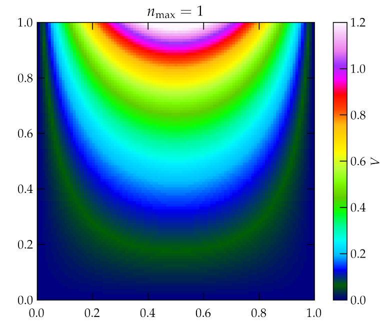
Problem Set 7 — Solution

Monday, 9 March 2026

Physics 64



Problem 1 – General properties of Fourier transforms and convolutions Assuming that the Fourier transform of $f(t)$ is $\tilde{f}(\omega)$,

(a) show that if $f(t)$ is real, then $\tilde{f}(-\omega) = [\tilde{f}(\omega)]^*$.

(b) show that if $f(t)$ is even or odd, then so is its Fourier transform.

(c) if $f(t)$ is imaginary and odd, what can you deduce about its Fourier transform?

(d) show that $f * g = g * f$, meaning that the convolution of f and g is commutative.

(a) By definition,

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,\mathrm{d}t$$

If $f(t)$ is real, then $f(t) = [f(t)]^*$. So

$$\left[\tilde{f}(\omega)\right]^* = \int_{-\infty}^{\infty} e^{i\omega t} f^*(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} e^{i\omega t} f(t)\,\mathrm{d}t = \tilde{f}(-\omega)$$

(b) By definition,

$$\int_{-\infty}^{\infty} (\cos\omega t - i\sin\omega t) f(t)\,\mathrm{d}t$$

If f is even, $f(t) = f(-t)$, the sine integral vanishes by symmetry and

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t\,\mathrm{d}t$$

Since $\cos\omega t = \cos(-\omega t)$, $\tilde{f}(\omega)$ is even, too. Conversely, if $f(t)$ is odd, the cosine integral vanishes and

$$\tilde{f}(\omega) = -i\int_{-\infty}^{\infty} \sin(\omega t)\, f(t)\,\mathrm{d}t$$

If we let $\omega \to -\omega$, $\sin\omega t \to \sin(-\omega t) = -\sin(\omega t)$, so $\tilde{f}(\omega)$ is odd. **Hence, $\tilde{f}(\omega)$ has the same even/odd symmetry as $f(t)$.**

(c) If $f(t)$ is imaginary and odd, we already know that $\tilde{f}(\omega)$ is odd and since $\sin\omega t$ is real, so is $\tilde{f}(\omega)$. That is, $\tilde{f}(\omega)$ is both real and odd.

(d) By definition

$$(f * g)(t) \equiv \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\,\mathrm{d}\tau$$

Let $t' = t - \tau$. Then

$$(f * g)(t) = \int_{+\infty}^{-\infty} f(t - t')g(t')\,\mathrm{d}(-t') = \int_{-\infty}^{\infty} f(t - t')g(t')\,\mathrm{d}t' = (g * f)(t)$$

Problem 2 – Fourier transform of a triangle Consider the Fourier transform $\tilde{f}(k)$ of the function

$$f(x) = \begin{cases} A(a - |x|) & -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

(a) Should $\tilde{f}(k)$ be even or odd? real or imaginary?

(b) Compute $\tilde{f}(k)$.

(c) Explain why the way $\tilde{f}(k)$ changes with increasing a makes sense. Hint: you may find it useful to plot $\tilde{f}(k)$.

(a) $f(x)$ is real and even, so $\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,\mathrm{d}x$ should only involve the $\cos(kx)$ part of e^{-ikx} and so be real and even.

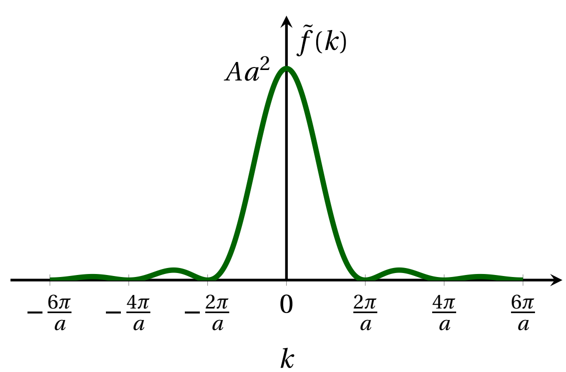
(b) We now evaluate $\tilde{f}(k)$:

