Homework 1

Due: Monday, 1/26/26, 23:59:59

Problem 1 — Summing a series

Consider the infinite series

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \tag{1}$$

(a) Without explicitly summing the series, use an integral test to determine whether it converges.

(b) Sum the series.

Problem 2 — Paramagnetism

In the Langevin model of paramagnetic behavior, the magnetization takes the form

$$M(x) = M_0 \left[\frac{\cosh x}{\sinh x} - \frac{1}{x}\right]$$

where M_0 is a constant and x is proportional to the applied magnetic field.

(a) What is the limiting value of the magnetization as $x \to \infty$?

(b) How does the magnetization depend on x as $x \to 0$? **Note**: I'm not looking for the value of $M(0)$ but the way $M(x)$ **depends on** x for small values of $|x|$.

Problem 3 — Limits

Find the following limits:

(a) $\lim_{x\to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right)$

(b) $\lim_{x\to 0} \left(\frac{2}{x} + \frac{1}{1-\sqrt{1+x}}\right)$

(c) $\lim_{x\to 0} \left(\frac{1-\cos kx}{1-\cosh kx}\right)$

Problem 4 — Numerically summing a series

The Riemann zeta function is defined by

$$\zeta(\nu) = \sum_{n=1}^{\infty} \frac{1}{n^\nu} \tag{2}$$

When $\nu = 1$, it is equal to the harmonic series, which we showed does not converge. For $\nu > 1$, the series *does* converge, although convergence can be slow for values of ν that are not large.

(a) For $\nu = 2$, the series converges to $\pi^2/6 \approx 1.64493$. About how many terms do you need to sum to achieve an accuracy of 0.01%? (Use Python and NumPy; include your commented code in your solution.)

(b) Now consider a way of estimating the series as

$$S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(n-1)^2} + \sum_{j=n}^{\infty} \frac{1}{j^2} \tag{3}$$

where we explicitly sum the first $n-1$ terms and then approximate the remaining infinite sum via an integral. About how many terms do you need to sum *explicitly* to achieve the same 0.01% accuracy using this method? Comment.

Problem 5 — Division of series

One way to develop the Taylor series expansion of $\tan x$ about $x = 0$ is by taking derivatives. An alternative is to divide the series for $\sin x$ by the series for $\cos x$ and to use the binomial expansion to "bring the denominator to the numerator." That is, the denominator will have the form

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots = 1 - q$$

where the term $-q$ is the sum of all but the first term. Therefore,

$$\tan x = \frac{\sin x}{\cos x} = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(1 + q + q^2 + \cdots\right)$$

since $1/(1-q) = 1 + q + q^2 + q^3 + \cdots$.

Use this fact to develop the Taylor series for $\tan x$ through at least x^5 and compare your result to

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots$$

Use matplotlib to prepare a plot comparing your approximation to $\tan x$, and estimate the range over which your expression agrees with the true value within 0.03%.