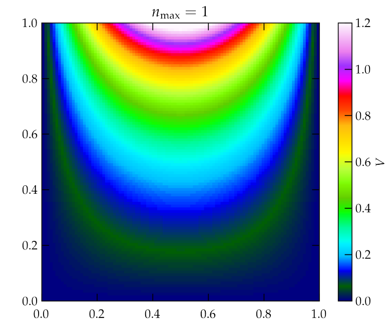
Practice with Contour Integration and Fourier Transforms

Thursday, 12 March 2026

Physics 64



Problem 1 – Pythagoras While $\sin^2\theta + \cos^2\theta = 1$ is undoubtedly true for real values of θ, is it still true when θ can be complex?

> yes

Problem 2 – Confirmation By evaluating

$$I_n = \oint z^n \,\mathrm{d}z$$

around a circle of radius R centered on the origin, show that for $n \in \mathbb{Z}$, $I_n = 0$ unless $n = -1$, in which case $I_n = 2\pi i$.

> We did this together. Check your notes.

Problem 3 – Poles and Residues For each of the following functions determine the poles and the corresponding residues: (a) $\dfrac{2z+1}{z^2-z-2}$, (b) $\left(\dfrac{z+1}{z-1}\right)^2$, (c) $\dfrac{\sin z}{z^2}$, (d) $\operatorname{sech} z$, (e) $\cot z$.

> (a) The poles are where $z^2 - z - 2 = (z-2)(z+1) = 0$, so at -1 and 2, where the residues are $1/3$ and $5/3$, respectively.
> (b) It is a second-order pole at $z = 1$, whose residue is 4.
> (c) A simple pole at $z = 0$ with residue 1.
> (d) Poles at $z = i\pi\left(\frac{1}{2} + n\right)$ for $n \in \mathbb{Z}$ with residues $\frac{i}{2}(-1)^{n+1}$.
> (e) Poles at $z = n\pi$ for $n \in \mathbb{Z}$ with residue 1.

Problem 4 – Our Friend Gauss

(a) Show that the Fourier transform of a gaussian is a gaussian. That is, compute the Fourier transform of

$$f(t) = Ae^{-t^2/\tau^2}$$

(b) If $f(t) = \psi(t)$ represents the wave function of a particle, what must A be for the state to be normalized?

(c) The time uncertainty of $f(t)$ is defined by

$$(\Delta t)^2 = \int_{-\infty}^{\infty} (t - \bar{t})^2 |f(t)|^2 \,\mathrm{d}t \tag{1}$$

where $\bar{t}$ is the mean value of t (which is zero in our case). What is Δt for our gaussian? For its transform, $\tilde{f}(\omega)$? What, then, is $\Delta t\,\Delta\omega$?

> (a) $\tilde{f}(\omega) = A\tau\sqrt{\pi}\exp\left[-(\omega\tau/2)^2\right]$, which is also a gaussian.
> (b) $A = \left(\frac{2}{\pi}\right)^{1/4}\tau^{-1/2}$.
> (c) $\Delta t = \frac{1}{2}\tau$ and $\Delta\omega = \frac{1}{\tau}$. Therefore, $(\Delta t)(\Delta\omega) = \frac{1}{2}$. Since $E = \hbar\omega$, this result implies that $(\Delta E)(\Delta t) = \hbar/2$ for a gaussian, which is the limit imposed by the Heisenberg uncertainty principle.

Problem 5 – Sanity check Show that

$$\int_0^i \frac{\mathrm{d}z}{1 - z^2} = \frac{i\pi}{4}$$

$$\begin{aligned}\int_0^i \frac{1}{(1+z)(1-z)}\,\mathrm{d}z &= \frac{1}{2}\int_0^i \left[\frac{1}{1+z}+\frac{1}{1-z}\right]\mathrm{d}z \\ &= \frac{1}{2}\left[\ln(1+z)-\ln(1-z)\right]_0^i \\ &= \frac{1}{2}\ln\left(\frac{1+i}{1-i}\right) = \frac{1}{2}\ln\left(\frac{e^{i\pi/4}}{e^{-i\pi/4}}\right) \\ &= \frac{i\pi}{4}\end{aligned}$$

Problem 6 – A Useful Trick Consider the integral

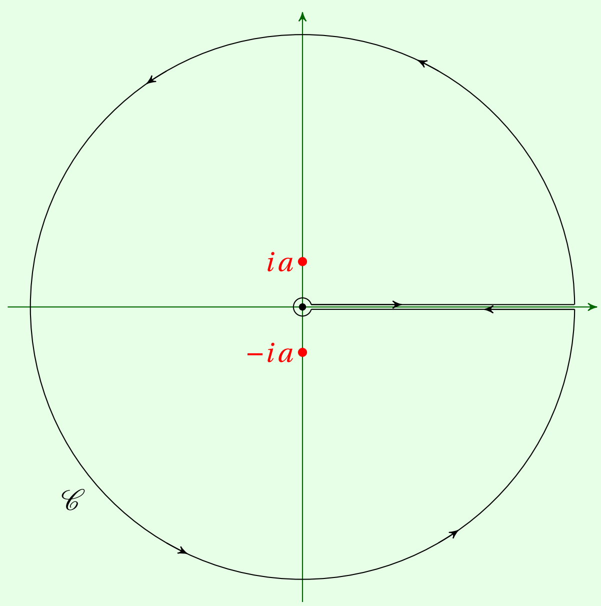
$$I = \int_0^\infty \frac{\ln x}{a^2+x^2}\,\mathrm{d}x$$

