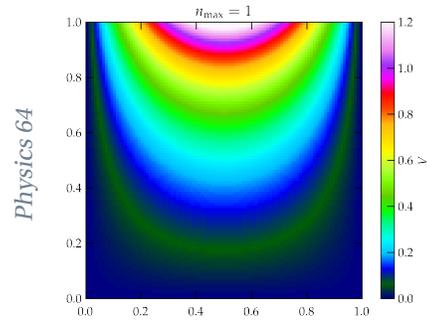


Problem Set 4

due: Monday, 16 February 2026



Problem 1 – Visiting τ Ceti In *Project Hail Mary*, by Andy Weir, a desperate Earth confronts a dimming Sun caused by waves of microscopic creatures (dubbed Astrophage) that absorb energy at the solar surface and then speed off to use the carbon dioxide in the Venusian atmosphere to procreate. In a heartwarming nod to international cooperation, scientists and engineers develop the ability to harness the energy stored in Astrophage to power a rocket to send three astronauts to visit the only star in the nearby galaxy that isn't experiencing exponential dimming, in hopes of learning how to stop the Astrophage catastrophe.

The ship accelerates at a constant rate (I'll call it g) for half the voyage to the star τ Ceti, then flips around and accelerates at g (to slow down), arriving in the neighborhood of τ Ceti approximately at rest with respect to the Sun and τ Ceti. Let's work out how to compare time on the astronauts' clocks to time on Earth clocks.

- (a) Recall that “moving clocks run slow” by $\sqrt{1 - v^2/c^2}$. If the ship happens to be moving at speed v with respect to the Earth at some point in the voyage, what is the relationship between $d\tau$ (a small interval of time on the ship's clock) and dt (the corresponding small interval of time on Earth clocks)?
- (b) Because the ship accelerates, it is not in an inertial frame and we cannot use $g = \frac{dv}{dt}$. However, if you imagine an inertial frame S' that happens to be at rest with respect to the ship at time τ on the ship (and t) on Earth, then for the next little while, the ship's behavior in S' is perfectly classical, since nothing is moving fast at all. That is, in S' there would be no relativistic funny business until the ship's speed in S' gets to be an appreciable fraction of the speed of light.

Let the ship accelerate for $d\tau$ at g . At the end of $d\tau$ we know how fast the ship is moving in S' and we know how fast S' is moving with respect to the Earth. Use the velocity transformation equation,

$$v = \frac{V + v'}{1 + Vv'/c^2}$$

to figure out how much the ship's velocity changes in the Earth frame.

- (c) You should now have a differential equation relating the change in the ship's velocity in Earth's frame to the change in proper time aboard the ship. Combining this equation with the time dilation equation will allow you to determine a relationship between time elapsed since departure aboard the ship and time elapsed since departure on the Earth for the first phase of the journey. *Hint:* At some point you may find it helpful to define $v/c = \tanh \phi$ (ϕ is known as the *velocity parameter*).
- (d) The second half of the journey just mirrors the first half. If the ship accelerated at $g = 15\text{ m/s}^2$ and it arrived at τ Ceti at the end of 3 years and 9 months of travel, how far did the ship travel from launch to arrival?
- (e) How much time elapsed on Earth between the ship's launch at its arrival at τ Ceti?

Problem 2 – Symmetries (Nearing 5.11) Represent a function f on the interval $-L < x < L$ by a Fourier series using periodic boundary conditions

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{n\pi i x/L}$$

- (a) If the function f is odd, prove that for all n , $a_{-n} = -a_n$.
- (b) If the function f is even, prove that all $a_{-n} = a_n$.
- (c) If the function f is real, prove that all $a_{-n} = a_n^*$.
- (d) If the function is both real and even, characterize a_n .
- (e) If the function is imaginary and odd, characterize a_n .

Problem 3 – Parabola (10 points) Develop a Fourier series for the parabola

$$f(x) = x(L-x) \quad 0 \leq x < L$$

using functions u_n that vanish at $x = 0$ and $x = L$. Evaluate the series at $x = L/2$ to show that

$$\pi^3 = 32 \left(1 - \frac{1}{27} + \frac{1}{125} - \frac{1}{343} + \dots \right)$$

Then use Parseval's identity to develop a series representation for π^6 . That is, evaluate $\langle f, f \rangle = \sum_n |a_n|^2 \langle u_n, u_n \rangle$, and rearrange as appropriate.