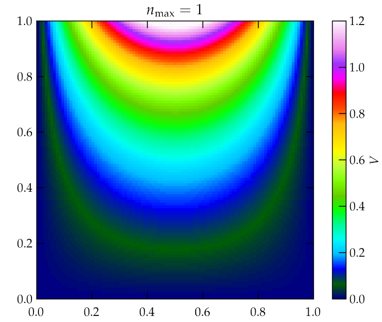
Practice with Contour Integration and Fourier Transforms

Thursday, 12 March 2026

Physics 64



- A complex variable z may be represented in rectangular form $(x + iy)$ or in polar form $(re^{i\phi})$, where $r = \sqrt{x^2 + y^2}$ and $\tan\phi = y/x$.
- Euler's identity is $e^{i\phi} = \cos\phi + i\sin\phi$, which means that

$$\cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2} \quad \text{and} \quad \sin\phi = \frac{e^{i\phi} - e^{-i\phi}}{2i}$$

  You should know these in your sleep.

- For a function of a complex variable $f(z) = u(x, y) + iv(x, y)$ to be differentiable at a point z_0, it must satisfy the Cauchy-Riemann conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1}$$

- Cauchy's integral theorem holds that the integral of a function around a closed contour on and inside of which it is differentiable is zero.
- The square root of a real number is a multivalued function: $\sqrt{1} = \pm 1$. We often have to pick which root we mean from the context. Similarly, $\ln z$ is a multivalued function, since if we use a polar representation for $z = re^{i\phi + i2\pi n}$, then $\ln z = \ln r + i(\phi + 2\pi n)$ where $n \in \mathbb{Z}$.
- When you integrate along a contour, you *always* know what phase that the complex number has based on the phase it had at the point you just came from. If your path circles the origin, then the phase changes by 2π.
- The integral $\oint z^n \, dz$ (n is an integer) around a closed path that encircles the origin is zero unless $n = -1$, in which case it is $2\pi i$. This is the basis of the residue theorem, which holds that the integral of a function $f(z)$ around a closed path in the complex plane is equal to $2\pi i$ times the sum of the residues at the enclosed poles. The **residue** is the coefficient of the z^{-1} term.
- If a pole lies along the path of integration, it contributes only $\pi i a_{-1}$ to the Cauchy principal value, which is defined as $\mathrm{P}\int_a^b = \lim_{\epsilon\to 0}\left[\int_a^{z_0-\epsilon} + \int_{z_0+\epsilon}^b\right]$ for a pole at z_0.
- You may compute the residue of a pole at z_0 by substituting $z = z_0 + \zeta$ and looking for the coefficient of the term proportional to ζ^{-1}. You may also use

$$a_{-1} = \frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}\left[(z - z_0)^n f(z)\right] \tag{2}$$

  for a pole of order n.

- The Fourier transform is defined by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx$$
$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} \, dk$$

  which requires that

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} \, dk$$

- It is also defined by

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} \, dt$$
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega t} \, d\omega$$

Problem 1 – Pythagoras While $\sin^2\theta + \cos^2\theta = 1$ is undoubtedly true for real values of θ, is it still true when θ can be complex?

Problem 2 – Confirmation By evaluating

$$I_n = \oint z^n \, dz$$

around a circle of radius R centered on the origin, show that for $n \in \mathbb{Z}$, $I_n = 0$ unless $n = -1$, in which case $I_n = 2\pi i$.

Problem 3 – Poles and Residues For each of the following functions determine the poles and the corresponding residues: (a) $\dfrac{2z+1}{z^2-z-2}$, (b) $\left(\dfrac{z+1}{z-1}\right)^2$, (c) $\dfrac{\sin z}{z^2}$, (d) $\operatorname{sech} z$, (e) $\cot z$.

