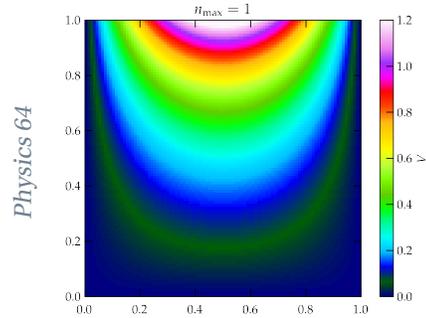


Problem Set 6 — Solution

Monday, 2 March 2026



Problem 1 – Differentiable? Show that the following functions are differentiable functions of $z = x + iy$ ($x, y \in \mathbb{R}$):

- (a) e^{ikz}
- (b) $1/z$
- (c) \sqrt{z}

For a function $f(z = x + iy) = u(x, y) + iv(x, y)$ to be differentiable, it must satisfy the Cauchy-Riemann conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad (1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (2)$$

(a) Let's first express e^{ikz} in terms of its real and imaginary parts:

$$e^{i(x+iy)} = e^{ix} e^{-y} = e^{-y} (\cos x + i \sin x)$$

which allows us to identify

$$u(x, y) = e^{-y} \cos x \quad \text{and} \quad v(x, y) = e^{-y} \sin x$$

Computing the partials, we have

$$\begin{aligned} u_x &= -e^{-y} \sin x & v_x &= e^{-y} \cos x \\ v_y &= -e^{-y} \sin x & u_y &= -e^{-y} \cos x \end{aligned}$$

which shows indeed that $u_x = v_y$ and $u_y = -v_x$, as required.

(b) For $1/z$, $\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$, so

$$u(x, y) = \frac{x}{x^2 + y^2} \quad \text{and} \quad v(x, y) = -\frac{y}{x^2 + y^2}$$

which means that

$$\begin{aligned} u_x &= \frac{y^2 - x^2}{(x^2 + y^2)^2} & v_x &= \frac{2xy}{(x^2 + y^2)^2} \\ v_y &= -\frac{x^2 - y^2}{(x^2 + y^2)^2} & u_y &= -\frac{2xy}{(x^2 + y^2)^2} \end{aligned}$$

again consistent with the Cauchy-Riemann conditions.

- (c) For the function $f(z) = \sqrt{z}$, it seems much easier to think about the behavior of this function in polar coordinates:

$$f(z) = \sqrt{re^{i\phi}} = \sqrt{r}e^{i\phi/2}$$

For the derivative of $f(z = re^{i\phi})$ to exist, we must have

$$\frac{df}{dz} = \lim_{\delta z \rightarrow 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

regardless of direction. Since $z = re^{i\phi}$, $\delta z = e^{i\phi}\delta r + rie^{i\phi}\delta\phi$, we must have

$$\frac{df}{dz} = \lim_{\delta r, \delta\phi \rightarrow 0} \frac{f(re^{i\phi} + e^{i\phi}\delta r + rie^{i\phi}\delta\phi) - f(re^{i\phi})}{e^{i\phi}(\delta r + ir\delta\phi)}$$

independent of the direction of approach. Consider first when $\delta\phi = 0$:

$$\frac{df}{dz} = \lim_{\delta r \rightarrow 0} \frac{e^{i\phi/2}\sqrt{r+\delta r} - e^{i\phi/2}\sqrt{r}}{e^{i\phi}\delta r} = e^{-i\phi/2} \frac{1}{2\sqrt{r}}$$

On the other hand, if we take $\delta r = 0$, we get

$$\frac{df}{dz} = \lim_{\delta\phi \rightarrow 0} \frac{\sqrt{r}e^{i(\phi+\delta\phi)/2} - \sqrt{r}e^{i\phi/2}}{rie^{i\phi}\delta\phi} = \frac{1}{i\sqrt{r}} \frac{i\phi/2 e^{i\phi/2}}{e^{i\phi}} = \frac{e^{-i\phi/2}}{2\sqrt{r}}$$

Since these give the same expression, both real and imaginary parts are equal and the function is differentiable, except at $r = 0$.

