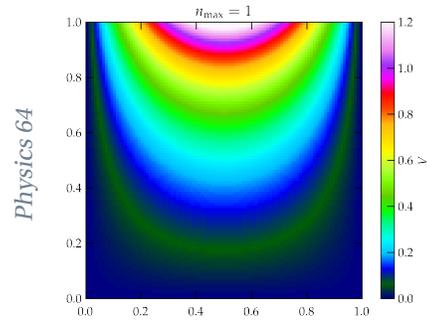


Problem Set 4 — Solution

Monday, 16 February 2026



Problem 1 – Visiting τ Ceti In *Project Hail Mary*, by Andy Weir, a desperate Earth confronts a dimming Sun caused by waves of microscopic creatures (dubbed Astrophage) that absorb energy at the solar surface and then speed off to use the carbon dioxide in the Venusian atmosphere to procreate. In a heart-warming nod to international cooperation, scientists and engineers develop the ability to harness the energy stored in Astrophage to power a rocket to send three astronauts to visit the only star in the nearby galaxy that isn't experiencing exponential dimming, in hopes of learning how to stop the Astrophage catastrophe.

The ship accelerates at a constant rate (I'll call it g) for half the voyage to the star τ Ceti, then flips around and accelerates at g (to slow down), arriving in the neighborhood of τ Ceti approximately at rest with respect to the Sun and τ Ceti. Let's work out how to compare time on the astronauts' clocks to time on Earth clocks.

- (a) Recall that “moving clocks run slow” by $\sqrt{1 - v^2/c^2}$. If the ship happens to be moving at speed v with respect to the Earth at some point in the voyage, what is the relationship between $d\tau$ (a small interval of time on the ship's clock) and dt (the corresponding small interval of time on Earth clocks)?
- (b) Because the ship accelerates, it is not in an inertial frame and we cannot use $g = \frac{dv}{d\tau}$. However, if you imagine an inertial frame S' that happens to be at rest with respect to the ship at time τ on the ship (and t on Earth), then for the next little while, the ship's behavior in S' is perfectly classical, since nothing is moving fast at all. That is, in S' there would be no relativistic funny business until the ship's speed in S' gets to be an appreciable fraction of the speed of light.

Let the ship accelerate for $d\tau$ at g . At the end of $d\tau$ we know how fast the ship is moving in S' and we know how fast S' is moving with respect to the Earth. Use the velocity transformation equation,

$$v = \frac{V + v'}{1 + Vv'/c^2}$$

to figure out how much the ship's velocity changes in the Earth frame.

- (c) You should now have a differential equation relating the change in the ship's velocity in Earth's frame to the change in proper time aboard the ship. Combining this equation with the time dilation equation will allow you to determine a relationship between time elapsed since departure aboard the ship and time elapsed since departure on the Earth for the first phase of the journey. *Hint:* At some point you may find it helpful to define $v/c = \tanh \phi$ (ϕ is known as the *velocity parameter*).
- (d) The second half of the journey just mirrors the first half. If the ship accelerated at $g = 15 \text{ m/s}^2$ and it arrived at τ Ceti at the end of 3 years and 9 months of travel, how far did the ship travel from launch to arrival?
- (e) How much time elapsed on Earth between the ship's launch at its arrival at τ Ceti?

- (a) The ship's clock runs slow according to folks on Earth:

$$dt = \frac{d\tau}{\sqrt{1 - v^2/c^2}} \quad (1)$$

- (b) The small change in the velocity of the rocket in S' is just $v' = g d\tau$. The corresponding change in velocity in Earth's frame that takes place while the ship clock goes from τ to $\tau + d\tau$ is

$$\begin{aligned} dv &= \frac{V + g d\tau}{1 + Vg d\tau/c^2} - V \\ &= (V + g d\tau)(1 - Vg d\tau/c^2 + \dots) - V \\ &= V + g d\tau - V^2 g d\tau/c^2 + \mathcal{O}(d\tau^2) - V \\ dv &= g d\tau \left(1 - \frac{V^2}{c^2}\right) \end{aligned}$$

- (c) Note that the rocket was moving at $v(t) = V$ at t and is moving at $v(t + dt) = V + dv$ at $t + dt$. So, the change in the rocket's velocity with time is

$$dv = g \left(1 - \frac{v^2}{c^2}\right) d\tau = g \left(1 - \frac{v^2}{c^2}\right) dt \sqrt{1 - v^2/c^2} = g \left(1 - \frac{v^2}{c^2}\right)^{3/2} dt$$

Separating, we get

$$\frac{dv}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} = g dt \quad (2)$$

