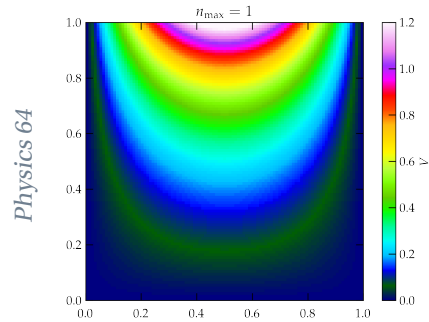


Problem Set 9

due: Monday, 6 April 2026



Problem 1 – Hyperspheres (10 points) The surface area and volume of an n -dimensional sphere of radius r may be expressed

$$A = S_n r^{n-1} \quad V = C_n r^n$$

In three dimensions, we know that the area and volume of a “normal” sphere are given by $4\pi r^2$ and $\frac{4}{3}\pi r^3$, respectively, so that $S_3 = 4\pi$ and $C_3 = \frac{4}{3}\pi$. We now seek to develop expressions for arbitrary dimension n .

(a) Express S_n in terms of C_n .

(b) To evaluate S_n we use the same trick we used to evaluate the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

That is, we multiply together two Gaussian integrals and express the result in polar coordinates:

$$\begin{aligned} \left\{ \int_{-\infty}^{\infty} e^{-\alpha x^2} dx \right\} \left\{ \int_{-\infty}^{\infty} e^{-\alpha y^2} dy \right\} &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{-\alpha(x^2+y^2)} \\ &= \int_0^{\infty} e^{-\alpha r^2} 2\pi r dr = \frac{\pi}{\alpha} \end{aligned}$$

where $S_2 = 2\pi$. Use this same method to show that

$$\int_0^{\infty} e^{-\alpha r^2} S_n r^{n-1} dr = \left(\frac{\pi}{\alpha}\right)^{n/2}$$

(c) Solve for C_n and S_n in terms of the gamma function, $\Gamma(n+1) = \int_0^{\infty} x^n e^{-x} dx$. Check your expressions for $n=2$ and $n=3$, and evaluate C_6 and S_6 .

(d) Use a Monte Carlo method to estimate C_6 . Make a log-log plot of the absolute value of the difference between the numerical estimate and the analytic value as a function of the number N of “darts” you throw. Does the trend agree with expectations?

Problem 2 – Blackbody Radiation An oven with thick, opaque walls is heated to a uniform temperature T . Through a tiny hole in one wall emerges light of a broad range of angular frequencies ω , which is called **cavity radiation** or more commonly **blackbody radiation**. Its spectrum is given by

$$P(\omega) d\omega = \frac{\hbar}{4\pi^2 c^2} \frac{\omega^3}{e^{\hbar\omega/k_B T} - 1} d\omega$$

where $P(\omega)$ is the power per unit area per unit angular frequency. Planck used a computational trick to obtain this result in 1900, but Einstein took it seriously and used the idea that light energy came in discrete packets (which we now call photons) to explain the **photoelectric effect**. This work earned him the Nobel Prize.

- (a) The spectrum above is reported as a function of angular frequency, but experiments typically measure wavelengths, not frequencies. Of course, $c = \lambda\nu = \frac{\lambda\omega}{2\pi}$. Show that the spectral distribution as a function of wavelength is

$$\mathcal{P}(\lambda) = \frac{2\pi hc^2}{\lambda^5} \frac{1}{\exp(hc/\lambda k_B T) - 1}$$

and confirm that it has the appropriate dimensions.

- (b) Show that if one is only able to measure the long-wavelength portion of the spectrum coming from a distant star, it is impossible to determine the star's surface temperature. *Think very carefully; this is physics, not math!*

Problem 3 – Two-Dimensional Random Walk A random walker starts at the origin and takes steps of length 1 in a random direction in the xy plane. Simulate the progress of such a walker and compare your results to the expectation that the walker's typical distance from the origin after n steps is \sqrt{n} . Use your judgment on how best to make this comparison, but please explain your choices.