MOTION I

T. M. Helliwell © 2009

What do we mean by "understanding" something?

We can imagine that this complicated array of moving things which constitutes "the world" is something like a great chess game being played by the gods, and we are observers of the game. We do not know what the rules of the game are; all we are allowed to do is to *watch* the playing. Of course, if we watch long enough, we may eventually catch on to a few of the rules. *The rules of the game* are what we mean by *fundamental physics*. Even if we knew every rule, however, we might not be able to understand why a particular move is made in the game, merely because it is too complicated and our minds are limited. If you play chess you must know that it is easy to learn all the rules, and yet it is often very hard to select the best move or to understand why a player moves as he does. So it is in nature, only much more so; but we may be able at least to find all the rules. Actually, we do not have all the rules now. (Every once in a while something like castling is going on that we still do not understand.) Aside from not knowing all of the rules, what we really can explain in terms of those rules is very limited, because almost all situations are so enormously complicated that we cannot follow the plays of the game using the rules, much less tell what is going to happen next. We must, therefore, limit ourselves to the more basic question of the rules of the game. If we know the rules, we consider that we "understand" the world.

R. P. Feynman

PREFACE

Motion. How do things move, and why do they move that way? That's what this book is about.

We will explore the motion of objects of about our size that move with everyday speeds. Such objects are said to obey the classical laws of motion, or *classical mechanics*. These laws turn out to work very well also in many situations not part of common experience, from the motion of electrons in TV picture tubes to the motion of planets around the Sun; from the motion of bacteria in a fluid to the motion of galaxies in a cluster of galaxies.

Beginning textbooks rarely show how the classical laws were discovered historically or even what they mean exactly. There is good reason for this: their discovery was difficult and circuitous, requiring centuries. Textbooks can't be expected to follow the historical twists and turns, blind alleys and misconceptions, without confusing the reader and leaving insufficient time to work with the laws as they eventually emerged. But if all reference to discovery is left out, the laws seem to burst forth as revealed truth. How could mere mortals have discovered them?

There are several reasons why it was so difficult to figure out how and why things move as they do. One was *gravitation*, which helped complicate motion. It was a force that couldn't be turned off, so the simplicity of force-free motion was obscured. Another was *friction*, both air resistance and surface friction. Objects rolling along the ground slowed down and came to rest, so it was easy to view such behavior as simple and natural. Again, the greater simplicity and naturalness of force-free motion was obscured. Still another reason was that calculus had to be invented to describe motion. It took people of extraordinary ability, with both physical and mathematical insight, to invent the assortment of ideas and techniques needed to understand motion.

For these reasons among others, the laws were hard to discover. And they do not seem easy as we learn them ourselves. They might seem less mysterious and difficult if we could discover them in a straightforward way ourselves. If we ourselves could do the experiments and try to interpret them, we might come closer to understanding them.

Motion is an attempt to carry out this program of discovery. We will undertake an expedition to discover the laws by ourselves. We are initially freed from the confusions of gravity and friction, so we can make rapid progress. We also happen to already know some calculus and are prepared to work with vectors, so we can describe our discoveries in a precise language.

Motion therefore purports to discover the physical laws from scratch, by placing us in situations where discovery is natural, given a healthy curiosity, some mathematical background, and a willingness to do experiments and reflect upon their outcomes.

We are not quite ordinary explorers, however. We have at least two unique traits, fortunate traits in the interests of space, clarity, and our patience. First, we invent notation and concepts which are more or less the conventional ones. Second, we generally do experiments in the right order, get proper results the first time, and interpret them correctly from the outset. We manage to avoid doing the wrong experiment, falling into conceptual traps, or making mathematical mistakes.

Real physicists, of course, make lots of mistakes. Part of the challenge is to figure out when we are wrong, to check and double-check, to do an experiment or calculation in more than one way, to expose our results to others, to be prepared to be shot down, and to keep plugging until we get it right. Toward the end of his very productive life, the British physicist William Thompson (Lord Kelvin) said:

> One word characterizes the most strenuous of the efforts for the advancement of science that I have made persevering during fifty-five years; that word is *failure*.

Much more recently, Virginia Trimble, Professor of Physics at the University of California, Irvine, has written in *Visit to a Small Universe* (American Institute of Physics):

> Apart from anything else, scientific research can be rip-roaring good fun . . . When a new experiment or calculation comes out right, one shares a bit of the triumphal exultation common to all creators. When things go wrong, there is at any rate the consolation of communal suffering. Description of my most recent failure (". . .and, as a result of a small error in setting up the observing files, I am now the world's foremost expert on molecular gas ten minutes of arc south of the Crab Nebula") has elicited a similar anecdote from every astronomer I've told it to.

Motion is primarily concerned with fundamental classical mechanics, the basic ideas and guiding principles of the mechanics of Isaac Newton and others. In some of the *Sample Problems* and *Special Topics*, the text brings forward applications and extensions of the laws, to topics like rocket motion, rotating space stations, decays of atomic nuclei, astrophysics, and cosmology. Much of today's physics depends upon or is at least related to the classical laws of motion.

Motion differs from most other textbooks in a number of ways besides those already mentioned.

(1) Reference frames come early on. Where do observers stand to observe the physics?

(2) Concepts are emphasized. Which laws come directly from experiment? What kinds of experiments are needed? Which ideas are more abstract?

(3) Conservation laws come before the causes of nonconservation are identified. Momentum conservation comes before force, angular momentum comes before torque.

(4) Dynamics comes before kinematics. Uniformly accelerated motion follows from the dynamical equation ($\mathbf{F} = m\mathbf{a}$) as a special case.

(5) The importance of drawing *isolation diagrams* and how to deduce the laws of motion from them is emphasized, with a *template* of the necessary steps presented.

(6) Dimensional reasoning and dimensional analysis are used throughout.

(7) Elastic collisions are discussed in the context of time reversibility, and a pictorial method is introduced for finding the outcomes of one-dimensional elastic collisions.

(8) Except for calculus, where some knowledge is assumed, math topics are introduced as needed in the context of physics, rather than as a prologue. It was in the context of physics that the math was developed in the first place.