$$\begin{aligned}
\tilde{f}(k) &= \int_{-a}^{a} A(a - |x|)e^{-ikx}\,\mathrm{d}x \\
&= A\int_{-a}^{0}(a + x)e^{-ikx}\,\mathrm{d}x + A\int_{0}^{a}(a - x)e^{-ikx}\,\mathrm{d}x \\
&= A\left\{ \frac{(a+x)e^{-ikx}}{-ik}\Big|_{-a}^{0} - \int_{-a}^{0}\frac{e^{-ikx}}{-ik}\,\mathrm{d}x + \frac{(a-x)e^{-ikx}}{-ik}\Big|_{0}^{a} + \int_{0}^{a}\frac{e^{-ikx}}{-ik}\,\mathrm{d}x \right\} \\
&= A\left\{ \frac{a}{-ik} - \frac{e^{-ikx}}{-k^2}\Big|_{-a}^{0} + \frac{a}{ik} + \frac{e^{-ikx}}{-k^2}\Big|_{0}^{a} \right\} \\
&= A\left\{ \frac{e^{ika} - 1}{-k^2} + \frac{e^{-ika} - 1}{-k^2} \right\} \\
&= A\left\{ \frac{2 - e^{ika} - e^{-ika}}{k^2} \right\} = \frac{2A}{k^2}(1 - \cos ka) = \frac{4A}{k^2}\sin^2(ka/2)
\end{aligned}$$

(c) It may be helpful to rewrite slightly:

$$\tilde{f}(k) = Aa^2\left(\frac{\sin ka/2}{ka/2}\right)^2$$

A plot is shown at right. The central peak extends between $-\frac{2\pi}{a}$ and $\frac{2\pi}{a}$ and has height Aa^2. As a gets large, the width of $f(x)$ gets large in position space and small in k space.



Problem 3 – Fourier transform of a SHO eigenfunction Calculate the Fourier transform of the first excited state of a simple harmonic oscillator, which has the form

$$\psi(x) = Axe^{-\alpha x^2}$$

where A is a normalization constant and α is a positive constant. Does it have the same property as the ground state—that the functional form of the Fourier transform is the same as the function itself?

Taking the Fourier transform of the first excited state:

$$\begin{aligned}
\tilde{\psi}(k) &= \int_{-\infty}^{\infty} Axe^{-\alpha x^2} e^{-ikx}\,\mathrm{d}x = A\int_{-\infty}^{\infty}\left(i\frac{\partial}{\partial k}\right)e^{-\alpha x^2 - ikx}\,\mathrm{d}x \\
&= A\left(i\frac{\partial}{\partial k}\right)\sqrt{\frac{\pi}{\alpha}}e^{-k^2/4\alpha} = Ai\sqrt{\frac{\pi}{\alpha}}\left(-\frac{k}{2\alpha}\right)e^{-k^2/4\alpha} \\
&= -iA\sqrt{\frac{\pi}{2\alpha^3}}ke^{-k^2/4\alpha}
\end{aligned}$$

which indeed has the same form ($xe^{-\beta x^2} \leftrightarrow ke^{-\gamma k^2}$).

Problem 4 – Calling Hendrik and Augustin-Louis Calculate the Fourier transform of

$$f(x) = Ae^{-\alpha|x|}$$

where A is a normalization constant and α is a positive constant.

This function $f(x)$ is an even function, so we anticipate that $\tilde{f}(k)$ will be real and even. Let's turn the crank:

$$\begin{aligned}
\tilde{f}(k) &= \int_{-\infty}^{\infty} Ae^{-\alpha|x|}e^{-ikx}\,\mathrm{d}x \\
&= A\int_{-\infty}^{0} e^{\alpha x - ikx}\,\mathrm{d}x + A\int_{0}^{\infty} e^{-\alpha x - ikx}\,\mathrm{d}x \\
&= A\frac{e^{(\alpha - ik)x}}{\alpha - ik}\bigg|_{-\infty}^{0} + A\frac{e^{-(\alpha+ik)x}}{-(\alpha+ik)}\bigg|_{0}^{\infty} \\
&= A\left[\frac{1}{\alpha - ik} + \frac{1}{\alpha + ik}\right] = A\frac{2\alpha}{\alpha^2 + k^2}
\end{aligned}$$

This is a Lorentzian (Cauchy) function of k, which falls off rather slowly in the wings.