

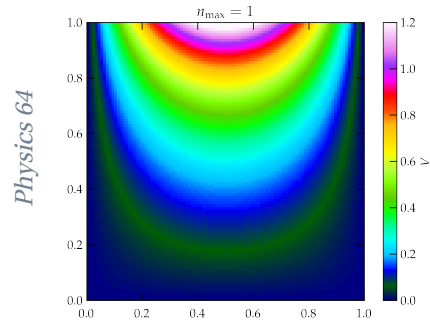
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## Midterm Examination 2

31 March 2026

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Name Key



1. Time: 75 minutes
2. No books, notes, calculators, or computers.
3. Three problems of unequal weight for a total of 100 points.
4. *Read through the whole exam before starting.* Work carefully, check your algebra, and explain what you are doing with brief English phrases.
5. This test is given under the HMC Honor Code.
6. If you don't understand what a question is asking, please ask for clarification.

*Good luck, everyone!*

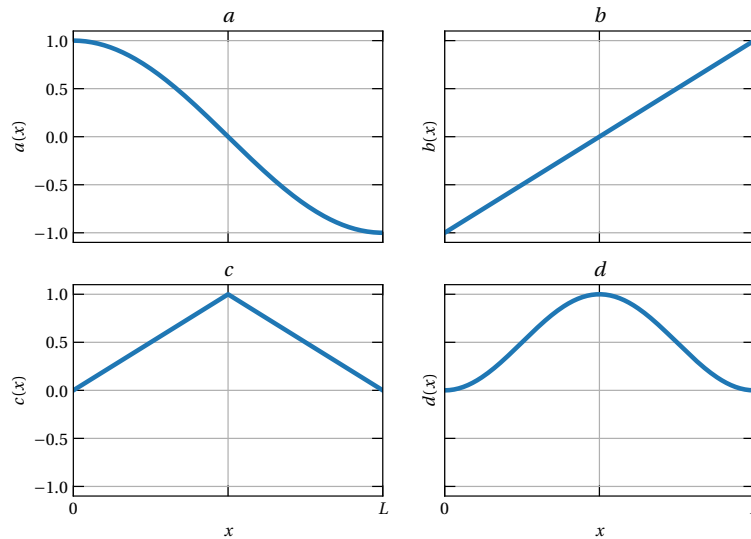
I will respect the HMC Honor Code in taking this exam

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*signed*

$$\Gamma(x+1) = \int_0^{\infty} t^x e^{-t} dt \quad \Gamma(n+1) = n\Gamma(n) \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad (a+b)^n = \sum_{i=0}^n \binom{n}{i} a^{n-i} b^i$$
$$e \approx 2.718 \quad e^{i\phi} = \cos \phi + i \sin \phi \quad \cos \phi = \frac{e^{i\phi} + e^{-i\phi}}{2} \quad \sin \phi = \frac{e^{i\phi} - e^{-i\phi}}{2i}$$
$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b \quad \cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$
$$\sum_{n=0}^{N-1} \rho^n = \frac{1 - \rho^N}{1 - \rho} \quad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad |x| < 1$$
$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$
$$\oint f(z) dz = 2\pi i \sum_n a_{-1}(z_n) \quad a_{-1}(z_0) = \lim_{z \rightarrow z_0} \frac{1}{(v-1)!} \frac{d^{v-1}}{dz^{v-1}} [f(z)(z-z_0)^v]$$
$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega t} d\omega \quad \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk$$
$$\int_{-\infty}^{\infty} e^{-\alpha x^2 - \beta x - \gamma} dx = \sqrt{\frac{\pi}{\alpha}} e^{\beta^2/4\alpha - \gamma}$$

**Problem 1 – Pin the Tail on the Fourier Series (24 points)** Fourier series expansions on  $0 \leq x < L$  of the four functions illustrated in the figures on the left are shown in algebraic form on the right. Match the series to the original function and write a *short* rationale.



$$f(x) = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi nx/L)}{n}$$

$$g(x) = \frac{1}{2} - \frac{1}{2} \cos(2\pi x/L)$$

$$h(x) = -\frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{1-4n^2} \sin(2\pi nx/L)$$

$$j(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n \text{ odd}} \frac{\cos(2\pi nx/L)}{n^2}$$

For example, if you think  $a(x) = f(x)$ , then write  $f$  after " $a(x) =$ " below and a phrase to explain your reasoning.

$a(x) = h(x)$

$b(x) = f(x)$

$c(x) = j(x)$

$d(x) = g(x)$

$c(x) + d(x)$  are continuous, but  $c(x)$  has a discontinuous derivative — so its coefficients should fall off like  $1/n^2$ .

