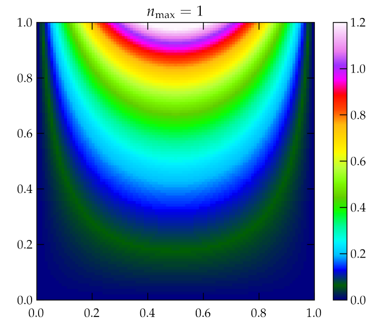
Practice with Contour Integration and Fourier Transforms

Thursday, 26 March 2026

Physics 64



Problem 1 – Femtosecond Laser Pulses Short-pulse laser systems often produce pulses with a temporal shape

$$\mathcal{E}(t) = E_0 \operatorname{sech}(t/\tau) e^{-i\Omega t} = \frac{E_0 e^{-i\Omega t}}{\cosh(t/\tau)} = \frac{2E_0 e^{-i\Omega t}}{e^{t/\tau} + e^{-t/\tau}} \tag{1}$$

where τ determines the pulse duration, and $\Omega = \frac{2\pi c}{\lambda}$ is the mean angular frequency of the "carrier" wave. The pulse's frequency spectrum is given by the Fourier transform,

$$E(\omega) = \int_{-\infty}^{\infty} \mathcal{E}(t) e^{i\omega t}\,\mathrm{d}t = \int_{-\infty}^{\infty} \frac{2E_0 e^{-i\Omega t} e^{i\omega t}}{e^{t/\tau} + e^{-t/\tau}}\,\mathrm{d}t \tag{2}$$

[Don't be confused: I'm leaving off the tilde to simplify the notation.]

(a) Rewrite Eq. (2) in terms of a dimensionless time variable defined by $x = t/\tau$.

(b) Locate the poles of the integrand.

(c) With the goal of evaluating $E(\omega)$ via contour integration, *briefly* explain why it is unhelpful to seek to close the contour using a semicircular arc at radius R in the upper half-plane, where $R \to \infty$.

(d) Instead, close the contour by integrating around a rectangle whose other long side goes from $x = \infty + i\pi$ to $x = -\infty + i\pi$. What is the value of the integrand along this long side? On the short vertical segments?

(e) Use the residue theorem to evaluate $E(\omega)$. Be sure to write your final answer in simplest form. Comment.

(f) The pulse duration, Δt, is just the quantum-mechanical uncertainty:

$$(\Delta t)^2 = \frac{\displaystyle\int_{-\infty}^{\infty} t^2 |\mathcal{E}(t)|^2\,\mathrm{d}t}{\displaystyle\int_{-\infty}^{\infty} |\mathcal{E}(t)|^2\,\mathrm{d}t} = \tau^2 \frac{\displaystyle\int_{-\infty}^{\infty} x^2 \operatorname{sech}^2 x\,\mathrm{d}x}{\displaystyle\int_{-\infty}^{\infty} \operatorname{sech}^2 x\,\mathrm{d}x}$$

where the integral in the denominator is there for normalization. To evaluate the uncertainty, we need the integrals

$$A = \int_{-\infty}^{\infty} x^2 \operatorname{sech}^2 x\,\mathrm{d}x \quad \text{and} \quad B = \int_{-\infty}^{\infty} \operatorname{sech}^2 x\,\mathrm{d}x$$

Can you evaluate them? *Hint:* do A first. If you have time, calculate the uncertainty product, $(\Delta t)(\Delta\omega)$, for the pulse, and compare to the value we got for a gaussian, which was $1/2$.