To integrate, we need to make a substitution to simplify the denominator. Let

$$\frac{v}{c} = \tanh \phi \quad \Rightarrow \quad dv = c \operatorname{sech}^2 \phi d\phi \quad (3)$$

Substituting this result into Eq. (2), we get

$$\frac{c \operatorname{sech}^2 \phi d\phi}{\operatorname{sech}^3 \phi} = \boxed{c \cosh \phi d\phi = g dt} \quad (4)$$

But from Eq. (1),

$$dt = \frac{d\tau}{\sqrt{1 - v^2/c^2}} = \frac{d\tau}{\sqrt{1 - \tanh^2 \phi}} = \frac{d\tau}{\operatorname{sech} \phi} = \cosh \phi d\tau \quad (5)$$

Therefore,

$$\cosh \phi d\phi = \frac{g}{c} dt = \frac{g}{c} \cosh \phi d\tau \quad \Rightarrow \quad d\phi = \frac{g}{c} d\tau \quad \Rightarrow \quad \boxed{\phi = \frac{g\tau}{c}}$$

Very interesting: the variable ϕ grows linearly with proper time τ , just the way a nonrelativistic velocity would grow with time at constant acceleration. The quantity ϕ is called the *velocity parameter*. Since the rocket velocity is $v = c \tanh \phi = c \tanh(g\tau/c)$, we see that the longer the rocket accelerates, the larger the argument of the hyperbolic tangent, but the velocity approaches c slower and slower.

Substituting into Eq. (4), we have

$$\begin{aligned}
 g dt &= c \cosh(g\tau/c) \frac{g}{c} d\tau \\
 \int_0^t dt' &= \int_0^\tau \cosh(g\tau'/c) d\tau' \\
 t &= \frac{c}{g} \sinh(g\tau/c)
 \end{aligned} \tag{6}$$

Equation (6) is the relation we seek between the time elapsed on the ship, τ , and the time elapsed on Earth, t . If $g = 15 \text{ m/s}^2$ and $\tau = \frac{1}{2}(3.75 \text{ y})$, then the time to cover half the distance to τ Ceti is

$$t_{1/2} = \frac{3 \times 10^8 \text{ m/s}}{15 \text{ m/s}^2} \sinh \left[\frac{(15 \text{ m/s}^2)(1.875 \text{ y})}{3 \times 10^8 \text{ m/s}} \right] = (2 \times 10^7 \text{ s}) \sinh [2.9625] = 6.11 \text{ y}$$

So, the journey to τ Ceti takes 12.2y, according to people who remain on Earth.

(d) How far did the ship travel? We have to integrate:

$$\begin{aligned}
 x_{1/2} &= \int_0^t v(t') dt' = \int_0^{\tau/2} c \tanh(\phi) [\cosh \phi d\tau'] \\
 &= \int_0^{\phi_{\max}} c \sinh \phi \frac{c}{g} d\phi = \frac{c^2}{g} (\cosh \phi_{\max} - 1) \\
 &= \frac{c^2}{g} \left[\cosh \left(\frac{g\tau}{2c} \right) - 1 \right]
 \end{aligned}$$

We already evaluated the maximum velocity parameter and found 2.9625. So,

$$x = 2x_{1/2} = \frac{2c^2}{g} [\cosh(2.9625) - 1] = 11 \text{ cy}$$

Problem 2 – Symmetries (Nearing 5.11) Represent a function f on the interval $-L < x < L$ by a Fourier series using periodic boundary conditions

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{n\pi i x/L}$$

- If the function f is odd, prove that for all n , $a_{-n} = -a_n$.
- If the function f is even, prove that all $a_{-n} = a_n$.
- If the function f is real, prove that all $a_{-n} = a_n^*$.
- If the function is both real and even, characterize a_n .
- If the function is imaginary and odd, characterize a_n .

Let's first confirm that the functions $u_n(x) = e^{n\pi i x/L}$ are orthogonal on the interval $-L < x < L$:

$$\begin{aligned}\langle u_m, u_n \rangle &= \int_{-L}^L u_m^*(x) u_n(x) dx \\ &= \int_{-L}^L e^{-m\pi i x/L} e^{n\pi i x/L} dx \\ &= \int_{-L}^L e^{(n-m)\pi i x/L} dx = \begin{cases} 2L & n = m \\ \frac{L}{(n-m)\pi i} e^{(n-m)\pi i x/L} \Big|_{-L}^L = 0 & n \neq m \end{cases}\end{aligned}$$

Because they are orthogonal, we can isolate a unique coefficient a_n in the sum by taking the inner product with u_n , which will collapse the infinite sum to a single term.