(a) For real a, should this integral be finite? That is, is the divergence as $x \to 0$ integrable? And does the integrand vanish sufficiently strongly as $x \to \infty$ to yield a finite integral?

(b) Knowing that the value of $\ln z$ infinitesimally above the x axis can be different from its value infinitesimally below the x axis, if your contour takes you around the origin, can you devise a contour to allow you to use the residue theorem to evaluate the integral, or does the integral along the path $x + i(2\pi - \epsilon)$ cancel the integral along $x + i\epsilon$ as $\epsilon \to 0$?

(c) Now consider $\oint \frac{(\ln z)^2}{a^2+z^2}\,\mathrm{d}z$ around the same contour as in the previous part. It should yield a path to evaluating the original integral I. See if you can show that $I = \frac{\pi \ln a}{2a}$.

(a) As $x \to 0$, the integrand looks like $\frac{\ln x}{a^2}$, unless $a = 0$. Assuming $|a| > 0$, then $\int \frac{\ln x}{a^2}\,\mathrm{d}x = \frac{x\ln x - x}{a^2}$. As $x \to 0$ this goes to zero (which you can verify with l'Hôpital). As $x \to \infty$, the x^2 in the denominator also drives the integrand to zero, so there is no convergence problem and the integral should be finite for $a \neq 0$.
(b) We seek to evaluate $I = \int_0^\infty \frac{\ln z}{a^2+z^2}\,\mathrm{d}z$. The poles are at $z = \pm i a$. Consider contour $\mathscr{C}$ shown below:



It starts at $z = \varepsilon$, heads out the real axis to R, goes counterclockwise around the circle $z = Re^{i\theta}$ as θ goes from 0 to 2π, goes from right to left along $z = re^{2\pi i}$, and finally goes clockwise around the tiny circle at radius ε. So,

$$\begin{aligned}\oint_{\mathscr{C}} \frac{\ln z}{a^2+z^2}\,\mathrm{d}z &= \int_\varepsilon^\infty \frac{\ln x}{a^2+x^2}\,\mathrm{d}x + \int_0^{2\pi} \frac{\ln\left(Re^{i\theta}\right) Ri e^{i\theta}}{a^2+R^2 e^{2i\theta}}\,\mathrm{d}\theta + \int_\infty^\varepsilon \frac{\ln\left(xe^{2\pi i}\right)}{a^2+x^2 e^{4\pi i}}\,\mathrm{d}x + \int_{2\pi}^0 \frac{\ln\left(\varepsilon e^{i\theta}\right)}{a^2+\varepsilon^2 e^{2i\theta}}\varepsilon i e^{i\theta}\,\mathrm{d}\theta \\ &= \int_\varepsilon^\infty \frac{\ln x\,\mathrm{d}x}{a^2+x^2} + \frac{\ln R}{R}\int_0^{2\pi} \frac{i\theta i e^{i\theta}}{a^2/R^2+e^{2i\theta}}\,\mathrm{d}\theta - \int_\varepsilon^\infty \frac{\ln x + 2\pi i}{a^2+x^2}\,\mathrm{d}x - \varepsilon\int_0^{2\pi} \frac{\ln\varepsilon + i\theta}{a^2+\varepsilon^2 e^{2i\theta}} i e^{i\theta}\,\mathrm{d}\theta\end{aligned}$$

The second and fourth integrals go to zero as $R \to \infty$ and $\varepsilon \to 0$. The first integral (which we care about) gets wiped out by the first term in the numerator of the third integral, leaving a term that we don't care about (but could evaluate with the residue theorem). Darn!

