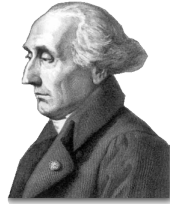


Rutherford Scattering

Wednesday, 16 October 2013

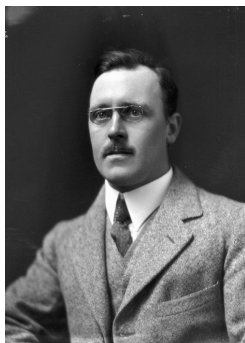
Rutherford scattering is an example of orbital motion in a repulsive $1/r$ potential, so we can carry over all the theory we have developed for planetary motion.

Physics 111



Ernest Rutherford was studying alpha radiation (which we have come to understand is the emission of a helium nucleus comprising two protons and two neutrons when an unstable heavy nucleus decays) when an enterprising undergraduate, Ernest Marsden, asked for a research position. Rutherford paired him with graduate student Hans Geiger and took advantage of Marsden's young eyes to look for the faint flashes of light from a phosphor screen caused by incident alpha particles. To keep Marsden busy, Rutherford told him to look for scattering of alpha particles from a gold foil in the backward direction, suspecting that it would take Marsden some time to be convinced that nothing was happening. After all, the prevailing view of the structure of matter was J.J. Thomson's plum-pudding model, in which electrons were the low-mass plums swimming in a sea of positive pudding. To Rutherford's astonishment, Marsden was fairly quickly able to observe back-scattered alpha particles.

It was quite the most incredible event that has ever happened to me in my life. It was almost as incredible as if you fired a 15-inch shell at a piece of tissue paper and it came back and hit you.



(a) Ernest Marsden, 1921



(b) Ernest Rutherford and Hans Geiger, 1913

Figure 1: The discoverers of the nucleus, somewhat after the 1909 gold-foil experiment.

Plan of attack

An alpha particle (mass m , charge $Z'e$, speed v_0) is incident at impact parameter b on a (gold) nucleus of charge Ze at rest in the lab frame.

1. Use μ , v_0 , b , Z , and Z' to find p_θ and E .
2. Solve $u(\theta)$ for $u \rightarrow 0$ (i.e., $r \rightarrow \infty$), which should give two angles.
3. Find how the range of b from b to $b + db$ maps into scattering angle Θ to $\Theta + d\Theta$.
4. The differential cross section is then defined by $2\pi b db = \sigma(\Theta) d\Omega = \sigma(\Theta) 2\pi \sin\Theta d\Theta$.

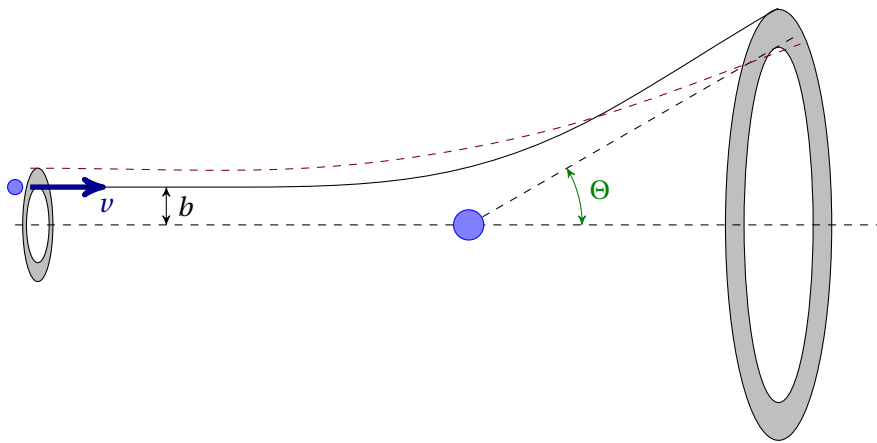


Figure 2: Scattering geometry

Attack!

Far from the nucleus, we may safely approximate that the alpha particle moves freely with energy and angular momentum given by

$$E_L = \frac{1}{2} m v_0^2 \quad \text{and} \quad p_\theta = m b v_0$$

in the lab frame. In the center of mass frame, however, the nucleus moves to the left while the alpha particle moves to the right and the total energy and angular momentum are given by

$$E = \frac{1}{2} \mu v_0^2 \quad \text{and} \quad p_\theta = \mu b v_0$$

We will solve the problem in the center of mass frame, and then worry about what implication for the scattering angle it might have when we transform back to the lab.

As it approaches the nucleus (really, after it has penetrated most of the electron cloud surrounding the nucleus), the alpha particle is subject to a potential of interaction given by

$$U(r) = \frac{ZZ'e^2}{4\pi\epsilon_0 r} = \frac{k}{r} = ku$$

where $u = 1/r$ and $k = ZZ'e^2/4\pi\epsilon_0$. For an α particle scattering off a gold nucleus, $Z = 2$ and $Z' = 79$.