(a) An odd function satisfies $f(-x) = -f(x)$; then

$$\begin{aligned}f(x) &= \sum_{n=-\infty}^{\infty} a_n e^{n\pi i x/L} \\ -f(-x) &= \sum_{n=-\infty}^{\infty} -a_n e^{n\pi i (-x)/L} \quad \text{let } n \rightarrow -m \\ &= \sum_{m=-\infty}^{\infty} -a_{-m} e^{m\pi i x/L} \quad \text{order of summation is irrelevant} \\ -f(-x) = f(x) &= \sum_{m=-\infty}^{\infty} (-a_{-m}) e^{m\pi i x/L}\end{aligned}$$

Since m is a dummy index of summation; we see that $a_n = -a_{-n}$.

(b) Even functions satisfy $f(-x) = f(x)$:

$$\begin{aligned}f(-x) &= \sum_{n=-\infty}^{\infty} a_n e^{n\pi i (-x)/L} \\ &= \sum_{n=-\infty}^{\infty} a_n e^{(-n)\pi i x/L} \\ &= \sum_{n=-\infty}^{\infty} a_{-n} e^{n\pi i x/L}\end{aligned}$$

Hence, $a_{-n} = a_n$.

(c) If $f(x)$ is real, then $f(x) = f^*(x)$:

$$\begin{aligned}f^*(x) &= \sum_{n=-\infty}^{\infty} (a_n)^* e^{n\pi (-i)x/L} \\ &= \sum_{m=-\infty}^{\infty} (a_{-m})^* e^{m\pi i x/L} = \sum_{n=-\infty}^{\infty} a_{-n}^* e^{n\pi i x/L} = \sum_{n=-\infty}^{\infty} a_n e^{n\pi i x/L}\end{aligned}$$

Hence, $a_{-n}^* = a_n$.

(d) If $f(x)$ is both real and even, then $a_{-n} = a_n$ and $a_{-n} = a_n^*$. So $a_n = a_n^*$, meaning that a_n is real and equal to a_{-n} .

(e) If $f(x)$ is both imaginary and odd, then $a_{-n} = -a_n$ (since it is odd). For $f(x)$ to be imaginary, $[f(x)]^* = -f(x)$. So,

$$\begin{aligned}\left[\sum_{n=-\infty}^{\infty} a_n e^{n\pi i x/L} \right]^* &= \sum_{n=-\infty}^{\infty} a_n^* e^{n\pi (-i)x/L} \\ &= \sum_{m=-\infty}^{\infty} a_{-m}^* e^{m\pi i x/L} = \sum_{n=-\infty}^{\infty} (-a_n) e^{n\pi i x/L}\end{aligned}$$

Therefore, $a_{-n}^* = -a_n = a_{-n}$, which means that $a_{-n} = -a_n$ is real.

Problem 3 – Parabola (10 points) Develop a Fourier series for the parabola

$$f(x) = x(L-x) \quad 0 \leq x < L$$

using functions u_n that vanish at $x = 0$ and $x = L$. Evaluate the series at $x = L/2$ to show that

$$\pi^3 = 32 \left(1 - \frac{1}{27} + \frac{1}{125} - \frac{1}{343} + \dots \right)$$

Then use Parseval's identity to develop a series representation for π^6 . That is evaluate $\langle f, f \rangle = \sum_n |a_n|^2 \langle u_n, u_n \rangle$, and rearrange as appropriate.

Trigonometric functions that vanish at $x = 0$ are sines: $u_n(x) = \sin(n\pi x/L)$. However, $f(x)$ is even with respect to $x = L/2$, whereas $\sin(n\pi x/L)$ is odd with respect to $L/2$ for even n . So, we look for coefficients a_n in

$$f(x) = \sum_{n \text{ odd}}^{\infty} a_n \sin(n\pi x/L)$$

Multiply both sides by $\sin(m\pi x/L)$ and integrate from 0 to L :

$$\int_0^L x(L-x) \sin(m\pi x/L) dx = \sum_{n=1}^{\infty} \int_0^L dx \sin(m\pi x/L) a_n \sin(n\pi x/L)$$

