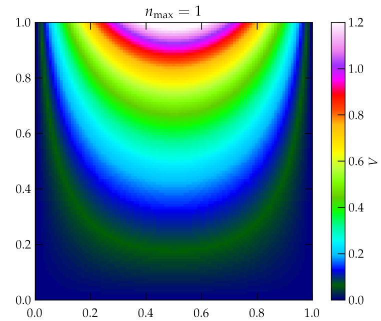
Problem Set 10 — Solution

Monday, 13 April 2026

Physics 64



Problem 1 – Euler's (Equidimensional) Equation The ordinary differential equation

$$x^2 y''(x) + (2\alpha + 1)xy'(x) + \beta y(x) = 0 \tag{1}$$

for dimensionless constants α and β, is **equidimensional**, since each term has whatever dimensions y has.

(a) Look for a solution of the form $y = x^r$. What is the quadratic equation that r must solve? (This is called the **indicial equation**.)

(b) What are the two linearly independent solutions if $\alpha = 1$ and $\beta = -1$?

(c) What is the condition for there to be a single value of r?

(d) When there is only one value for r, the second linearly independent solution is not a simple power of x. Instead, show that $y = x^r \ln x$ is a second solution to the differential equation.

(a) Substituting $y = x^r$ into the differential equation gives

$$r(r-1)x^r + (2\alpha+1)rx^r + \beta x^r = 0$$

As advertised, each term is proportional to x^r, which we cannot rely upon to make the left-hand side zero. Therefore, we must have

$$(r^2 - r + 2\alpha r + r + \beta)x^r = 0 \tag{2}$$

which is a quadratic equation for r with solution

$$\boxed{r = -\alpha \pm \sqrt{\alpha^2 - \beta}} \tag{3}$$

(b) For $\alpha = 1$ and $\beta = -1$, the indicial equation gives

$$r = -1 \pm \sqrt{1+1} = -1 \pm \sqrt{2}$$

Since $y_1 = x^{-1+\sqrt{2}}$ is linearly independent from $y_2 = x^{-1-\sqrt{2}}$, we have identified the two solutions.

(c) If $\alpha^2 - \beta = 0$, then the two roots are the same, and we have identified only one solution in the form of $y = x^r$.

(d) Assuming that $\alpha^2 = \beta$, we investigate a solution

$$y_2 = x^{-\alpha} \ln x$$

Calculating derivatives gives

$$y_2' = -\alpha x^{-\alpha-1}\ln x + x^{-\alpha-1} = x^{-\alpha-1}(1-\alpha\ln x)$$
$$y_2'' = x^{-\alpha-2}[-(\alpha+1)(1-\alpha\ln x)-\alpha] = x^{-\alpha-2}\left[-(2\alpha+1)+(\alpha^2+\alpha)\ln x\right]$$

which we can substitute into the differential equation to get

$$\begin{aligned} x^{-\alpha}\left[-(2\alpha+1)+(\alpha^2+\alpha)\ln x+(2\alpha+1)(1-\alpha\ln x)+\beta\ln x\right] &= x^{-\alpha}\left[\alpha^2+\alpha-2\alpha^2-\alpha+\beta\right]\ln x \\ &= (-\alpha^2+\beta)x^{-\alpha}\ln x = 0 \end{aligned}$$

Since the condition of getting a single solution is that $\alpha^2-\beta=0$, we see that the left-hand side is indeed zero, so $x^{-\alpha}\ln x$ is a second solution to the differential equation.

Problem 2 – Make up your mind! (after Nearing 10.12) A thick slab of material of density ρ and specific heat capacity c is alternately heated and cooled at its top surface ($x=0$) so that the surface temperature oscillates between T_1 and $-T_1$ with period τ according to

$$T(0,t) = \begin{cases} T_1 & 0 < t \bmod \tau < \tau/2 \\ -T_1 & \tau/2 < t \bmod \tau < \tau \end{cases}$$

Find the temperature inside the material for $x>0$ (x is oriented positive inside the material away from the top surface).