(9) There are two self-quizzes at the end of each "Day", with full solutions given at the end of the text. One is a "picture quiz", emphasizing concepts and pictures; the other is a "problem quiz", with more emphasis on analysis.

(10) Days *Seven* and *Fourteen* in *Motion* I, and Day *Twenty-One* in *Motion* II, are summaries of what has gone before. They are often structured in a more deductive manner than the main text, to give a different review and overview of the material.

Solving problems is an important feature of learning physics. Everybody has had the experience of hearing a clear lecture or reading a clear section of text and believing that they understand the material. However, we often find that we don't *really* understand it until we can use it in some way, such as in solving problems. Real problems are those whose answers (if they have any!) are unknown. We gain confidence in a possible answer by having to check it in as many ways as we

can, without the crutch of looking up the "correct" answer. The even-numbered problems in the text are like that. It is however also useful to practice on problems whose answers *can* be looked up - these are the odd-numbered problems, with answers given at the end of the text. These problems can help the reader gain confidence that he or she understands what is going on.

The text contains a large number of worked sample problems to help the reader understand how the concepts are applied. Some of the problems at the end of each *Day* are quite similar to the sample problems, and others are *not* so similar. They can all be solved, however, using the principles and techniques already developed. The experience of most people is that the sometimes difficult physics can only be *thoroughly* understood by solving some problems without a step-by-step template to relieve most of the confusion and anguish. Expect that some of the problems will be *hard*, and that you will understand the material better after figuring them out. If none of the assigned problems seem hard to you, find some harder ones.

Computers are now used pretty much everywhere in science. In classical mechanics they are used primarily to solve equations of motion that cannot be solved analytically (i. e., by common functions), or to calculate an integral. The most common approach is to use finite-difference numerical algorithms. In *Motion I* such numerical techniques are introduced in Appendixes in *Days One*, *Nine*, and *Eleven.*

The author greatly appreciates the many suggestions on *Motion I* made by colleagues, especially to Professors Chen, Eckert, Haskell, Lyzenga, Petersen, Rudolph, Sparks, and Townsend. The many direct and indirect suggestions made by students have also been very helpful. The goal is to make the text correct, clear, readable, and interesting. Please let me know of failures (or even successes) in reaching these goals . Physics is not easy, so some parts may require two or more readings, a careful study of the Sample Problems, trying the Picture Quizzes and Problem Quizzes, and working additional problems.

I am greatly indebted to Sally Arroyo in College Relations for the cover design, and to Kathy Morrison for production.

I hope some readers will see reflected in the text a little of the fascination with understanding the world that is at the core of doing physics. I also hope the reader will enjoy this odyssey, and that he or she comes to some understanding of the laws of motion and what they mean. That is not to say the reader will gain an ultimate understanding of why things behave as they do. That is a deep and ongoing topic of interest, and if such an understanding is possible, nobody is there yet.

Motion I

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On Solving Problems

A problem solution should be written so that another student in the course, who is unfamiliar with the problem, can readily understand both the problem and its solution.

1. Begin by writing down the problem statement, either verbatim or adequately paraphrased. Then when the homework is graded and returned, you will understand the problem and the solution when you look at it again.

2. Think about the problem before beginning to solve it. Picture the physical situation, think of techniques you might use, and guess how it should come out.

3. Draw as many diagrams of the situation described in the problem statement as are needed to make the problem and your analysis clear.

4. Write in a clear, organized way, using words, mathematics, and figures as appropriate. *It is common to use too few words and too few figures.* The solution should be an essay, with mathematics included as part of the English language.

5. *Solve a problem algebraically first; only substitute in numbers at the end.* Your approach is more general this way, and you are more likely to catch errors by taking limiting cases or by checking dimensions as you go, for example.

6. Box-in or double-underline the final answer, if it is a reasonably short expression. For numerical answers, be sure to include units (e.g., kg m/s) as appropriate. Don't include more significant figures than is justified by the information given. (See next page)

7. Check the answer in one or more ways. For example,
   (a) is the answer reasonable (is it about the size you had expected?) If it is an algebraic expression, is it right in special cases where you already know the answer?
   (b) Are the dimensions correct?

8. *Ask yourself what you have learned.*

9. It is all right to work by yourself or with other people. If you work with others, be sure everyone is contributing. Do not merely copy someone else's homework, or allow anyone else to copy yours. If some person or book (other than the instructor or textbook) helped you substantially in solving a problem, give credit to the source in a brief note at the beginning or end of the solution.

10. If there are several pages in a homework set, staple them together. Put your name, the date, and the course number (and instructor if appropriate) at the top of the first page.

Significant Figures

Many problems in this text use made-up numbers to illustrate concepts and formulas. When calculating some quantity using these numbers, there is no need to keep an unreasonable number of significant figures in your answer. For example, suppose an object covers a distance $d = 4.2$ meters in a time interval $t = 1.7$ seconds, at constant velocity v. What is v? A calculator gives $v = d/t = 1.4705882..$ m/s. In your answer, write instead $v = 1.47$ m/s, or even 1.5 m/s. That is sufficient to show that you understand how to do the calculation.

If a problem uses real data, found by you or others in the laboratory or by observation, it is essential not to exaggerate the number of significant figures. The final answer to a calculation should not claim greater accuracy than is consistent with the data from which the answer was derived: That makes the answer *false*! Examples:

(1) Find the distance $a = b + c$, where one has measured the distances b and c to be $b = 4.2 \cdot 10^3$ m and $c = 3.7 \cdot 10^2$ m.

Answer: A calculator gives $a = 4.57 \cdot 10^3$ m. A better answer, however, is $\underline{4.6 \cdot 10}^3$ m, since the data was only given to two significant figures. If you claim $a = 4.57 \cdot 10^3$ m, that implies you *believe* the last digit, the "7". There is no justification for this belief, however, because the data as given is consistent with $b = 4.24 \cdot 10^3$ m, for example, which would give $a = 4.61 \cdot 10^3$ m.

(2) Find the distance $d = vt$ where one has measured a velocity $v = 3.12 \cdot 10^6$ m/s for a measured time interval $t = 4.1$ s.

