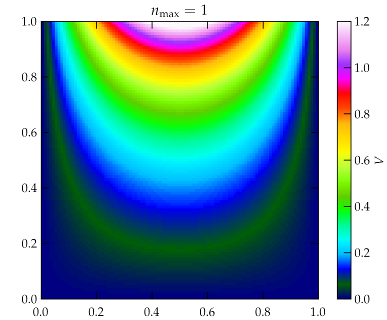
Physics 64



Midterm Examination 1 Solution

12 February 2026

Problem 1 – Holy Hyperbole (25 points) The hyperbolic trigonometric functions are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \tanh x = \frac{\sinh x}{\cosh x}$$

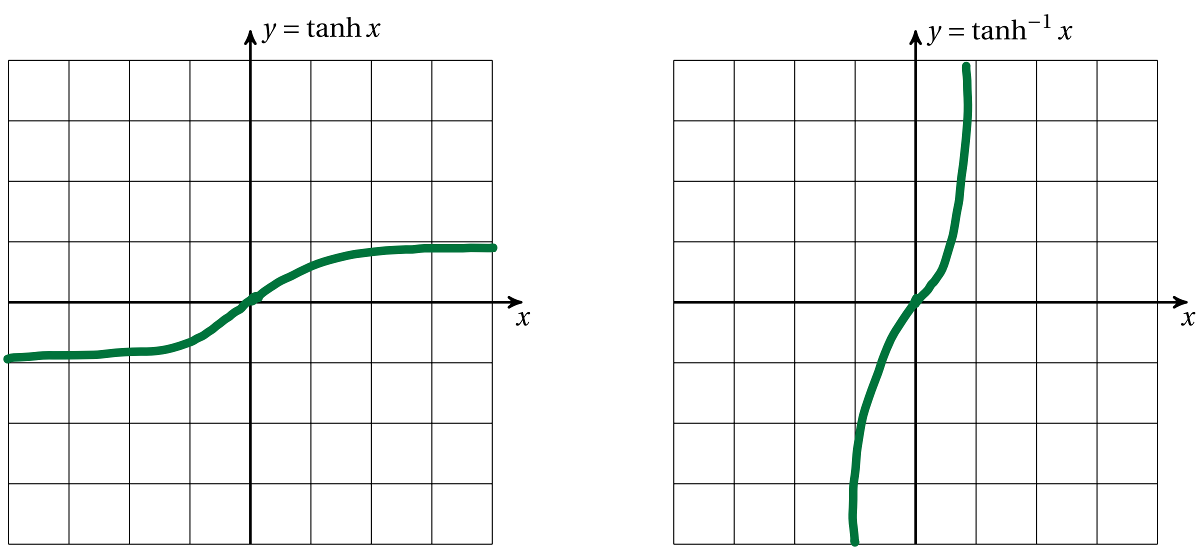
(a) **(5 points)** Show by explicit differentiation that $\frac{\mathrm{d}}{\mathrm{d}x} \tanh x = \operatorname{sech}^2 x$, where $\operatorname{sech} x = 1/\cosh x$.

$$\begin{aligned}
\frac{\mathrm{d}\tanh x}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sinh x}{\cosh x}\right) &= \frac{\cosh x}{\cosh x} - \frac{\sinh x}{\cosh^2 x}\sinh x \\
&= 1 - \tanh^2 x = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \\
&= \operatorname{sech}^2 x
\end{aligned}$$

Issues

- Many people tore apart the hyperbolic trig functions. That makes for more work. The derivative of sinh is cosh and the derivative of cosh is sinh.

(b) **(10 points)** Sketch $y = \tanh x$ on the left axes below, and its inverse on the right axes, treating the grid as comprising unit squares.



At $x = 0$, $\sinh x = 0$ and $\cosh x = 1$, so $\tanh x = 0$. It is an odd function, since $\sinh x$ is odd and $\cosh x$ is even. For large x, both numerator and denominator approach $e^x/2$, so their ratio approaches 1.

Issues

- To go from the function to its inverse, reflect across the line $y = x$. Note: the inverse is not the reciprocal.
- $\tanh x$ is odd. Therefore, its inverse is, too.

(c) **(10 points)** If $y = \tanh^{-1} x$, then $x = \tanh y$ and

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \operatorname{sech}^2 y = 1 - \tanh^2 x = 1 - x^2$$

so

$$\mathrm{d}y = \frac{\mathrm{d}x}{1 - x^2}$$

Expand the right-hand side for $|x| < 1$ and integrate to obtain the Taylor series for $y(x) = \tanh^{-1} x$.

$$\begin{aligned} \mathrm{d}y &= \mathrm{d}x \left[1 + x^2 + x^4 + x^6 + \cdots\right] \\ \int_0^y \mathrm{d}y' &= \int_0^x \left[1 + x'^2 + x'^4 + x'^6 + \cdots\right] \mathrm{d}x' \\ \tanh^{-1} x &= x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots \end{aligned}$$

Issues

- The purpose of this exercise was to develop the Taylor series for $\tanh^{-1} x$ without having to deal with all the derivatives. If your final answer didn't start with $\tanh^{-1} x = \cdots$???

