Problem 1 – Femtosecond Laser Pulses (35 points) Short-pulse laser systems often produce pulses with a temporal shape

$$\mathcal{E}(t) = E_0 \operatorname{sech}(t/\tau) = \frac{E_0}{\cosh(t/\tau)} = \frac{2E_0}{e^{t/\tau} + e^{-t/\tau}} \tag{1}$$

The pulse's frequency spectrum is given by the Fourier transform,

$$E(\omega) = \int_{-\infty}^{\infty} \mathcal{E}(t) e^{i\omega t}\, \mathrm{d}t = \int_{-\infty}^{\infty} \frac{2E_0 e^{i\omega t}}{e^{t/\tau} + e^{-t/\tau}}\, \mathrm{d}t \tag{2}$$

[Don't be confused: I'm leaving off the tilde to simplify the notation.]

(a) **(2 points)** Rewrite Eq. (2) in terms of a dimensionless time variable defined by $x = t/\tau$.

$$E(\omega) = 2E_0\tau \int_{-\infty}^{\infty} \frac{e^{i\omega\tau x}}{e^x + e^{-x}}\, \mathrm{d}x$$

(b) **(5 points)** Locate the poles of the integrand.

$$e^x + e^{-x} = 0 \qquad \Longrightarrow \qquad e^{2x} = -1 = e^{i\pi + i2\pi n}$$

Poles at $x = e^{i\pi/2 + i\pi n}$ for $n \in \mathbb{Z}$.

(c) **(5 points)** With the goal of evaluating $E(\omega)$ via contour integration, *briefly* explain why it is unhelpful to seek to close the contour using a semicircular arc at radius R in the upper half-plane, where $R \to \infty$.

There is an infinite series of poles along the imaginary axis, so such a contour would enclose all of them and possibly lead to an alternating series.

(d) **(10 points)** Instead, close the contour by integrating around a rectangle whose other long side goes from $x = \infty + i\pi$ to $x = -\infty + i\pi$. What is the value of the integrand along this long side? On the short vertical segments?

Along $z = x + i\pi$, the integral would be

$$2E_0\tau \int_{\infty}^{-\infty} \frac{e^{i\omega\tau x} e^{-\omega\tau\pi}}{e^{i\pi + x} + e^{-i\pi - x}}\, \mathrm{d}x = 2E_0\tau e^{-\omega\tau\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega\tau x}}{e^x + e^{-x}}\, \mathrm{d}x$$

where I have used $e^{i\pi} = -1$ in the denominator *and* I reversed the limits of the integral. This is just $e^{-\omega\tau\pi}$ times the integral we are trying to evaluate, $E(\omega)$.
Along the vertical segment going from R to $R + i\pi$ as $R \to \infty$, the denominator diverges exponentially, so the integral along that segment vanishes. The same argument applies on the segment from $-R + i\pi$ to $-R$. Therefore, the integral around the rectangle is equal to $E(\omega)[1 + e^{-\omega\tau\pi}]$.

(e) **(13 points)** Use the residue theorem to evaluate $E(\omega)$. Be sure to write your final answer in simplest form. Comment.

By the residue theorem, the integral around the closed rectangle is equal to $2\pi i \times$ residue at $i\pi/2$.
Let's write the integrand for $z = i\pi/2 + \zeta$:

$$2E_0\tau \frac{e^{i\omega\tau(i\pi/2+\zeta)}}{e^{i\pi/2+\zeta} + e^{-i\pi/2-\zeta}} = 2E_0\tau \frac{e^{-\pi\omega\tau/2}}{ie^{\zeta} - ie^{-\zeta}} = \frac{2E_0\tau e^{-\pi\omega\tau/2}}{i(1+\zeta) - i(1-\zeta)} = \frac{2E_0\tau e^{-\pi\omega\tau/2}}{2i\zeta}$$

So, the residue is $\dfrac{E_0\tau e^{-\pi\omega\tau/2}}{i}$.
Therefore, the residue theorem gives

$$\begin{aligned}
E(\omega)[1 + e^{-\pi\omega\tau}] &= 2\pi i \frac{E_0\tau e^{-\pi\omega\tau/2}}{i} \\
E(\omega) &= \frac{2\pi E_0\tau e^{-\pi\omega\tau/2}}{1 + e^{-\pi\omega\tau}} \\
&= \pi E_0\tau \frac{2}{e^{\pi\omega\tau/2} + e^{-\pi\omega\tau/2}} \\
E(\omega) &= \pi E_0\tau \operatorname{sech}(\pi\omega\tau/2)
\end{aligned}$$

Wow! Just like the gaussian, the Fourier transform of a hyperbolic secant is also a hyperbolic secant!