Problem 4 – Our Friend Gauss

(a) Show that the Fourier transform of a gaussian is a gaussian. That is, compute the Fourier transform of

$$f(t) = Ae^{-t^2/\tau^2}$$

(b) If $f(t) = \psi(t)$ represents the wave function of a particle, what must A be for the state to be normalized?

(c) The time uncertainty of $f(t)$ is defined by

$$(\Delta t)^2 = \int_{-\infty}^{\infty} (t-\bar{t})^2 |f(t)|^2 \, \mathrm{d}t \tag{3}$$

where $\bar{t}$ is the mean value of t (which is zero in our case). What is Δt for our gaussian? For its transform, $\tilde{f}(\omega)$? What, then, is $\Delta t \; \Delta\omega$?

Problem 5 – Sanity check Show that

$$\int_0^i \frac{\mathrm{d}z}{1-z^2} = \frac{i\pi}{4}$$

Problem 6 – A Useful Trick Consider the integral

$$I = \int_0^\infty \frac{\ln x}{a^2+x^2}\,\mathrm{d}x$$

(a) For real a, should this integral be finite? That is, is the divergence as $x \to 0$ integrable? And does the integrand vanish sufficiently strongly as $x \to \infty$ to yield a finite integral?

(b) Knowing that the value of $\ln z$ infinitesimally above the x axis can be different from its value infinitesimally below the x axis, if your contour takes you around the origin, can you devise a contour to allow you to use the residue theorem to evaluate the integral, or does the integral along the path $x + i(2\pi - \epsilon)$ cancel the integral along $x + i\epsilon$ as $\epsilon \to 0$?

(c) Now consider $\oint \frac{(\ln z)^2}{a^2+z^2}\,\mathrm{d}z$ around the same contour as in the previous part. It should yield a path to evaluating the original integral I. See if you can show that $I = \frac{\pi \ln a}{2a}$.

Convolution

The **convolution** of functions $f(t)$ and $g(t)$ is defined by

$$(f * g)(t) \equiv \int_{-\infty}^{\infty} g(t')f(t-t')\,\mathrm{d}t' \tag{4}$$

and represents how one signal gets smeared out by another (such as a system response function). The convolution of two functions has a particularly straightforward representation in terms of their Fourier transforms. Given

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}\,\mathrm{d}t$$
$$\tilde{g}(\omega) = \int_{-\infty}^{\infty} g(t)e^{i\omega t}\,\mathrm{d}t$$

we can write Eq. (4) in terms of the Fourier transforms:

$$\begin{aligned}
(f * g)(t) &= \int_{-\infty}^{\infty} \mathrm{d}t' \, \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \tilde{f}(\omega)e^{-i\omega t'} \, \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\Omega\, \tilde{g}(\Omega)e^{-i\Omega(t-t')} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \tilde{f}(\omega) \int_{-\infty}^{\infty} \mathrm{d}\Omega\, \tilde{g}(\Omega)e^{-i\Omega t} \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t'\, e^{-i(\omega-\Omega)t'}}_{\delta(\omega-\Omega)} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \tilde{f}(\omega)\tilde{g}(\omega)e^{-i\omega t}
\end{aligned}$$

That is, **the Fourier transform of the convolution is the product of the Fourier transforms**. This result is known as the **convolution theorem**.

The **correlation** of functions $f(t)$ and $g(t)$ is closely related to the convolution operation just described. It is defined by

$$\mathrm{Corr}(f,g)(t) = \int_{-\infty}^{\infty} f(\tau)g(\tau+t)\,\mathrm{d}\tau \qquad (5)$$

You may readily confirm that the Fourier transform of the correlation is the product of $\tilde{f}(\omega)[\tilde{g}(\omega)]^*$, which is known as the **correlation theorem**.

By extension, the autocorrelation of a function with itself satisfies

$$\mathrm{FT}[\mathrm{Corr}(f,f)] = |\tilde{f}(\omega)|^2 \qquad (6)$$

which is known as the **Wiener-Khinchin theorem**. The autocorrelation of a function describes how self-similar it is at various delays.