$\therefore c(x) = j(x)$  and  $d(x) = g(x)$

Both  $a(x) + b(x)$  fall off like  $1/n$ , but  $b(x)$  is negative for small  $x$ . All the coefficients of  $f(x)$  are negative while  $\frac{\sin(2\pi nx/L)}{n} > 0$  for small  $x$

$\therefore b(x) = f(x)$

so  $a(x) = h(x)$

**Problem 2 – A Nice Little Contour Integral (36 points)** Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{1}{1+x^3} dx$$

(a) (4 points) Should this integral be positive, negative, complex, or undefined? Briefly explain your reasoning.

- The integrand is real, so we can rule out complex
- No convergence issues as  $|x| \rightarrow \infty$
- The integrand diverges at  $x = -1$ ; it is  $+\infty$  for  $-1+\epsilon$  and  $-\infty$  for  $-1-\epsilon$ . The explosion is odd, so mostly cancels.

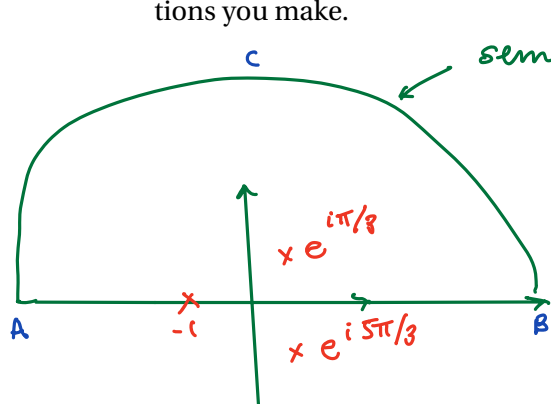
(b) (4 points) Locate the poles.

Since the integrand is more positive than negative, it should be positive

$$1+x^3=0 \rightarrow x^3 = -1 = e^{i\pi + 2\pi ni}$$

$$x = e^{\frac{i\pi}{3}(1+2n)} \quad n=0, 1, 2$$

(c) (12 points) Identify a closed contour you can use to evaluate  $I$ , carefully justifying any assumptions you make.



semicircle at  $z = Re^{i\theta}$

On semicircle,  $dz = Ri e^{i\theta} d\theta$ , so

$$\int_C \frac{1}{1+z^3} dz = \int_0^\pi \frac{1}{1+R^3 e^{3i\theta}} Ri e^{i\theta} d\theta = \frac{Ri}{R^3} \int_0^\pi \frac{e^{i\theta}}{e^{3i\theta} + 1/R^3} d\theta$$

goes to 0 as  $R \rightarrow \infty$       finite

so, closing along the semicircle adds nothing to the integral.

$$\text{Hence, } I = \int_{-\infty}^{\infty} \frac{1}{1+x^3} dx = \oint_{ABCA} \frac{1}{1+z^3} dz$$

(d) (16 points) Evaluate  $I$  and compare your result to the expectations you set in part (a).

By the residue theorem,  $I = \pi i a_{-1}(-1) + 2\pi i a_{-1}(e^{i\pi/3})$

half normal value for a pole on the path of integration.

$$\boxed{a_{-1}(-1)} = \lim_{z \rightarrow -1} \frac{1}{1+z^3} (z+1) \xrightarrow{\text{l'Hôp}} \frac{1}{3z^2} \Big|_{z=-1} = \frac{1}{3}$$

$$\boxed{a_{-1}(e^{i\pi/3})} = \lim_{z \rightarrow e^{i\pi/3}} \frac{1}{1+z^3} (z - e^{i\pi/3}) \xrightarrow{\text{l'Hôp}} \frac{1}{3z^2} \Big|_{z=e^{i\pi/3}} = \frac{1}{3} e^{-i2\pi/3}$$

$$\text{So, } I = \pi i \left[ \frac{1}{3} + \frac{2}{3} e^{-i2\pi/3} \right]$$

$$\cos\left(\frac{2\pi}{3}\right) - i \sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} - i \frac{\sqrt{3}}{2}$$

$$= \pi i \left[ \frac{1}{3} - \frac{1}{3} - \frac{i\sqrt{3}}{3} \right] = \frac{\pi}{\sqrt{3}}, \text{ which indeed is positive. } \checkmark$$

**Problem 3 – A Resonant Cavity (40 points)** In a resonant cavity, formed by (parallel) high-reflectivity mirrors facing one another, an electromagnetic oscillation of frequency  $\omega_0$  dies out as

$$A(t) = \begin{cases} A_0 e^{-\omega_0 t/2Q} e^{-i\omega_0 t} & t > 0 \\ 0 & t < 0 \end{cases}$$

The parameter  $Q$  (the *quality* of the cavity) is a measure of the ratio of the energy stored in the cavity to the energy lost in one cycle.