Answer: A calculator gives $d = 1.2792 \cdot 10^7$ m. However, the data do not justify so many decimal places. When multiplying two numbers, the percentage uncertainty in the product cannot be less than the percentage uncertainty in the least certain factor. Note that the velocity might be anywhere between 3.115 and 3.124, an uncertainty of about $\pm$ 0.15%, while the time interval might be anywhere between 4.05 and 4.14, an uncertainty of about $\pm$ 1% . Therefore the distance d can only be found within about $\pm$ 1%, so a more reasonable answer is $d = \underline{1.3 \cdot 10}^7$ m, because even $1.28 \cdot 10^7$ m is unreliable by itself. Better, one can write $d = (1.28 \pm .01) \cdot 10^6$ m to indicate the uncertainty of about $\pm$ 1%. (It is o.k. here to keep all digits throughout a calculation, as long as you do not claim at the end greater accuracy than is justified.)

Techniques for finding and representing errors will be discussed in greater detail in lab courses, where real data is typically used.

Units

The fundamental quantities in mechanics (often called the fundamental *dimensions*) are *mass*, *length*, and *time*.

In the International System of Units (abbreviated *SI Units* from the French *Le Systeme International d'Unites*) mass is measured in *kilograms* (kg), length is measured in *meters* (m) and time is measured in *seconds* (s). These are the primary units we will use in this text.

All other quantities in mechanics can be built upon these three. Velocity is measured in meters/second, acceleration in meters/second^2, momentum in kilogram · meter/second, and so on.

Some combinations are given their own names.

Force is expressed in *Newtons* (N), where 1 N = 1 kg m/s^2.

Energy is expressed in *Joules* (J), where 1 J = 1 kg m^2/s^2.

Power is measured in *Watts* (W), where 1 W = 1 J/s = 1 kg m^2/s^3.

Some factors of the primary units are given standard prefixes. (Those most commonly used are set in bold face type.)

Factor	*Prefix*	*Symbol*	*Factor*	*Prefix*	*Symbol*
10^{-18}	atto	a	10^{1}	deka	da
10^{-15}	**femto**	**f**	10^{2}	hecto	h
10^{-12}	**pico**	**p**	**10^{3}**	**kilo**	**k**
10^{-9}	**nano**	**n**	**10^{6}**	**mega**	**M**
10^{-6}	**micro**	**μ**	**10^{9}**	**giga**	**G**
10^{-3}	**milli**	**m**	**10^{12}**	**tera**	**T**
10^{-2}	**centi**	**c**	10^{15}	peta	P
10^{-1}	deci	d	10^{18}	exa	E

A millimeter is 10^{-3} meters, a femtosecond is 10^{-15} seconds, and a gigawatt is 10^{9} Watts.

Another commonly-used system of units is the cgs (Gaussian) system, in which length is measured in *centimeters* (cm), mass is measured in *grams* (g), and time is measured in *seconds* (s). Here 1 cm = 10^{-2} m and 1 g = 10^{-3} kg. This system is commonly used in physics research.

In the British system of units the fundamental dimensions are not mass, length, and time, but *force*, length, and time. In the British system force is measured in *pounds* (lb), length is measured in *feet* (ft), and time is measured in *seconds* (s). Conversion factors between these units and SI units are given in a table at the end of the text. A one-kilogram mass on the Earth weighs about 2.2 lbs, and one meter is about 3.28 feet.

THE BUBBLE

Awaking from a deep sleep, we find ourselves floating around inside a large spherical transparent *Bubble*. This is a surprise; we don't remember being here before. Outside, in every direction, myriad stars shine, stationary on a deep-black sky. There are no familiar constellations. There is no nearby sun and no visible planets. We seem to be isolated, stranded in space.

Day One

An Inertial Frame

"You can observe a lot just by watching." - Yogi Berra

1.1 Inertial motion

Floating slowly toward a wall of the Bubble, we discover a light switch. An inspection of the interior discloses all the comforts of home: food, water, string, stopwatches, a few meter sticks, some large metal balls, and even assorted springs · · · just the things most needed when marooned on an island in space.

For an hour or so we fritter away the time sucking breakfast from plastic tubes and playing catch with the metal balls. They seem to move in straight lines; there is no gravity here to make them fall. When we push a ball in some direction, it starts moving in that direction and keeps moving in the same direction until it hits something. To check that the balls really do move straight, we tie some strings taut from one side of the Bubble to the opposite side, running in different directions, to serve as straight-line standards. When we set a ball at rest next to a string, it stays put; when we push off a ball parallel to a string, it keeps moving right alongside.

We grab some stopwatches, and with nothing but time on our hands decide to see whether the balls also move at constant *speed*, as they appear to do. We find a meter stick and coordinate our efforts: One of us holds the stick next to a string while someone else marks off every meter on the string with a pen. Then we set a ball moving and record times as the ball passes meter marks. A plot of distance x *vs* time t gives the graph shown in Fig. 1. The ratio of distance/time is constant, within experimental error, so the speed of the ball seems to be constant.

Fig. 1. The position x of a ball as a function of time t. The slope is the change in x (written Δx) divided by the corresponding time interval Δt, in the limit of small Δt. The slope is also the ball's speed v, which is constant since $\Delta x/\Delta t$ is the same everywhere.

If we had started the ball at some position $x = x_0$ (instead of at $x = 0$), the graph would have looked as shown in Fig. 2 instead. In that case the straight-line function is

$$x = x_0 + vt \tag{1}$$

where the speed $v = \dfrac{x_f - x_0}{t_f}$ is the constant slope of the line. Here t_f is the final time, and x_f is the ball's final position.

Fig. 2. A plot of position x *vs* time t of a ball moving in the Bubble, if it starts at $x = x_0$ at time $t = 0$. Its speed v is constant, equal to the slope of the graph.

It is entertaining to launch *ourselves* off the Bubble wall, floating alongside a string while registering the times when our eyes pass meter marks. We find that we also move in a straight line at constant speed.