The partial differential equation we must solve is

$$u_t = Du_{xx}$$

where $D=\kappa/c\rho$ and we assume that the thermal conductivity κ, the specific heat capacity c, and the mass density ρ are uniform in the slab. We look for a separated variable solution of the form $u(x,t)=X(x)T(t)$, which we substitute in the differential equation to get

$$X(x)T'(t) = DX''(x)T(t)$$

Dividing by $u=XT$ yields

$$\frac{T'}{T} = D\frac{X''}{X} = k$$

where the left-hand side is a function of time only and the right-hand side is a function only of depth x. For this equation to hold at all times, both sides must be a constant (which I have provisionally called k).

To decide what "sign" to assign the constant, let us look to the boundary conditions. The temporal behavior oscillates with period τ, so we would like a constant k that yields oscillatory $T'(t)=kT(t)$. Letting $T(t)=e^{-i\omega_n t}$, with $\omega_n = 2\pi n/\tau = n\omega$ and $n\in\mathbb{Z}$, we get

$$-in\omega e^{-in\omega t} = ke^{-in\omega t} \qquad \Longrightarrow \qquad k = -in\omega$$

Substituting this value into the equation for $X(x)$ gives

$$DX'' = -in\omega X \qquad \Longrightarrow \qquad X'' = \frac{-in\omega}{D}X$$

which has solution $X = e^{\lambda_n x}$ with $\lambda_n^2 = -in\omega/D$. So,

$$\lambda_n = \pm\frac{1-i}{\sqrt{2}}\sqrt{\frac{\omega n}{D}} = \pm(1-i)\sqrt{\frac{\omega n}{2D}}$$

When $n > 0$, the positive sign gives solutions that grow exponentially as we move into the bulk (x grows); we must discard these as unphysical. The negative sign gives solutions that decay exponentially, which *are* physical. When $n < 0$, $\lambda_n = \pm\sqrt{i|n|\omega/D} = \pm\sqrt{1+i}\sqrt{|n|\omega/2D}$ and we again must choose the negative sign. So, the general form of solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n e^{(-1+i)\sqrt{n\omega/D}x} e^{-in\omega t} + b_n e^{(-1-i)\sqrt{n\omega/D}x} e^{in\omega t}\right) \tag{4}$$

We have a temporal boundary condition at $x = 0$:

$$u(0,t) = T(0,t) = \sum_{n=1}^{\infty}\left(a_n e^{-in\omega t} + b_n e^{in\omega t}\right)$$

To solve for the unknown coefficients, we multiply by $e^{im\omega t}$ and integrate over a period:

$$\begin{aligned}
\int_0^{\tau/2} T_1 e^{im\omega t}\,\mathrm{d}t - \int_{\tau/2}^{\tau} T_1 e^{im\omega t}\,\mathrm{d}t &= \sum_{n=1}^{\infty}\int_0^{\tau}\left(a_n e^{-in\omega t} + b_n e^{in\omega t}\right)e^{im\omega t}\,\mathrm{d}t \\
T_1\left[\frac{e^{im\omega t}}{im\omega}\bigg|_0^{\tau/2} - \frac{e^{im\omega t}}{im\omega}\bigg|_{\tau/2}^{\tau}\right] &= (a_n\delta_{m,n} + b_n\delta_{m,-n})\tau \\
\frac{T_1}{im\omega}\left[e^{im\pi} - 1 - 1 + e^{im\pi}\right] &= (a_n\delta_{m,n} + b_n\delta_{m,-n})\tau \\
\frac{2T_1}{im\omega}[(-1)^m - 1] &= (a_n\delta_{m,n} + b_n\delta_{m,-n})\tau
\end{aligned}$$

Therefore,

$$a_m = \begin{cases}\frac{4T_1 i}{m\omega\tau} = \frac{2T_1 i}{m\pi} & m \text{ odd} \\ 0 & m \text{ even}\end{cases} \quad \text{and} \quad b_m = \begin{cases}-\frac{2T_1 i}{m\pi} & m \text{ odd} \\ 0 & m \text{ even}\end{cases}$$