(a) (20 points) Calculate the frequency distribution of the power emerging from the cavity,

$$S(\omega) = a^*(\omega)a(\omega)$$

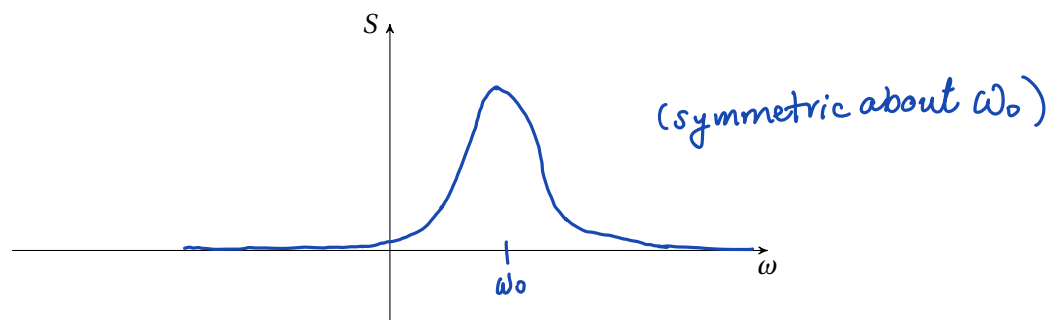
where  $a(\omega)$  is the Fourier transform of  $A(t)$ .

$$\begin{aligned} a(\omega) &= \int_{-\infty}^{\infty} e^{i\omega t} A_0 e^{-\omega_0 t/2Q} e^{-i\omega_0 t} dt \\ &= A_0 \int_0^{\infty} e^{i(\omega - \omega_0 + i\omega_0/2Q)t} dt = A_0 \left. \frac{e^{i(\omega - \omega_0 + i\omega_0/2Q)t}}{i(\omega - \omega_0 + i\omega_0/2Q)} \right|_0^{\infty} \\ &= \frac{A_0 i}{\omega - \omega_0 + \frac{i\omega_0}{2Q}} \end{aligned}$$

$$S(\omega) = a^*(\omega)a(\omega) = \frac{A_0^2}{(\omega - \omega_0)^2 + \frac{\omega_0^2}{4Q^2}}$$

minimum denom at  $\omega = \omega_0$

(b) (5 points) Sketch  $S(\omega)$  on the axes below.



(c) (5 points) Compute the normalized spectral distribution function, defined by

$$\mathcal{P}(\omega) \equiv \frac{S(\omega)}{\int_{-\infty}^{\infty} S(\omega) d\omega} \quad \omega' \equiv \omega - \omega_0$$

$$\int_{-\infty}^{\infty} S(\omega) d\omega = \int_{-\infty}^{\infty} \frac{A_0^2 d\omega}{(\omega - \omega_0)^2 + \frac{\omega_0^2}{4Q^2}} = A_0^2 \int_{-\infty}^{\infty} \frac{d\omega'}{\omega'^2 + \frac{\omega_0^2}{4Q^2}}$$

$$\left. \begin{aligned} \text{Let } \omega' &\equiv \frac{\omega_0}{2Q} \tan \phi \\ d\omega' &= \frac{\omega_0}{2Q} \sec^2 \phi d\phi \end{aligned} \right\} = A_0^2 \int_{-\pi/2}^{\pi/2} \frac{\frac{\omega_0}{2Q} \sec^2 \phi d\phi}{\frac{\omega_0^2}{4Q^2} \sec^2 \phi} = A_0^2 \frac{2Q}{\omega_0} \cdot \pi$$

$$\text{So, } \mathcal{P}(\omega) = \frac{A_0^2}{(\omega - \omega_0)^2 + \frac{\omega_0^2}{4Q^2}} \bigg/ \frac{A_0^2 2\pi Q}{\omega_0} = \frac{\omega_0 / 2\pi Q}{(\omega - \omega_0)^2 + (\omega_0 / 2Q)^2} = \frac{1}{\pi} \frac{\beta}{(\omega - \omega_0)^2 + \beta^2}$$

where  $\beta \equiv \frac{\omega_0}{2Q}$

(d) (10 points) The spectral width  $\sigma$  of the distribution is defined by

$$\sigma^2 \equiv \int_{-\infty}^{\infty} \mathcal{P}(\omega) (\omega - \bar{\omega})^2 d\omega$$

where  $\bar{\omega} = \int_{-\infty}^{\infty} \omega \mathcal{P}(\omega) d\omega$  is the mean value of  $\omega$ . For example, for the gaussian distribution,

$$p(\omega) = \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left[-\frac{\omega^2}{2\sigma_0^2}\right]$$

the spectral width is  $\sigma_0$ . What is the spectral width  $\sigma$  of the light emerging from the cavity? Comment.

$$\sigma^2 = \int_{-\infty}^{\infty} \frac{\beta}{\pi} \frac{(\omega - \omega_0)^2}{(\omega - \omega_0)^2 + \beta^2} d\omega = \frac{\beta}{\pi} \int_{-\infty}^{\infty} \frac{d\omega}{1 + \left(\frac{\omega - \omega_0}{\beta}\right)^2}$$

This integral diverges, since it goes to  $\int \frac{d\omega}{1}$  as  $\omega \rightarrow \infty$

The variance of  $\mathcal{P}(\omega)$  [a Lorentzian] diverges — the distribution falls off too slowly away from the peak at  $\omega_0$ .