1.2 General one-dimensional motion

A ball launched in the Bubble and then left alone drifts in a straight line at constant velocity. We can also speed up a ball or slow it down so that it continues to move in a straight line, but no longer at constant speed. An illustration of this more general motion is shown in Fig. 3(a), a plot of its position x as a function of time t. The ball is moving to the right where the slope is positive, and to the left where the slope is negative. The ball is instantaneously at rest where the slope is zero.

The *velocity* $v(t)$ of the ball is the derivative

$$v(t) = \frac{dx(t)}{dt}, \tag{2}$$

the slope of $x(t)$. The velocity generally varies with time, as in the motion shown in Fig. 3. The *acceleration* $a(t)$ of the object is given by a second derivative,

$$a(t) = \frac{dv(t)}{dt} = \frac{d^2x(t)}{dt^2}. \tag{3}$$

The velocity of the ball, the slope of Fig. 3(a), is graphed in Fig. 3(b), and the acceleration of the ball is graphed in Fig. 3(c).

The velocity is positive and constant from $t = 0$ to $t = t_1$, when it starts to decrease. The velocity is zero at t_2, and then becomes negative until time t_4, meaning that the ball moves to the left during that time interval. At time t_4 the ball again comes instantaneously to rest. The slope then becomes positive again beyond t_4; the ball moves to the right faster and faster as the slope increases.*

Fig. 3(a) The position $x(t)$ of a ball (b) The ball's velocity $v(t)$ (c) The ball's acceleration $a(t)$

*The speed of the ball is the magnitude of its velocity, so the speed is always positive or zero. The speed of the ball is zero at t_2 and t_4; everywhere else it is positive.

The ball has zero acceleration between $t = 0$ and $t = t_1$ because the velocity is constant between these times. The acceleration is negative between t_1 and t_3, since the velocity of the ball is decreasing. The acceleration is instantaneously zero at t_3, and then it starts to increase, so the acceleration is positive after t_3. We note that x, v, and a are all continuous functions of time, with no sharp corners.

1.3 Coordinates and the position vector

How can we define the position of a ball? We often find it convenient to use Cartesian coordinates, a rectangular set of three mutually perpendicular axes (x, y, z) as shown in Fig. 4(a), with an origin located where the axes intersect. The x, y, and z coordinates of the ball are the x, y, and z distances out from the origin, as illustrated.

Alternatively, we can define the ball's position by an arrow whose tail is at the origin of coordinates and whose tip is at the center of the ball, as shown in Fig. 4(b). We call the arrow a *vector* **r**, such that **r** has three *components* x, y, z, equal to the three perpendicular distances of the ball from the origin.*

Fig. 4. (a) Cartesian coordinates (x, y, z) of a point P. (b) A vector **r**, with components x, y, and z, pointing from the origin to P.

Why represent position by a *vector*?

(1) Vectors are *pictorial*: We can "see" the ball's position to be at the tip of the vector.

(2) The notation is *compact*: A single vector **r** represents three numbers x, y, and z.

(3) Vectors are *independent of a rotation of coordinates*: If we had used a different set of Cartesian axes (x', y', z'), with the same origin but tilted with respect to our original set, as shown in Fig. 5, the coordinates of a ball would be *different* than before. But the vector **r**, represented by an arrow from the origin to the ball, would be unchanged. It is true that the components (x', y', z') of **r** along the new axes are different, but the vector itself is the same as before.

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*Vectors are usually printed in bold face (**r**), and written by hand using an arrow over the symbol ($\vec{r}$).

Fig. 5. Two different sets of Cartesian axes, with the same origin but rotated relative to one another. The position vector **r** of a ball is the same for both coordinate systems, but the components of **r** are different.

A vector **r** has both *direction* and *magnitude*. The direction is the way the arrow points, and the magnitude is the length of the vector, the square root of the sum of squares of its components, $|\mathbf{r}| \equiv r = \sqrt{x^2 + y^2 + z^2}$ according to the Pythagorean theorem in three dimensions. If we use the primed axes instead, the components x', y', z' of **r** are different, but the magnitude of **r** is the same; i.e.,

$$|\mathbf{r}| = \sqrt{x'^2 + y'^2 + z'^2} = \sqrt{x^2 + y^2 + z^2}\,. \tag{4}$$

The magnitude of a non-zero vector is always taken to be *positive*.

Sample Problem 1. A ball is placed at position $\mathbf{r} = (x, y, z) = (0,\ 2\text{ m},\ 1\text{ m})$ using the Cartesian coordinates shown. A primed set of coordinates has the same origin, but the y' axis passes right through the ball. What are the coordinates of the ball in the primed coordinates?

Solution: The ball is a distance $r = \sqrt{x^2 + y^2 + z^2} = \sqrt{0 + 2^2 + 1^2}\text{ m} = \sqrt{5}\text{ m}$ from the origin. So in the primed coordinates, $\mathbf{r} = (x', y', z') = (0, \sqrt{5}\text{ m}, 0)$, since the ball is located on the y' axis. (We use the symbol "m" to represent meters.)

1.4 Displacement vectors

Set a ball initially at position $\mathbf{r}_1 = (x_1, y_1, z_1)$, and then move it by the distances Δx, Δy, Δz, to a new position $\mathbf{r}_2$. (We use the symbol Δ ("delta") to indicate a *change* in a quantity; for example, Δx means a change in x.) Define the *displacement vector* $\mathbf{D}$ of the ball to be $\mathbf{D} = (\Delta x, \Delta y, \Delta z)$ as shown in Fig. 6(a). Note that $\mathbf{D}$ points from the old position to the new position of the ball, and its magnitude is $|\mathbf{D}| = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$.

The new position vector

$$\mathbf{r}_2 = (x_2, y_2, z_2) = (x_1 + \Delta x, y_1 + \Delta y, z_1 + \Delta z) \tag{5}$$

is the sum of $\mathbf{r}_1$ and $\mathbf{D}$, as shown in Fig. 6(b). That is, the sum $\mathbf{r}_2 = \mathbf{r}_1 + \mathbf{D}$ of the two vectors $\mathbf{r}_1$ and $\mathbf{D}$ is the vector whose components are the sum of the components of $\mathbf{r}_1$ and $\mathbf{D}$. Pictorially, the vector $\mathbf{r}_2$ extends from the *tail* of $\mathbf{r}_1$ to the *tip* of $\mathbf{D}$.