(c) Following the hint, we now try again after squaring the logarithm:

$$
\begin{aligned}
J \equiv \oint_{\mathscr{C}} \frac{(\ln z)^2}{a^2+z^2}\,\mathrm{d}z &= \int_\varepsilon^\infty \frac{(\ln x)^2}{a^2+x^2}\,\mathrm{d}x + \underbrace{\int_0^{2\pi} \cdots \mathrm{d}\theta}_{\text{vanishes}} + \int_\infty^\varepsilon \frac{[\ln\left(x e^{2\pi i}\right)]^2}{a^2+x^2 e^{4\pi i}}\,\mathrm{d}x + \underbrace{\int_{2\pi}^{0} \cdots \mathrm{d}\theta}_{\text{vanishes}} \\
&= \int_\varepsilon^\infty \frac{(\ln x)^2}{a^2+x^2}\,\mathrm{d}x + \int_\infty^\varepsilon \frac{(\ln x)^2 + 4\pi i \ln x - 4\pi^2}{a^2+x^2}\,\mathrm{d}x \\
&= \int_\varepsilon^\infty \frac{4\pi^2}{a^2+x^2}\,\mathrm{d}x - \int_\varepsilon^\infty \frac{4\pi i \ln x}{a^2+x^2}\,\mathrm{d}x
\end{aligned}
$$

We have done the first integral before; you can use the substitution $x = a\tan\phi$ to get $4\pi^2 \times \pi/2a = 2\pi^3/a$. The second integral is $-4\pi i I$. So, if we can evaluate the residues, we can determine I.
First, the residue at $z = ia$. I will use the series approach by letting $z = ia + \zeta$ and looking for the term in the integrand proportional to $1/\zeta$.

$$
\frac{[\ln(ia+\zeta)]^2}{a^2+(ia+\zeta)^2} = \frac{[\ln a + i\pi/2]^2}{a^2 + -a^2 + 2ia\zeta + \zeta^2} = \frac{(\ln a)^2 + i\pi \ln a - \pi^2/4}{2ia\zeta + \zeta^2}
$$

So

$$
a_{-1}(ia) = \frac{(\ln a)^2 + i\pi \ln a - \pi^2/4}{2ia}
$$

For the pole at $z = -ia$ it is critical that we represent $-i$ as $e^{3\pi i/2}$, since that is the phase we have assumed for the contour that surrounds this pole. Taking $z = ae^{3\pi i/2} + \zeta$, we get

$$
\frac{[\ln\left(ae^{3\pi i/2}+\zeta\right)]^2}{a^2+(ae^{3\pi i/2}+\zeta)^2} = \frac{(\ln a + 3\pi i/2)^2}{a^2 + a^2 e^{3\pi i} + 2ae^{3\pi i/2}\zeta + \zeta^2} = \frac{(\ln a)^2 + 3\pi i \ln a - 9\pi^2/4}{-2ai\zeta + \zeta^2}
$$

so the residue is

$$
a_{-1}(-ia) = -\frac{(\ln a)^2 + 3\pi i \ln a - 9\pi^2/4}{2ai}
$$

By the residue theorem,

$$
J = 2\pi i \frac{-2\pi i \ln a + 2\pi^2}{2ai} = \frac{2\pi}{a}(\pi^2 - i\pi \ln a)
$$

which also equals

$$
J = \frac{2\pi^3}{a} - 4\pi i I
$$

Hence,

$$
I = \frac{\pi \ln a}{2a}
$$

Convolution

The **convolution** of functions $f(t)$ and $g(t)$ is defined by

$$
(f * g)(t) \equiv \int_{-\infty}^{\infty} g(t') f(t-t')\,\mathrm{d}t' \tag{2}
$$

and represents how one signal gets smeared out by another (such as a system response function). The convolution of two functions has a particularly straightforward representation in terms of their Fourier transforms. Given

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} \, \mathrm{d}t$$
$$\tilde{g}(\omega) = \int_{-\infty}^{\infty} g(t) e^{i\omega t} \, \mathrm{d}t$$

we can write Eq. (2) in terms of the Fourier transforms:

$$\begin{aligned}
(f * g)(t) &= \int_{-\infty}^{\infty} \mathrm{d}t' \, \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \tilde{f}(\omega) e^{-i\omega t'} \, \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\Omega \, \tilde{g}(\Omega) e^{-i\Omega(t-t')} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \tilde{f}(\omega) \int_{-\infty}^{\infty} \mathrm{d}\Omega \, \tilde{g}(\Omega) e^{-i\Omega t} \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t' \, e^{-i(\omega-\Omega)t'}}_{\delta(\omega-\Omega)} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \tilde{f}(\omega) \tilde{g}(\omega) e^{-i\omega t}
\end{aligned}$$

That is, **the Fourier transform of the convolution is the product of the Fourier transforms**. This result is known as the **convolution theorem**.

The **correlation** of functions $f(t)$ and $g(t)$ is closely related to the convolution operation just described. It is defined by

$$\mathrm{Corr}(f, g)(t) = \int_{-\infty}^{\infty} f(\tau) g(\tau + t) \, \mathrm{d}\tau \tag{3}$$

You may readily confirm that the Fourier transform of the correlation is the product of $\tilde{f}(\omega)[\tilde{g}(\omega)]^*$, which is known as the **correlation theorem**.

By extension, the autocorrelation of a function with itself satisfies

$$\mathrm{FT}\big[\mathrm{Corr}(f, f)\big] = |\tilde{f}(\omega)|^2 \tag{4}$$

which is known as the **Wiener-Khinchin theorem**. The autocorrelation of a function describes how self-similar it is at various delays.