite side is $\ell \cos 30^o = (\sqrt{3}/2)\ell$, and so the distance from a vertex to the CM of the system is $(2/3)(\sqrt{3}/2)\ell = \ell/\sqrt{3}$. Therefore for mass No. 1 the product of its displacement times the perpendicular lever arm is $a \times \ell/\sqrt{3} = \ell/\sqrt{3}$, in the clockwise direction, which we take to be the positive direction. Adding in these products for each displacement component of masses Nos. 2 and 3, the sum of products is

$$\ell \left(\frac{1}{\sqrt{3}} - \frac{1}{2\sqrt{3}}|c| - \frac{1}{2}|d| - \frac{1}{2\sqrt{3}}|e| + \frac{1}{2}|f| \right) = 0. \tag{13.90}$$

Substituting in the known values of c, e, the fact that $f = -d$, and solving for d, we find that $d = \sqrt{3}/2$. So finally the third normal mode is

$$E(3) = \begin{pmatrix} 1 & , & 0 \\ -1/2 & , & \sqrt{3}/2 \\ -1/2 & , & -\sqrt{3}/2 \end{pmatrix}. \tag{13.91}$$

which is illustrated in Figure 13.10(c). Figure 13.10 shows all three normal modes we have chosen. They are not unique! Any oscillation of the system, with no motion of the CM and no angular momentum about the CM, can be represented as a linear combination of these three.

Problems

PROBLEM 13-1: Two blocks, of masses m and M, are connected by a single spring of force-constant k. The blocks are free to slide on a frictionless table. Beginning with the Lagrangian, find the oscillation frequency of the system. Show that for the special case $M = m$, the frequency is what you would expect when the center of the spring remains at rest.

PROBLEM 13-2: Two blocks, of different masses m and M, are connected together linearly by three springs of equal force-constants k. The outer springs are also attached to stationary walls, while the middle spring connects the two masses. Find the normal mode frequencies and the corresponding normal mode eigenvectors.

PROBLEM 13-3: Reconsider the problem of two equal-mass blocks and three springs, in a straight line with the outer springs attached to stationary walls. Now suppose all the springs have different force constants. Find the eigenfrequencies and eigenvectors.

PROBLEM 13-4: A hypothetical linear molecule of four atoms is free to move in three dimensions. How many degrees of freedom are there? How many translational modes? How many rotational modes? How many vibrational modes? Then suppose instead that the four atoms are all in the same plane but not lined up, still free to move in three dimensions. How many degrees of freedom are there in this case, and how many are there of each kind of translational, rotational, and vibrational modes?

PROBLEM 13-5: Find the normal modes of oscillation (both eigenfrequencies and eigenvectors) for small-amplitude motions of a double pendulum (a lower mass m hanging from an upper mass M) where the pendulum lengths are equal. Suppose $m \neq M$. In that case, show that if one of the two masses is initially displaced while the other is at its equilibrium position, and both are then released, energy is transferred from one to the other and then back again. How does the period of this exchange depend upon the relative values of m and M?

PROBLEM 13-6: A uniform horizontal rod of mass m and length ℓ is supported against gravity by two identical springs, one at each end of the rod. Assuming the motion is confined to the vertical plane, find the normal modes and frequencies of the system. Then find the motion in case just one end of the rod is displaced from equilibrium and released from rest.

PROBLEM 13-7: The voltage across a capacitor is $V_C = q/C$, where C is the capacitance and q is the charge on the capacitor. The voltage across an inductor is $V_L = dI/dt$, where L is the inductance and I is the current through the inductor. A wire attached to a capacitor whose charge is changing carries a current $I = dq/dt$. The net voltage drop around any closed circuit is zero, so a simple electrical L, C circuit obeys $L\ddot{q} + q/C = 0$, and so oscillates with frequency $\omega = 1/\sqrt{LC}$. Find the normal mode oscillation frequencies and eigenvectors of each of the two-loop circuits shown below.
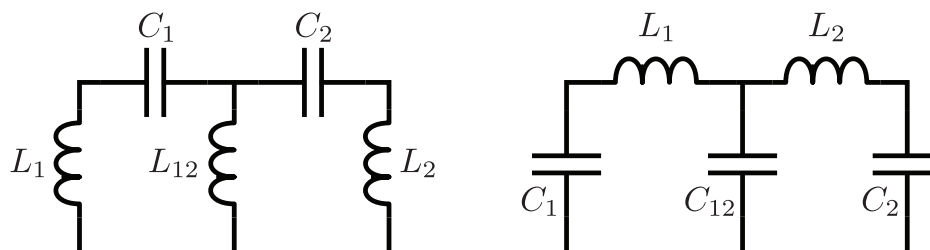
FIGURE 13.11

PROBLEM 13-8: The voltage across a capacitor is $V_C = q/C$, where C is the capacitance and q is the charge on the capacitor. The voltage across an inductor is $V_L = dI/dt$, where L is the inductance and I is the current through the inductor. A wire attached to a capacitor whose charge is changing carries a current $I = dq/dt$. The net voltage drop around any closed circuit is zero, so a simple electrical L, C circuit obeys $L\ddot{q} + q/C = 0$, and so oscillates with frequency $\omega = 1/\sqrt{LC}$. Find the normal mode oscillation frequencies and eigenvectors of the circuit shown below.

PROBLEM 13-9: A block of mass M can move without friction on a horizontal rail. A simple pendulum of mass m and length ℓ hangs from the block. Find the normal modes of oscillation, including the oscillation frequencies and relative amplitudes.

PROBLEM 13-10: A block of mass M can move without friction on a horizontal rail. A horizontal spring of force-constant k connects one end of the block to a stationary wall. A simple pendulum of mass m and length ℓ hangs from the block. Find the normal modes of oscillation of the system, including the oscillation frequencies and relative amplitudes.

PROBLEM 13-11: In Example 14-5 three degenerate normal modes were derived for the case of three equal masses at the vertices of an equilateral triangle, where the springs form the sides of the triangle. Any other oscillation in which the CM remains at rest and the system has no angular momentum must be a linear combination of these three modes. In particular, consider an oscillation identical to that of the second normal mode of Example 14-5, except that it has been rotated to the right by 120°, so for example mass No. 2 at the lower right now oscillates directly toward and away from the CM, rather than mass No. 1. By symmetry with the second normal mode, this mode should be possible, and should have the same frequency. Find the linear combination of the normal modes of Example 14-5 which is equal to the oscillation
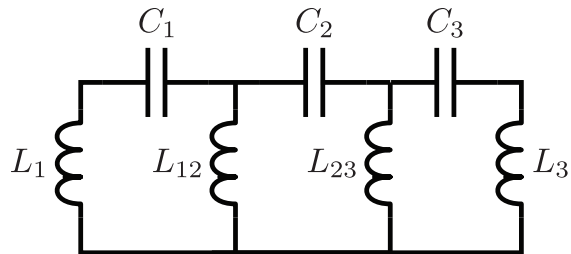
FIGURE 13.12

described above.

PROBLEM 13-12: A molecule is modeled by three atoms of equal mass at the vertices of a 45° right triangle connected by springs of equal force constant. Find a complete set of normal modes of oscillation.

Aleksandr Lyapunov (1857-1918)

Life of ?