Fig. 6. (a) The displacement vector has components Δx, Δy, *and* Δz, the coordinate changes of a ball. (b) The new position vector $\mathbf{r}_2$ of a ball is the vector sum of the old position vector $\mathbf{r}_2$ and the displacement vector $\mathbf{D}$. Note that the tail of $\mathbf{r}_2$ is at the tail of $\mathbf{r}_1$, and the tip of $\mathbf{r}_2$ is at the tip of $\mathbf{D}$.

If $\mathbf{r}_2 = \mathbf{r}_1 + \mathbf{D}$ is the vector *sum* of $\mathbf{r}_1$ and D, then also $D = \mathbf{r}_2 - \mathbf{r}_1$ is the vector *difference* of $\mathbf{r}_2$ and $\mathbf{r}_1$. The components of $\mathbf{D}$ are

$$\mathbf{D} = (\Delta x, \Delta y, \Delta z) = (x_2 - x_1, y_2 - y_1, z_2 - z_1). \tag{6}$$

Pictorially, note from Fig. 6(b) that $\mathbf{D}$ points from the *tip* of r_1 to the *tip* of r_2. Note also that if we choose a different origin of coordinates, the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are different, but the displacement vector $\mathbf{D}$ is the same.

Sample Problem 2. A ball moves from the location $\mathbf{r}_1 = (1, 2, 3)$ to $\mathbf{r}_2 = (4, 1, -1)$, and then to $\mathbf{r}_3 = (3, 2, 1)$, where distances are in meters. (a) Find the net displacement vector of the ball. (b) Find the length of the vector.

Solution: (a) The net displacement vector is

$$\mathbf{D} = \mathbf{r}_3 - \mathbf{r}_1 = (3-1,\ 2-2,\ 1-3)\ \text{m} = (2, 0, -2)\ \text{m},$$

as shown. [It doesn't matter what the ball does *between* $\mathbf{r}_1$ and $\mathbf{r}_3$, for purposes of finding *net* displacement: For example, the displacement in going from $\mathbf{r}_1$ to $\mathbf{r}_2$ is $(\mathbf{r}_2 - \mathbf{r}_1)$, and the displacement in going from $\mathbf{r}_2$ to $\mathbf{r}_3$ is $(\mathbf{r}_3 - \mathbf{r}_2)$. The *net* displacement is

$$(\mathbf{r}_2 - \mathbf{r}_1) + (\mathbf{r}_3 - \mathbf{r}_2) = \mathbf{r}_3 - \mathbf{r}_1,$$

the same as if the ball went directly from $\mathbf{r}_1$ to $\mathbf{r}_3$.]

(b) The length of $\mathbf{D}$ is

$$|\mathbf{D}| = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2} = \sqrt{2^2 + 0 + (-2)^2} = 2\sqrt{2}\ \text{m},$$

which is the straight-line distance between the initial and final points. [Note that this length is not the actual distance the ball moves, because the point $\mathbf{r}_2$ is not on the straight line between the initial and final points.]

1.5 Inertial and noninertial frames of reference

We tug on a panel lying against one wall. It pulls away, exposing a set of controls. Fiddling with various levers and switches, the Bubble wall suddenly starts to move. We manage to reach a wall strap and hang on. Now all of the gear not tied down seems to be orbiting around the center of the Bubble!

Stars visible through the opposite wall are now moving; in fact they rotate around us every couple of minutes. We must have triggered a burst from external jets, causing the Bubble to rotate, so the stars appear to rotate the opposite way from our point of view.

After awhile we get used to the rotation, although our environment has changed. When we grab a handle on the wall and hold on, we seem to be pulled toward the wall by some mysterious force. If we push off from a wall a little bit, we slowly float back to it. And when we return to our game of catch, the ball curves as we throw it, no longer traveling in a straight line.

If we gently push a ball in just the right direction, we can even get it to travel in a circular arc around the Bubble interior, returning to us like a boomerang a couple of minutes later. This circular behavior doesn't square with our past experience.

What's going on? The balls used to move in straight lines, but now they curve. Why is that? The answer is that *we* are now doing something different, not the balls. The ***Bubble*** is now rotating, and we're rotating with it, so balls curve from our point of view. In fact, the balls are still moving in straight lines relative to the *stars*. For some reason they like to move in straight lines at constant speed in a reference frame in which the stars are at rest, while in other frames they curve.

In our original, nonrotating frame, the balls showed the property of *inertia*. They just kept on doing what they started out to do. If they started at rest, they stayed at rest. If they started to move in a particular direction with a particular speed, they kept on moving in the same direction with the same speed.

We call the original, nonrotating frame of reference an *inertial frame*. In that frame an initially-moving ball keeps moving with constant speed in the same direction. The frame in which we now find ourselves, which is rotating relative to the inertial frame, is *noninertial*, because the balls no longer have that property of inertia. If we put one at rest it starts to drift toward the wall, and if we toss one, it moves in a curved path from our point of view.

Balls only move in a straight line at constant speed if we don't *touch* them. When we throw them or catch them, they obviously change speed.

Based on our observations so far we will say that our frame is inertial if balls move in a straight line at constant speed (or remain at rest if started at rest) so long as nothing touches them.

Returning to the controls we push the various levers and switches back into place and hang on to the wall. The Bubble shivers and gradually stops its rotation; we stop with it. We verify we've returned to the inertial frame by throwing balls; they once again move in straight lines whichever way we throw them.

Sample Problem 3. When we switched on external jets for a brief period, we caused the Bubble to rotate about an axis through its center once every two minutes. Suppose we throw a ball from the center of the Bubble, in a direction perpendicular to the axis of rotation, toward the rim a distance $R = 12$ m away. Sketch qualitatively the path of the ball in an inertial frame and in the rotating frame. If we aim the ball toward a target on the rim, by how much will it miss the target if it is thrown with speed $v = 2.0$ m/s?

Solution: Qualitative pictures of the path are shown below. The ball moves in a straight line at constant speed in the inertial frame. Because the Bubble rotates relative to the inertial frame, the path of the ball must be curved in the rotating frame of the Bubble. The dots represent the position of the ball at one-second time intervals.