From the previous notes, we had

$$\mp \frac{du}{d\theta} = \frac{\mu}{p_\theta} \sqrt{\frac{2}{\mu}(E - ku) - \frac{p_\theta^2}{\mu^2} u^2}$$

Squaring this expression and rearranging slightly, we get

$$\left(\frac{p_\theta}{\mu}\right)^2 (u'^2 + u^2) = \frac{2}{\mu}(E - ku)$$

where $u' = \frac{du}{d\theta}$. Differentiating with respect to θ yields

$$2u'u'' + 2uu' = -\frac{2\mu}{p_\theta^2} ku' \quad \Rightarrow \quad u'' + u = -\frac{\mu k}{p_\theta^2}$$

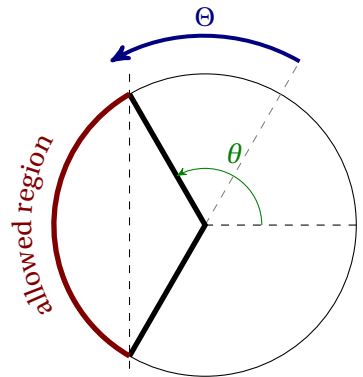
A possible solution to this equation is

$$u = -\frac{\mu k}{p_\theta^2}(1 + \epsilon \cos\theta)$$

where I have selected $\theta = 0$ as the symmetry line and expressed the amplitude of the sinusoidal portion in terms of the eccentricity, ϵ .

Now, $u = 1/r$ is necessarily positive, which means that we must have $\epsilon > 1$ and $\epsilon \cos\theta < 1$. The asymptotes are the values of θ at which r diverges, which means $u \rightarrow 0$. These occur at $\epsilon \cos\theta = -1$, or $\theta = \cos^{-1}(-1/\epsilon)$. The angle of deviation of the beam, Θ , is thus given by

$$\Theta = 2 \left[\cos^{-1}(-1/\epsilon) - \frac{\pi}{2} \right]$$



Rearranging slightly, we have

$$\begin{aligned}\frac{\Theta}{2} + \frac{\pi}{2} &= \cos^{-1}(-1/\epsilon) \\ \cos\left(\frac{\Theta}{2} + \frac{\pi}{2}\right) &= -\frac{1}{\epsilon} \\ \sin(\Theta/2) &= -1/\epsilon\end{aligned}$$

We have shown that

$$\epsilon^2 = 1 + \frac{2p_\theta^2 E}{\mu k^2} \approx 1 + \left(\frac{2Eb}{k}\right)^2$$

Combining these relations to eliminate the eccentricity gives

$$\frac{1}{\sin^2(\Theta/2)} = 1 + \left(\frac{2Eb}{k}\right)^2$$

Taking now the differential of each side gives

$$-\frac{\cos(\Theta/2)}{\sin^3(\Theta/2)} d\Theta = \left(\frac{2E}{k}\right)^2 2b db$$

As described in point 4 of the plan of attack, the differential cross section is given by

$$\sigma(\Theta) = \frac{2\pi b db}{2\pi \sin \Theta} = \frac{b}{2 \sin(\Theta/2) \cos(\Theta/2)} \frac{db}{d\Theta} = -\frac{1}{2} \left(\frac{k}{2E}\right)^2 \frac{1}{\sin^4(\Theta/2)}$$

or

$$\boxed{\sigma(\Theta) = \left(\frac{ZZ'e^2}{8\pi\epsilon_0 E}\right)^2 \csc^4(\Theta/2)} \quad (1)$$

For those of you who studied Rutherford scattering in Physics 54, this is the expression you put to the test. If you are somewhat discomfited by the strong singularity as $\Theta \rightarrow 0$, fret not. It says that the total cross section,

$$\sigma = \int \sigma(\Theta) d\Omega$$

diverges. This merely reports that there is no value of impact parameter b for which there is no deviation, since the Coulomb interaction does not have a hard cutoff. From a practical standpoint, however, this inconvenience of little consequence. We measure for appreciable angles of scattering and compare the relative intensity at these angles to Eq. (1).

Transformation Back to the Lab Frame

We have been working in the center of mass frame and have found the differential cross section for scattering, $\sigma(\Theta)$, in this frame, where Θ is the angle of deviation of the incident beam. Of course, in the center of mass frame, the two particles have to come in with equal and opposite momenta, and depart with the same. So, the “target” particle also rebounds at Θ .

With respect to the lab (the rest frame of the target particle, m_2), the center of mass moves at

$$\mathbf{V} = \mathbf{v}_0 \frac{m_1}{m_1 + m_2} = \mathbf{v}_0 \frac{\mu}{m_2}$$

where \mathbf{v}_0 is the velocity of the incident particle in the lab frame. It will turn out to simplify the algebra to define $\rho = m_1 / m_2$, in terms of which we have

$$\mathbf{V} = \mathbf{v}_0 \frac{\rho}{1 + \rho}$$

The incident particle in the center of mass frame, therefore, has velocity

$$\mathbf{v}'_1 = \mathbf{v}_0 - \mathbf{V} = \frac{\mathbf{v}_0}{1 + \rho}$$

We assume elastic scattering of the α particle, so that the outgoing velocity in the center of mass frame has magnitude $|\mathbf{v}'_1|$, but is directed away from the incident direction by angle Θ , as illustrated in Fig. 3a. Transforming the outgoing velocity back to the lab frame by adding \mathbf{V} as illustrated in Fig. 3b yields outgoing velocity \mathbf{v}_1 .