(f) **(Bonus 10 points)** The pulse duration, Δt, is just the quantum-mechanical uncertainty:

$$(\Delta t)^2 = \frac{\int_{-\infty}^{\infty} t^2 |\mathcal{E}(t)|^2 \, \mathrm{d}t}{\int_{-\infty}^{\infty} |\mathcal{E}(t)|^2 \, \mathrm{d}t} = \tau^2 \frac{\int_{-\infty}^{\infty} x^2 \operatorname{sech}^2 x \, \mathrm{d}x}{\int_{-\infty}^{\infty} \operatorname{sech}^2 x \, \mathrm{d}x}$$

where the integral in the denominator is there for normalization. To evaluate the uncertainty, we need the integrals

$$A = \int_{-\infty}^{\infty} x^2 \operatorname{sech}^2 x \, \mathrm{d}x \quad \text{and} \quad B = \int_{-\infty}^{\infty} \operatorname{sech}^2 x \, \mathrm{d}x$$

Can you evaluate them? *Hint:* do A first. If you have time, calculate the uncertainty product, $(\Delta t)(\Delta\omega)$, for the pulse, and compare to the value we got for a gaussian, which was $1/2$.

$$A = \int_{-\infty}^{\infty} \frac{4x^2}{(e^x + e^{-x})^2} \, \mathrm{d}x$$

To avoid the infinite line of poles along the imaginary axis, we again try to use the rectangular contour that comes back along the line with imaginary part $i\pi$. Along that path, we have

$$C = \int_{\infty}^{-\infty} \frac{4(x+i\pi)^2}{(e^{x+i\pi} + e^{-(x+i\pi)})^2} \, \mathrm{d}x = -\int_{-\infty}^{\infty} \frac{4(x^2 + 2i\pi x - \pi^2)}{(e^x + e^{-x})^2} \, \mathrm{d}x$$

First, note that the part of C arising from the x^2 term exactly cancels A. Darn! Second, the term proportional to x in the numerator is an odd function, so it integrates to zero. We're left with $\pi^2 B$.
Now we can use the residue theorem to evaluate the integral around the closed rectangle.

The integrand at $z = i\pi/2 + \zeta$ is

$$\frac{4(i\pi/2+\zeta)^2}{(e^{i\pi/2+\zeta}+e^{-i\pi/2-\zeta})^2} = 4\frac{-\pi^2/4+i\pi\zeta+\zeta^2}{(ie^{\zeta}-ie^{-\zeta})^2} = 4\frac{-\pi^2/4+i\pi\zeta+\zeta^2}{(2i\zeta)^2} = \frac{\pi^2/4-i\pi\zeta-\zeta^2}{\zeta^2}$$

so the residue is $-i\pi$ and the value of the integral is

$$\pi^2 B = 2\pi i(-i\pi) = 2\pi^2 \qquad \Longrightarrow \qquad \boxed{B=2}$$

So, we lost two powers of x on the reverse trajectory. Maybe lightning will strike twice. Let's try

$$\begin{aligned} C \equiv \oint_C \frac{z^4}{\cosh^2 z}\,\mathrm{d}z &= \int_{-\infty}^{\infty} x^4 \operatorname{sech}^2 x\,\mathrm{d}x - \int_{-\infty}^{\infty}(x+i\pi)^4\operatorname{sech}^2(x+i\pi)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty}\left[x^4 - x^4 + 6\pi^2x^2 - \pi^4\right]\operatorname{sech}^2 x\,\mathrm{d}x \\ &= 6\pi^2 A - \pi^4 B \end{aligned}$$

where I have used that $\operatorname{sech}(x+i\pi) = -\operatorname{sech}(x)$, which we showed earlier.
We can also evaluate C by the residue theorem. At $z = \frac{i\pi}{2}+\zeta$, the integrand is

$$\frac{4(i\pi/2+\zeta)^4}{(2i\zeta)^2} = -\frac{(i\pi/2)^4 + 4(i\pi/2)^3\zeta + \cdots}{\zeta^2}$$

where I have used the same expansion for the (same) denominator we had before. The residue is thus

$$a_{-1} = \frac{i\pi^3}{2}$$

so we get

$$2\pi i\frac{i\pi^3}{2} = 6\pi^2 A - 2\pi^4 \qquad \Longrightarrow \qquad \boxed{A = \frac{\pi^4}{6\pi^2} = \frac{\pi^2}{6}}$$

Therefore,

$$(\Delta t)^2 = \tau^2\frac{A}{B} = \tau^2\frac{\pi^2}{12} \qquad \Longrightarrow \qquad \boxed{\Delta t = \tau\frac{\pi}{2\sqrt{3}}}$$

For the width in frequency, we found $E(\omega) = \pi E_0\tau\operatorname{sech}(\pi\omega\tau/2)$. Let $x = \pi\omega\tau/2$. Then from the previous work,

$$\Delta x = \frac{\pi}{2\sqrt{3}} \qquad \Longrightarrow \qquad \Delta\omega = \frac{\pi}{2\sqrt{3}}\frac{2}{\pi\tau} = \frac{1}{\tau\sqrt{3}}$$

Finally, the uncertainty product for the hyperbolic secant pulse is

$$(\Delta t)(\Delta\omega) = \tau\frac{\pi}{2\sqrt{3}}\frac{1}{\tau\sqrt{3}} = \frac{\pi}{6} \approx 0.524$$

which just slightly greater than the value of 0.5 for a gaussian.