Ball's motion seen in an outside inertial frame
(In this frame the Bubble wall is seen to rotate)

Ball's motion seen inside the rotating Bubble frame
(In this frame the Bubble wall is stationary)

The miss-distance is most easily calculated in the inertial frame, where the ball moves for a time interval $t = R/v = (12\text{ m})/(2.0\text{ m/s}) = 6.0$ s in a straight line. During this time the Bubble rim, which turns once around every two minutes, turns through 1/20*th* of a full revolution, since $\frac{6\text{ seconds}}{2\text{ minutes}} = \frac{6\text{ s}}{120\text{ s}} = \frac{1}{20}$. The circumference is

$$C = 2\pi R = 24\pi \text{ m}$$

around, so the rim moves a distance $24\pi/20$ m $\approx$ 3.8 m, which is the miss-distance measured around the rim.

Ernst Mach (1836-1916), physicist and philosopher, believed that it cannot be a coincidence that inertial frames are nonrotating relative to the distant stars. In fact, he believed that the stars exert some sort of influence over local objects, causing them to move in straight lines relative to the stars. This belief is called "Mach's Principle". Suppose the universe is homogeneous and isotropic on a large scale (that is, suppose that when averaged over large enough volumes, the density of galaxies is independent of the distance from our own Milky Way galaxy, and independent of direction as well, as we look out towards them). If the Machian "influence" due to each galaxy is proportional to $1/d^n$, where d is the galaxy's distance from us, for what values of the exponent n would the net effect of distant galaxies be more important than that of nearby ones? How might we measure n experimentally? (A reference on Mach's Principle: D.W. Sciama, *The Physical Foundations of General Relativity*, Doubleday & Co. Garden City, NY, 1969.) Some physicists are attracted to Mach's idea, and others are repelled, believing it "smacks of astrology". The issue is not settled.

State of the Search

1. *Our goal is to describe and explain the motion of objects.* The first step in this quest is to choose the best rest-frame for observers. What frame of reference should they use?

   There is a local frame of reference in which objects are seen to move in especially simple ways, if nothing touches them. An object placed at rest stays at rest, and an object given an initial velocity moves in a straight line, retaining its initial velocity. That is, objects display the property of *inertia*: If nothing touches them, they keep on moving with their original velocity. This local frame of reference is called an *inertial frame.*

   If this local inertial frame is far from gravitating objects like stars or planets, then it does not rotate relative to the distant stars. Frames that *do* rotate relative to the distant stars are *noninertial*, in that objects observed from the point of view of such a frame do not behave so simply: Their paths are generally curved, for example, even though nothing touches them.

   Why does there exist an inertial frame, in which the natural motion of an object is a straight line at constant speed? No one knows. Why does this inertial frame, determined from purely local observations, not rotate relative to the distant stars? No one knows.

2. The position of an object and its displacement from one position to another can both be represented by *vectors*, each with a magnitude and a direction. The position vector of an object depends upon the choice of origin, but the displacement vector, pointing from the object's initial position to its final position, is independent of the choice of origin. The velocity of the object can also be represented by a vector, where the vector's length is the speed of the object, and the vector's direction is in the direction of motion.

3. The *sum* $\mathbf{A}+\mathbf{B}$ of two vectors $\mathbf{A}$ and $\mathbf{B}$ is found by first moving $\mathbf{B}$ parallel to itself until its tail is at the tip of $\mathbf{A}$, as illustrated; then the vector $\mathbf{A}+\mathbf{B}$ extends from the tail of $\mathbf{A}$ to the tip of $\mathbf{B}$.

   The *difference* $\mathbf{A}-\mathbf{B}$ of two vectors $\mathbf{A}$ and $\mathbf{B}$ is found by adding $-\mathbf{B}$ to $\mathbf{A}$, where $-\mathbf{B}$ has the same magnitude as $\mathbf{B}$, but points in the opposite direction. Alternatively, the difference can be found by first placing the tails of $\mathbf{A}$ and $\mathbf{B}$ together; then $\mathbf{A}-\mathbf{B}$ points from the tip of $\mathbf{B}$ to the tip of $\mathbf{A}$, as shown.

Appendix A: Vectors in general

Many physical quantities can be represented by vectors, so it is useful to summarize some of their general properties. An arbitrary vector $\mathbf{A}$ in three dimensions has components A_x, A_y, and A_z along Cartesian axes, so we can write $\mathbf{A} = (A_x, A_y, A_z)$. In the case where $\mathbf{A}$ has no z-component, the vector and its components are illustrated in Fig. A1.

Fig. A1. Components of a vector in two dimensions. The x-component of $\mathbf{A}$ is $A_x = A\cos\theta$ and the y-component of $\mathbf{A}$ is $A_y = A\sin\theta$, where A (without bold face) is the magnitude of $\mathbf{A}$, $A = \sqrt{A_x^2 + A_y^2}$.

We can also express $\mathbf{A}$ in terms of *unit vectors* pointing in the x, y, and z directions, respectively, as shown in Fig. A2(a). If $\mathbf{x}$ is a vector in the x-direction, and x (without bold face) is the *magnitude* (length) of the vector, we define the unit vector $\hat{\mathbf{x}}$ to be $\hat{\mathbf{x}} = \mathbf{x} / x$, so it clearly has magnitude "one" and points in the x-direction. We define $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ similarly. Then an arbitrary vector $\mathbf{A}$ can be decomposed into

$$\mathbf{A} = A_x\,\hat{\mathbf{x}} + A_y\,\hat{\mathbf{y}} + A_z\,\hat{\mathbf{z}}$$

where A_x, A_y, and A_z are the Cartesian components of $\mathbf{A}$. According to Eq. 7, $\mathbf{A}$ is the sum of the three vectors $A_x\hat{\mathbf{x}}$, $A_y\hat{\mathbf{y}}$, and $A_z\hat{\mathbf{z}}$, as shown in Fig. A2(b), where A_x is the length and $\hat{\mathbf{x}}$ the direction for the vector in the x-direction, for example. The magnitude of the vector $\mathbf{A}$ is

$$|\mathbf{A}| \equiv A = \sqrt{A_x^2 + A_y^2 + A_z^2}\,.$$

Fig. A2. (a) Unit vectors (b) The vector **A** and its components in three dimensions