If $m = n$, then

$$\int_0^L \sin^2(m\pi x/L) dx = \frac{L}{2}$$

since the average value of $\sin^2 \theta$ is $\frac{1}{2}$. If $m \neq n$ (but both are odd),

$$\int_0^L \sin(m\pi x/L) \sin(n\pi x/L) dx = \int_0^L \frac{\cos[(m-n)\pi x/L] - \cos[(m+n)\pi x/L]}{2} dx = 0$$

since $m-n$ and $m+n$ are even integers and so the cosines go through an integral number of full periods between 0 and L . So,

$$\begin{aligned} a_m \frac{L}{2} &= \int_0^L x(L-x) \sin(m\pi x/L) dx \\ &= x(L-x) \frac{-\cos(m\pi x/L)}{m\pi/L} \Big|_0^L + \int_0^L (L-2x) \frac{L}{m\pi} \cos(m\pi x/L) dx \\ &= (L-2x) \frac{L^2}{m^2\pi^2} \sin(m\pi x/L) \Big|_0^L - \int_0^L -2 \frac{L^2}{m^2\pi^2} \sin(m\pi x/L) dx \\ &= 2 \left(\frac{L}{m\pi} \right)^3 [-\cos(m\pi x/L)]_0^L = 2 \left(\frac{L}{m\pi} \right)^3 [1 - \cos(m\pi)] \end{aligned}$$

When m is odd, the term in brackets is 2, so

$$x(L-x) = \sum_{n=0}^{\infty} \frac{8L^2}{(2n+1)^3\pi^3} \sin[(2n+1)\pi x/L]$$

At $x = L/2$, this expression gives

$$\left(\frac{L}{2} \right)^2 = \frac{8L^2}{\pi^3} \sum_{n=0}^{\infty} \frac{\sin[(2n+1)\pi/2]}{(2n+1)^3}$$

or

$$\pi^3 = 32 \left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \dots \right)$$

Parseval's identity is $\langle f, f \rangle = \sum_n |a_n|^2 \langle u_n, u_n \rangle$. So

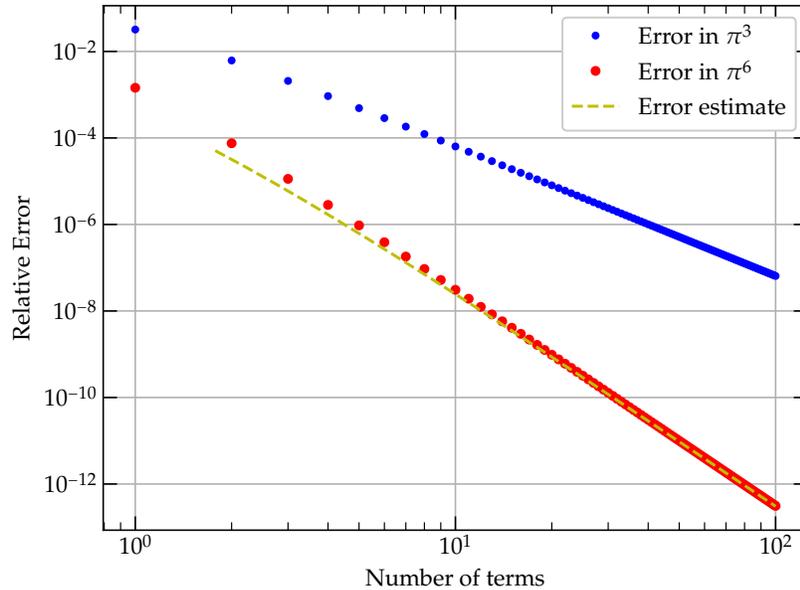
$$\begin{aligned} \int_0^L x^2(L-x)^2 dx &= \frac{64L^4}{\pi^6} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} \int_0^L \sin^2[(2n+1)\pi x/L] dx \\ \left[\frac{L^2 x^3}{3} - \frac{2Lx^4}{4} + \frac{x^5}{5} \right]_0^L &= \frac{32L^5}{\pi^6} \left(1 + \frac{1}{3^6} + \frac{1}{5^6} + \dots \right) \\ \frac{L^5}{30} &= \frac{32L^5}{\pi^6} \left(1 + \frac{1}{3^6} + \frac{1}{5^6} + \dots \right) \\ \pi^6 &= 960 \left(1 + \frac{1}{3^6} + \frac{1}{5^6} + \dots \right) \end{aligned}$$

This series converges *very rapidly*. If we sum the first N terms, the (absolute) error should be

$$\varepsilon_N = 960 \sum_{n=N}^{\infty} (2n+1)^{-6} \approx 960 \int_N^{\infty} (2x+1)^{-6} dx = 960 \left[\frac{(2x+1)^{-5}}{-5} \frac{1}{2} \right]_N^{\infty} = \frac{96}{(2N+1)^5}$$

Dividing this by the true value gives the relative error

$$\rho_N = \frac{96}{\pi^6 (2N+1)^5} \tag{7}$$



Relative error in the two series representations for powers of π , and the error estimate of Eq. (7).