If you are skeptical, we can work it out from Cauchy-Riemann. Our function is $f(z) = \sqrt{z} = u + iv$. Square both sides:

$$\begin{aligned} z &= (u + iv)^2 \\ x + iy &= (u^2 - v^2) + 2uvi \end{aligned}$$

The imaginary parts must be equal, so

$$y = 2uv \quad \implies \quad v = \frac{y}{2u}$$

The real parts also must be equal, so

$$u^2 - \left(\frac{y}{2u}\right)^2 = x$$

Multiply through by u^2 to get

$$u^4 - xu^2 - \frac{y^2}{4} = 0$$

This is a quadratic equation for u^2 with solution

$$u^2 = \frac{x \pm \sqrt{x^2 + y^2}}{2}$$

Since $v^2 = u^2 - x = \frac{-x \pm \sqrt{x^2 + y^2}}{2}$,

$$\begin{aligned} u(x, y) &= \left[\frac{x \pm \sqrt{x^2 + y^2}}{2} \right]^{1/2} \\ v(x, y) &= \left[\frac{-x \pm \sqrt{x^2 + y^2}}{2} \right]^{1/2} \end{aligned}$$

Time to evaluate some partial derivatives. Except, before we do, I'm going to take the positive square root, because I need v to be a real number. Furthermore, let $r = \sqrt{x^2 + y^2}$, so that $\frac{\partial r}{\partial x} = x/r$ and $\frac{\partial r}{\partial y} = y/r$. In terms of r , then,

$$u(x, y) = \frac{1}{\sqrt{2}} \sqrt{r+x} \quad (3)$$

$$v(x, y) = \frac{1}{\sqrt{2}} \sqrt{r-x} \quad (4)$$

So,

$$u_x = \frac{\partial u}{\partial x} = \frac{1}{\sqrt{2}} \frac{1}{2} \frac{1}{\sqrt{r+x}} \left(\frac{x}{r} + 1 \right) = \frac{1}{2\sqrt{2}} \frac{\sqrt{r+x}}{r}$$

and

$$v_y = \frac{\partial v}{\partial y} = \frac{1}{\sqrt{2}} \frac{1}{2} \frac{1}{\sqrt{r-x}} \frac{y}{r}$$

which admittedly doesn't look like u_x . However, multiply numerator and denominator by $\sqrt{r+x}$:

$$v_y = \frac{1}{2\sqrt{2}} \frac{\sqrt{r+x}}{\sqrt{r^2-x^2}} \frac{y}{r} = \frac{1}{2\sqrt{2}} \frac{\sqrt{r+x}}{r} = u_x$$

So, the first of the Cauchy-Riemann conditions is satisfied. You can use the same technique to confirm the second.

Problem 2 – Sanity Check Note that

$$\frac{1}{1+x^2} = \frac{i}{2} \left[\frac{1}{x+i} - \frac{1}{x-i} \right]$$

By integrating the right-hand side, show that

$$\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$$

We know we can integrate the left-hand side using the substitution $x = \tan \phi$:

$$\int_0^1 \frac{dx}{1+x^2} = \int_0^{\pi/4} \frac{\sec^2 \phi d\phi}{\sec^2 \phi} = \pi/4$$

so, we just need to integrate the right-hand side.

$$\begin{aligned} \text{RHS} &= \frac{i}{2} [\ln(x+i) - \ln(x-i)]_0^1 \\ &= \frac{i}{2} \left[\ln\left(\frac{1+i}{i}\right) - \ln\left(\frac{1-i}{-i}\right) \right] \\ &= \frac{i}{2} \left[\ln\left(\sqrt{2}e^{-i\pi/4}\right) - \ln\left(\sqrt{2}e^{i\pi/4}\right) \right] \\ &= \frac{i}{2} \left[\frac{1}{2} \ln 2 - i\pi/4 - \frac{1}{2} \ln 2 - i\pi/4 \right] = \frac{\pi}{4} \quad \blacksquare \end{aligned}$$

Problem 3 – A complex series Show that

$$\sum_{n=1}^{\infty} \frac{i^n}{n} = i\frac{\pi}{4} - \frac{1}{2}\ln 2$$

Hint: one clever way to sum a series such as $S = 1 - \frac{1}{3} + \frac{1}{5} - \dots$ is to define

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

and note that $f(1)$ is the desired sum S . On differentiating $f(x)$, we get

$$f'(x) = 1 - x^2 + x^4 - \dots$$

which is a geometric series that you can sum in closed form. On integrating,

$$\int_a^1 f'(x) dx = f(1) - f(a)$$

for a skillfully chosen a , you can show that $S = \frac{\pi}{4}$.