We can *add* the vectors **A** and **B** to form a vector **C** = **A** + **B**, as illustrated in Fig. A3(a). We place the tail of **B** at the tip of **A**; then **C** starts at the tail of **A** and ends at the tip of **B**. If **A** has components (A_x, A_y, A_z) and **B** has components (B_x, B_y, B_z), then **C** has components

$$(C_x,\ C_y,\ C_z) = (A_x + B_x ,\quad A_y + B_y ,\quad A_z + B_z).$$

Fig. A3. (a) Addition of vectors **A** and **B** (b) Finding the vector difference **D** = **A** − **B** by adding the vector − **B** to **A** (c) Subtraction of **B** from **A** by placing the tails of **A** and **B** together; the difference **D** = **A** − **B** then extends from the tip of **B** to the tip of **A**. (d) The results of (b) and (c) are the same because the two methods give opposite sides of a parallelogram.

We can also *subtract* a vector **B** from a vector **A** to form a vector **D** = **A** − **B**. A natural way to subtract **B** from **A** is to *add* the vector − **B** to the vector **A**, as illustrated in Fig. A3(b). The vector − **B** has the same length as **B**, but points in the opposite

direction, as shown. So the first step in finding $\mathbf{D}$ is to sketch $-\mathbf{B}$; this vector $-\mathbf{B}$ is then moved parallel to itself until its tail is at the same point as the tip of $\mathbf{A}$. The vector $\mathbf{D}$ then extends from the tail of $\mathbf{A}$ to the tip of $-\mathbf{B}$, as shown in Fig. A3(b).

An alternative, and equivalent, way to subtract $\mathbf{B}$ from $\mathbf{A}$ is to start by placing the tails of $\mathbf{A}$ and $\mathbf{B}$ together. Then $\mathbf{D} = \mathbf{A} - \mathbf{B}$ is the vector whose tail is at the tip of $\mathbf{B}$ and whose tip is at the tip of $\mathbf{A}$. This method of subtraction is illustrated in Fig. A3(c). Note that the result $\mathbf{D} = \mathbf{A} - \mathbf{B}$ is the same as in Fig. A3(b).

Why do the two different methods of subtraction give the same results? The reason is illustrated in Fig. A3(d). The line representing $-\mathbf{B}$ is parallel to the line representing $\mathbf{B}$, and the vectors are of equal length. Together with the vector $\mathbf{D} = \mathbf{A} + (-\mathbf{B})$, they form three sides of a parallelogram. The lower side (which represents subtraction by the first method) therefore has the same length (as well as the same direction) as the upper side, which represents subtraction by the second method.

Sample Problem (a) Sketch the vectors $\mathbf{A} = 2\hat{\mathbf{x}} + 3\hat{\mathbf{y}}$ and $\mathbf{B} = 3\hat{\mathbf{x}} - 2\hat{\mathbf{y}}$. (b) Find $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} - \mathbf{B}$, and sketch them as well.

Solution: (a) The vectors are as shown at the right, if we place their tails at the origin.

(b) The sum of the vectors is

$$\mathbf{A} + \mathbf{B} = (2+3)\hat{\mathbf{x}} + (3-2)\hat{\mathbf{y}} = 5\hat{\mathbf{x}} + \hat{\mathbf{y}}.$$

To construct the sum graphically, we can transport $\mathbf{B}$ parallel to itself so its tail is at the tip of $\mathbf{A}$, as shown. The sum is the vector extending from the tail of $\mathbf{A}$ to the tip of $\mathbf{B}$.

The *difference* between the two vectors is

$$\mathbf{A} - \mathbf{B} = (2-3)\hat{\mathbf{x}} + (3+2)\hat{\mathbf{y}} = -\hat{\mathbf{x}} + 5\hat{\mathbf{y}}.$$

To construct the difference graphically, we can leave the tails of $\mathbf{A}$ and $\mathbf{B}$ together; then $\mathbf{A} - \mathbf{B}$ is the vector extending from the tip of $\mathbf{B}$ to the tip of $\mathbf{A}$, as shown.

Alternatively, we can first draw the vector $-\mathbf{B}$ and then add it to $\mathbf{A}$, as shown at the bottom.

Appendix B: Derivatives and Finite Differences

The graph shows a function $y(x)$. The derivative of $y(x)$ is the slope

$$\frac{dy(x)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \equiv \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x},$$

which is the difference Δy between two y's divided by the difference Δx between two x's, in the limit as $\Delta x \to 0$. For example, if the function is $y(x) = x^2$ (which the graph above actually shows), then

$$\frac{dy(x)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - (x)^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x.$$

This is the *exact* derivative as a function of x. For example, at $x = 1$, $\frac{dy(x)}{dx} = 2$.

We can also find an *approximate* derivative of a smooth function $y(x)$ if Δx is small, where we do not (or for some reason *cannot*) take the limit $\Delta x \to 0$. Suppose we want the derivative at $x = 1$, and we set $\Delta x = 0.2$, as illustrated at the right. Then an *approximate* derivative at $x = 1$ is

$$\frac{dy(x)}{dx} \cong \frac{y(x+\Delta x) - y(x)}{\Delta x} = \frac{1.2^2 - 1.0^2}{0.2} = \frac{0.44}{0.2} = 2.2,$$

10% too large. If we shrink Δx to $\Delta x = 0.1$, then

$$\frac{dy(x)}{dx} \cong \frac{1.1^2 - 1.0^2}{0.1} = \frac{0.21}{0.1} = 2.1,$$

only 5% too large. We could obviously do still better by further shrinking Δx.

This method of finding derivatives by finite differences is very useful when we do not have a graph that matches a standard function like $y = x^2$. The graph might be found

from experimental data, for example, so would be unlikely to correspond to any standard function.