Analyzing the triangle formed by the velocity vectors in the lab frame, we have

$$v_1 \sin \vartheta = v'_1 \sin \Theta = \frac{v_0}{1 + \rho} \sin \Theta \quad (2)$$

$$v_1 \cos \vartheta = v'_1 \cos \Theta + V = v_0 \left(\frac{\cos \Theta}{1 + \rho} + \frac{\rho}{1 + \rho} \right) \quad (3)$$

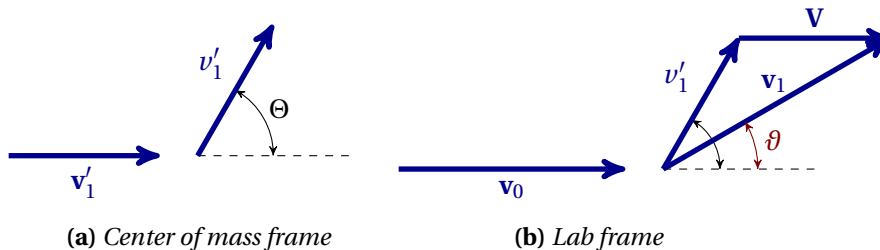


Figure 3: The collision as viewed in the center of mass and lab frames.

These two equations suffice to find the scattering angle in the laboratory, ϑ , in terms of the scattering angle in the center of mass frame, Θ . However, we need more than this: we want the differential scattering cross section in the lab frame. The particles that scatter into the angle range Θ to $\Theta + d\Theta$ is $\sigma(\Theta) 2\pi \sin\Theta d\Theta$. These same particles scatter into the solid angle $\Sigma(\vartheta) 2\pi \sin\vartheta d\vartheta$ in the lab. Setting these two expressions equal and solving for $\Sigma(\vartheta)$, we have

$$\Sigma(\vartheta) = \sigma(\Theta) \frac{\sin\Theta d\Theta}{\sin\vartheta d\vartheta} = \sigma(\Theta) \frac{d(\cos\Theta)}{d(\cos\vartheta)} \quad (4)$$

Hence, we need to compute $\cos\vartheta$, which we can do in a couple of steps. Then we can finish the computation by taking $d(\cos\vartheta)$.

First, divide Eq. (2) by Eq. (3) to get

$$\tan\vartheta = \frac{\sin\Theta}{\cos\Theta + \rho} \quad (5)$$

Next, square these same two equations and add them together to get

$$v_1^2 = \left(\frac{v_0}{1+\rho}\right)^2 [\sin^2\Theta + (\cos\Theta + \rho)^2] = \left(\frac{v_0}{1+\rho}\right)^2 [1 + 2\rho \cos\Theta + \rho^2] \quad (6)$$

Taking the square root of this equation and substituting it into Eq. (2) gives

$$\frac{v_0}{1+\rho} \sqrt{1 + 2\rho \cos\Theta + \rho^2} \sin\vartheta = \frac{v_0}{1+\rho} \sin\Theta$$

so

$$\sin\vartheta = \frac{\sin\Theta}{\sqrt{1 + 2\rho \cos\Theta + \rho^2}} \quad (7)$$

Dividing Eq. (2) by Eq. (5) gives

$$\cos\vartheta = \frac{\cos\Theta + \rho}{\sqrt{1 + 2\rho \cos\Theta + \rho^2}} \quad (8)$$

Taking the differential of both sides, we have

$$\begin{aligned} -\sin\vartheta d\vartheta &= \frac{-\sin\Theta d\Theta}{\sqrt{1 + 2\rho \cos\Theta + \rho^2}} + \frac{(\cos\Theta + \rho)(-\frac{1}{2})(-2\rho \sin\Theta) d\Theta}{(1 + 2\rho \cos\Theta + \rho^2)^{3/2}} \\ \sin\vartheta d\vartheta &= \frac{1 + 2\rho \cos\Theta + \rho^2 - \rho \cos\Theta - \rho^2}{(1 + 2\rho \cos\Theta + \rho^2)^{3/2}} \sin\Theta d\Theta \\ &= \frac{1 + \rho \cos\Theta}{(1 + 2\rho \cos\Theta + \rho^2)^{3/2}} \sin\Theta d\Theta \end{aligned}$$

Combining this result with Eq. (4) yields the desired relation for the scattering cross section in the laboratory frame,

$$\Sigma(\vartheta) = \left(\frac{ZZ' e^2}{8\pi\epsilon_0 E}\right)^2 \csc^4\left(\frac{\Theta}{2}\right) \left[\frac{(1 + 2\rho \cos\Theta + \rho^2)^{3/2}}{1 + \rho \cos\Theta}\right] \quad (9)$$

where ϑ and Θ are related by Eq. (5) and $\rho = m_1/m_2$. In the case of Rutherford scattering, the correction term in brackets is rather small, but if the masses of the two particles are close, it can be significant.