We would like to compute

$$S = \sum_{n=1}^{\infty} \frac{i^n}{n}$$

which looks like a geometric series, except the terms get divided by n . So, let us define

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$$

which means that

$$f'(z) = \sum_{n=1}^{\infty} z^{n-1} = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

If we now integrate,

$$f(i) - f(a) = \int_a^i \frac{dz}{z-1} = -\ln\left(\frac{i-1}{a-1}\right) = -\ln\left(\frac{1-i}{1-a}\right)$$

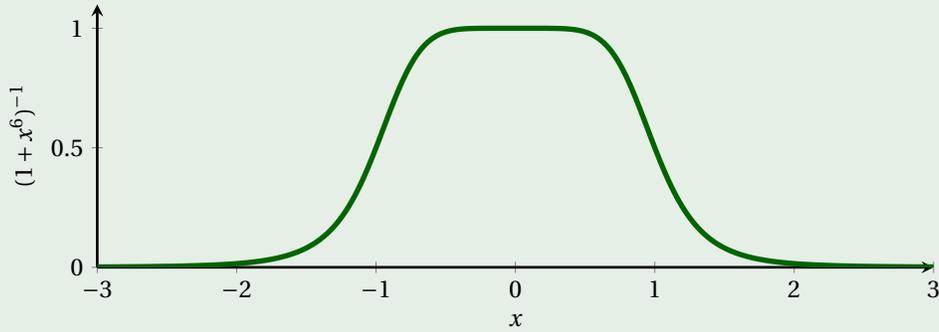
It would be nice if we picked a so that $f(a) = 0$, which means $a = 0$. Then

$$S = f(i) = -\ln(1-i) = -\ln[\sqrt{2}e^{-i\pi/4}] = -\left[\frac{1}{2}\ln 2 - i\pi/4\right] = i\frac{\pi}{4} - \frac{1}{2}\ln 2 \quad \blacksquare$$

Problem 4 – A Curious Integral Use the calculus of residues to evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{1}{1+x^6} dx$$

for which the integrand is shown below.



Check your answer by using `scipy.integrate.quad` to perform the integral numerically.

The integral runs from $x = -\infty$ to $x = \infty$ along the real axis. To use the residue theorem, we need to close the contour on the complex plane, which we do using a semicircle of radius R running from $\phi = 0$ to $\phi = \pi$. As we showed in class, the contribution along this path goes to zero as $R \rightarrow \infty$:

$$\begin{aligned} \int_0^\pi \frac{1}{1 + R^6 e^{6i\phi}} R i e^{i\phi} d\phi &= \frac{R}{R^6} \int_0^\pi \frac{i e^{i\phi}}{e^{6i\phi} + R^{-6}} d\phi \\ &= R^{-5} i \int_0^\pi \frac{i e^{i\phi}}{e^{6i\phi} + R^{-6}} d\phi \end{aligned}$$

The integrand is bounded, so the integral goes to zero as $R \rightarrow \infty$.

The residue theorem holds that the value of the integral is equal to $2\pi i$ times the sum of the enclosed residues. The residues occur at the zeros of $1 + z^6$ or

$$z^6 = e^{i\pi + 2\pi ni} \implies z = e^{i\pi/6 + \pi ni/3}$$

There are three residues in the upper half-plane, at $z \in e^{i\pi/6}, e^{i\pi/2}, e^{5i\pi/6}$. We can evaluate the residue at z_0 by finding

$$\lim_{z \rightarrow z_0} \frac{(z - z_0)}{1 + z^6} = \frac{1}{6z^5}$$

by l'Hôpital's rule. By the residue theorem, therefore,

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^6} dx = 2\pi i \sum_{n=0}^2 \frac{1}{6e^{i(1+2n)\pi/6}} = \frac{i\pi}{3} (e^{-i\pi/6} + e^{-i\pi/2} + e^{-5i\pi/6}) = \frac{i\pi}{3} \left(-\frac{i}{2} - i - \frac{i}{2} \right) = \frac{2\pi}{3}$$

Numerical check

Because the value of the integrand varies over several orders of magnitude, to achieve high accuracy it is necessary to break the range up into chunks over which the integral can be evaluated to avoid too much variation in scale. Since the integrand is symmetric, I will integrate from zero and double the result:

```
1 import numpy as np
2 from scipy.integrate import quad
3 intervals = (0, 1, 2, 5, 10, 25, 100, 1000)
4 f = lambda x: 1 / (1 + x**6)
5 subsums = np.array([quad(f, intervals[n], intervals[n+1]) for n in
6 range(len(intervals)-1)])
7 value = 2 * np.sum(res[:,0])
8 print(f"{value - 2 * np.pi / 3:.3g}")
9 -4.44e-16
```