Problem 2 – Ammonia (20 points) The matrix representing the hamiltonian of an ammonia molecule takes the form

$$\mathbb{H} = \begin{pmatrix} E_0 & -A \\ -A & E_0 \end{pmatrix}$$

Solve for the (energy) eigenvalues and normalized eigenvectors.

Let the energy eigenvalue be E. Then for a nontrivial solution we require that

$$\det\begin{pmatrix} E_0 - E & -A \\ -A & E_0 - E \end{pmatrix} = 0 \qquad \Longrightarrow \qquad E_0 - E = \pm A$$

Therefore, the eigenvalues are $E = E_0 \pm A$.

Corresponding to $E = E_0 - A$, the eigenvector satisfies

$$\begin{pmatrix} A & -A \\ -A & A \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0} \qquad \Longrightarrow \qquad x = y$$

Hence, the normalized eigenvector corresponding to the lower-energy eigenstate is

$$\psi_{\mathrm{I}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

When the energy is $E = E_0 + A$, we must solve

$$\begin{pmatrix}-A & -A\\-A & -A\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix} = \mathbf{0} \qquad \Longrightarrow \qquad x = -y$$

Hence, the normalized eigenvector corresponding to the higher-energy eigenstate is

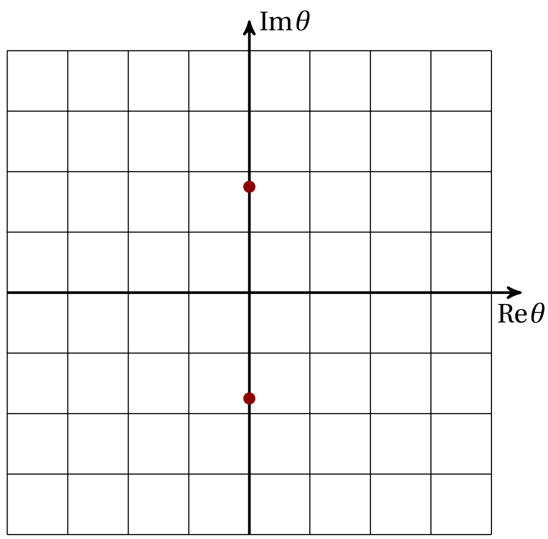
$$\psi_{\mathrm{II}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

Issues

- Some folks reversed the eigenvectors with respect to the eigenvalues. In general, lower-energy states have fewer nodes (hence, fewer negative signs).

Problem 3 – Dear Professor Euler (35 points)

(a) **(10 points)** If $\cos\theta = 3$, solve for θ [no fair using $\cos^{-1}(3)$; your answer should have a logarithm in it] and plot estimates for the values of θ on the complex plane below (again assume unit squares).



Let $z = e^{i\theta}$. Then

$$\begin{aligned}3 = \cos\theta &= \frac{e^{i\theta}+e^{-i\theta}}{2}\\ 6 &= z + z^{-1}\end{aligned}$$

Multiplying through by z and solving the quadratic equation, we get

$$e^{i\theta} = z = \frac{6 \pm \sqrt{36-4}}{2} = 3 \pm 2\sqrt{2}$$

Taking the logarithm,

$$i\theta = \ln\left(3 \pm 2\sqrt{2}\right) + 2\pi n i \qquad \Longrightarrow \qquad \theta = -i\ln\left(3 \pm 2\sqrt{2}\right) + 2\pi n$$

for $n \in \mathbb{Z}$. With the plus sign, we need to estimate $\ln 5.8$. Since $e \approx 2.7$, $\ln 5.8 \approx \ln(2e) \approx 1.7$. (On the scale of the grid, the points for $n \neq 0$ do not appear.)

Issues

- This one was almost identical to an exercise we did in class, using the same "trick" of setting $z = e^{i\theta}$. If you didn't put that one in your toolbox, now's the time.
- A unit square has sides of length 1.
- If you freeze when asked to think about numbers without a calculator, what can we do to overcome that panic?