Substituting these expressions into Eq. (4) gives

$$\begin{aligned}
u(x,t) &= \frac{2T_1}{\pi}\sum_{n\text{ odd}}\frac{i}{n}e^{(i-1)x\sqrt{n\omega/D}}e^{-in\omega t} - \frac{i}{n}e^{-(1+i)x\sqrt{n\omega/D}}e^{in\omega t} \\
&= \frac{2T_1}{\pi}\sum_{n\text{ odd}}\frac{i}{n}e^{-x\sqrt{2\pi n/D\tau}}\left[e^{i(x\sqrt{2\pi n/D\tau} - 2\pi nt/\tau)} - e^{-i(x\sqrt{2\pi n/D\tau} - 2\pi nt/\tau)}\right] \\
&= -\frac{4T_1}{\pi}\sum_{n\text{ odd}}\frac{1}{n}\sin\left(x\sqrt{2\pi n/D\tau} - 2\pi nt/\tau\right)\exp\left[-x\sqrt{2\pi n/D\tau}\right]
\end{aligned}$$

Problem 3 – Potential inside a cube Consider a cube of side L with conducting walls. One corner of the cube is at the origin; the other is at (L, L, L). All sides of the cube are grounded, except the top plate at $z = L$, which is held at potential V_0. Noting that the potential inside the cube satisfies Laplace's equation,

$$\nabla^2 V = 0$$

solve for the potential $V(\mathbf{r})$ inside the cube.

We look for a separated-variable solution of the form

$$V(x,y,z) = X(x)Y(y)Z(z)$$

and substitute into Laplace's equation to get

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

This can only work if each of the terms is a constant and if the sum of those constants is zero. That means our basis functions will be oscillatory in some directions and exponential in others. Since the sides are all grounded, I will chose negative separation constants for x and y, and positive for z.

In x and y we get

$$X(x) = \sin\left(\frac{m\pi x}{L}\right) \quad \text{and} \quad Y(y) = \sin\left(\frac{n\pi y}{L}\right)$$

for positive integers m and n. The equation for $Z(z)$ then becomes

$$Z'' = \frac{\pi^2}{L^2}(m^2+n^2)Z$$

Let $\boxed{k_{mn} \equiv \frac{\pi}{L}\sqrt{m^2+n^2}}$. Then the solutions for Z are $e^{\pm k_{mn}z}$. Since Z should vanish at $z=0$, we need the antisymmetric combination of these, which is $\sinh(k_{mn}z)$. So, our solution is a summation of the form

$$V(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}\sin(m\pi x/L)\sin\big(n\pi y/L\big)\sinh(k_{mn}z) \tag{5}$$

To determine the unknown coefficients c_{mn}, we apply the final boundary condition that $V = V_0$ on the top surface, where $z = L$:

$$V_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}\sin(m\pi x/L)\sin\big(n\pi y/L\big)\sinh(k_{mn}L)$$

Multiplying both sides by $\sin\big(m'\pi x/L\big)\sin\big(n'\pi y/L\big)$ and integrating over the top face, we get

$$\begin{aligned} V_0\left[L\frac{1-\cos m'\pi}{m'\pi}L\frac{1-\cos n'\pi}{n'\pi}\right] &= c_{mn}\sinh(k_{mn}L)\frac{L^2}{4}\delta_{mm'}\delta_{nn'} \\ \frac{4V_0}{\pi^2 m'n'} &= \frac{c_{m'n'}}{4}\sinh(k_{m'n'}L) \quad \text{when } m' \text{ and } n' \text{ are odd} \end{aligned}$$

Putting it all together, we find

$$V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{m\text{ odd}}\sum_{n\text{ odd}}\frac{\sin(m\pi x/L)\sin\big(n\pi y/L\big)\sinh[\pi\sqrt{m^2+n^2}\,\frac{z}{L}]}{mn\sinh[\pi\sqrt{m^2+n^2}]}$$