(b) **(15 points)** Evaluate $\int_0^\pi \sin^6\theta\,\mathrm{d}\theta$.

$$\begin{aligned}\int_0^\pi \sin^6\theta\,\mathrm{d}\theta &= \int_0^\pi \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^6 \mathrm{d}\theta \\ &= \frac{-1}{64}\sum_{n=0}^{6}\int_0^\pi (-1)^n \binom{6}{n} e^{i(6-2n)\theta}\,\mathrm{d}\theta\end{aligned}$$

The only term that survives is the one for $n = 3$, which gives

$$\int_0^\pi \sin^6\theta\,\mathrm{d}\theta = \frac{-1}{64}(-1)^3\binom{6}{3}\pi = \frac{5}{16}\pi$$

Issues

- Many attempted to use the Taylor series for $\sin\theta$. That leads to perdition.
- "Calling Prof. Euler" was supposed to be a hint.
- The formula for the binomial expansion was on the cover (although I debated whether it should be necessary). If you had trouble remembering how to compute a binomial coefficient, let's discuss. You should not need a calculator or other external aid to figure out what a binomial coefficient means—and therefore how to calculate them.
- While not wrong, it is not necessarily "right" to do all the integrals of cosines before noticing that they all cancel.
- If you got stuck computing, but were clever enough to notice that I gave you the answer for the general case in the next part, I gave you partial credit but the point was to leverage your understanding of Euler's formula to evaluate this integral.
- Many of you went back to the tried and true, leveraging $\sin^2\theta = \frac{1-\cos 2\theta}{2}$ and going to town. Not wrong, but not as easy as the method here, which also generalizes readily for arbitrary power $2n$. Time to learn something new!

(c) **(10 points)** Show that for positive integer n,

$$\int_0^\pi \sin^{2n}\theta \, d\theta = \frac{(2n-1)!!}{(2n)!!}\pi$$

By the same reasoning, the only term that survives is

$$\int_0^\pi \sin^{2n}\theta \, d\theta = \frac{(-1)^n}{(2i)^{2n}}\binom{2n}{n}\pi = \frac{1}{2^{2n}}\frac{(2n)!}{(n!)(n!)}\pi = \pi\frac{(2n)!}{[2^n n!]^2}$$

But

$$2^n n! = 2(n) \times 2(n-1) \times \cdots \times 2(2) \times 2(1) = (2n)!!$$

and

$$\frac{(2n)!}{2^n n!} = (2n-1) \times (2n-3) \times \cdots \times 1 = (2n-1)!!$$

Therefore,

$$\int_0^\pi \sin^{2n}\theta \, d\theta = \frac{(2n-1)!!}{(2n)!!}\pi$$

Issues

- If you understood the previous part, then the challenge here was figuring out how to keep track of the factors of 2 to make the double factorials work out.

Problem 4 – Period of a Pendulum (20 points) The period of a simple pendulum of length ℓ released from rest at θ_0 from the vertical is

$$T = 4\sqrt{\frac{\ell}{g}}\int_0^{\pi/2}\left[1 - \sin^2\left(\frac{\theta_0}{2}\right)\sin^2\phi\right]^{-1/2} d\phi$$

Develop an expression for T through terms of order $\sin^4(\theta_0/2)$ that is simplified to reduce to the expected $2\pi\sqrt{\ell/g}$ for $\theta_0 \ll 1$. You may assume that $|\theta_0| < \pi$.

Expand the integrand in a series using

$$(1-\epsilon)^{-1/2} = 1 + \frac{1}{2}\epsilon + \left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\frac{(-\epsilon)^2}{2!} = 1 + \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 + \cdots$$

In this case,

$$\epsilon = \sin^2\left(\frac{\theta_0}{2}\right)\sin^2\phi$$

so

$$\begin{aligned} T &= 4\sqrt{\frac{\ell}{g}}\int_0^{\pi/2}\left[1 + \frac{1}{2}\sin^2\left(\frac{\theta_0}{2}\right)\sin^2\phi + \frac{3}{8}\sin^4\left(\frac{\theta_0}{2}\right)\sin^4\phi + \cdots\right] d\phi \\ &= 4\sqrt{\frac{\ell}{g}}\left[\frac{\pi}{2} + \frac{1}{2}\sin^2\left(\frac{\theta_0}{2}\right)\frac{\pi}{4} + \frac{3}{8}\sin^4\left(\frac{\theta_0}{2}\right)\frac{3!!}{4!!}\frac{\pi}{2} + \cdots\right] \\ &= 2\pi\sqrt{\frac{\ell}{g}}\left[1 + \frac{1}{4}\sin^2\left(\frac{\theta_0}{2}\right) + \frac{9}{64}\sin^4\left(\frac{\theta_0}{2}\right) + \cdots\right] \end{aligned}$$

Issues

- Many decided to use the series for $(1-\epsilon)^{-1}$, which doesn't really apply.
- Work the series out slowly and carefully; I saw lots of sign errors.
- Way too many of you decided that what you needed to show was that if you subbed in $\theta_0 \to 0$, you got the familiar formula. Read again. That is *not* what you are asked to show.
- Many failed to factor out the leading dependence so that the series part starts with a 1. That's what you were asked to do. I you didn't understand that, it is time to learn that this is what series are about.
- The whole point of doing the series is to see how the period *differs* from the value you already knew for small-amplitude oscillation. Notice that you are solving for $\theta_0 < \pi$. That's not small!