The table below shows (hypothetical) measurements of the position x of an object at times separated by equal intervals $\Delta t = 1.0$ s. The points are displayed on a graph at the right. We would like to know: What is the (approximate) velocity of the object at $t = 5.0$ s?

t(s)	x(m)
0.0	0.00
1.0	1.10
2.0	2.15
3.0	3.05
4.0	3.98
5.0	4.80
6.0	5.51
7.0	6.11
8.0	6.60
9.0	6.98

We can use the finite difference method $v = \dfrac{dx(t)}{dt} \cong \dfrac{x(t+\Delta t) - x(t)}{\Delta t}$ to find out. At time $t = 5.0$ s,

$$\frac{dx(t)}{dt} \cong \frac{5.51\text{ m} - 4.80\text{ m}}{1.0\text{ s}} = \frac{0.71\text{ m}}{1.0\text{ s}} = 0.71\text{ m/s}.$$

Assuming the motion is smooth we could interpolate between points and refine our estimate. In that case, it is clear from the graph that the actual velocity of the object at $t = 5.0$ s should be somewhat larger than 0.71 m/s.

Finite difference methods are especially useful in solving equations containing derivatives; i.e., so-called "differential equations", using computers. We will discuss these methods at the end of *Day Nine*.

Appendix C: Velocity and Acceleration Vectors in Three Dimensions

When an object moves in three dimensions, its position is given by the vector $\mathbf{r} = (x, y, z)$, so its velocity has the three components

$$\mathbf{v} = (v_x, v_y, v_z) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right).$$

The acceleration of the object also has three components,

$$\mathbf{a} = (a_x, a_y, a_z) = \left(\frac{dv_x}{dt}, \frac{dv_y}{dt}, \frac{dv_z}{dt} \right).$$

Therefore if we know $\mathbf{r}(t) = (x(t), y(t), z(t))$, we can find the velocity and acceleration components in the same way as we find the velocity and acceleration in one dimension.

The velocity may also be thought of as the derivative

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t},$$

the difference between two position vectors divided by the time interval between them, in the limit of small Δt. Suppose for example the positions of a particle at two nearby times t and $t + \Delta t$ are as shown. Then the vector velocity $\mathbf{v}$ is the difference between the two position vectors (the little vector $\Delta \mathbf{r}$ that goes from the tip of $\mathbf{r}(t)$ to the tip of $\mathbf{r}(t + \Delta t)$, divided by the time interval Δt, in the limit as $\Delta t \to 0$.

The acceleration is similarly the difference between two velocity vectors, divided by Δt in the limit of small Δt,

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t}.$$

PICTURE QUIZ 1

Test your understanding. Answers are at the end of the text.

1. A ball is thrown in the Bubble, so that nothing touches it until it is caught. In each of the following pictures of the ball's motion as seen by observers within the Bubble, is the Bubble rotating clockwise, counterclockwise, or is it nonrotating relative to an inertial frame?

(a)

(b)

(c)

(d)

(e)

(f)

2. The position of a ball as a function of time is shown in the graph. At which times or time periods is the ball

   (a) at rest?
   (b) moving to the right?
   (c) moving to the left?
   (d) not accelerating?
   (e) accelerating to the right?
   (f) accelerating to the left?

3. Departing from Gamma Cepheus V, whose galactic location is specified by the position vector $\mathbf{r} = -27\hat{\mathbf{x}} + 32\hat{\mathbf{y}} - 79\hat{\mathbf{z}}$ light years (one light year is the distance light travels in one year), we move at constant speed $v = c/3$ (where c is the speed of light) for the time interval 360 years before arriving at our destination Eta Carina IV, whose position vector is $\mathbf{r} = 13\hat{\mathbf{x}} - 18\hat{\mathbf{y}} - 49\hat{\mathbf{z}}$ light years. To two significant figures, what is the magnitude of our displacement vector?

(a) 43 light years

(b) 71 light years

(c) 100 light years

(d) 120 light years

(e) none of the above

(4) Shown at the right are two vectors **A** and **B**. Which set of vectors below correspond to the correct sum $\mathbf{S} = \mathbf{A} + \mathbf{B}$ and difference $\mathbf{D} = \mathbf{A} - \mathbf{B}$?

(a)

(b)

(c)

(d)

PROBLEM QUIZ 1

Solve the problems. Solutions are given at the end of the text.

1. The graph at the right shows the position of an object as a function of time. Find the velocity of the object at the times

   (a) $t = 2$ s

   (b) $t = 6$ s

   (c) $t = 8$ s

   (d) $t = 11$ s

2. Which of the following vectors has the greatest magnitude? Which has the least?
   (a) $\hat{\mathbf{x}} + \hat{\mathbf{x}} + \hat{\mathbf{x}}$ (b) $\hat{\mathbf{x}} + \hat{\mathbf{y}}$ (c) $\hat{\mathbf{x}} + \hat{\mathbf{y}} - \hat{\mathbf{z}}$

3. Vector $\mathbf{A}$ has magnitude 3 and vector $\mathbf{B}$ has magnitude 2. What is the range of possible magnitudes of $\mathbf{A} + \mathbf{B}$? What is the range of possible magnitudes of $\mathbf{A} - \mathbf{B}$?

4. A fly flies around a cubical room of edge-length L, starting at the northwest corner of the ceiling. (a) If the fly flies successively to each of the eight corners of the room, landing finally at the southeast corner of the floor, what is the magnitude of its total displacement vector? (b) If the fly flies successively to each of the eight corners, and then ends by returning to its starting point, what is the magnitude of its total displacement vector?

Special Topic I: Geodesics

Geodesics (jē-ah-des′-iks) are shortest-distance paths in space or on surfaces. The geodesic between two given endpoints a and b is the curve from a to b which has the shortest length. On a flat surface, geodesics are straight lines. On the surface of a sphere, geodesics are great circles, the circles defined by the intersection of the spherical surface with planes passing through the center of the sphere. Longitude lines on Earth are arcs of great circles (if Earth is approximated as a sphere). The equator is also a great circle, but other lines of constant latitude are not great circles. Long-distance airline routes are typically portions of great circles, because these paths minimize the distance traveled, and therefore tend to minimize time and fuel use. For example, a non-stop flight from Los Angeles to London is typically a great-circle "polar" route over Hudson Bay in Canada and southern Greenland.

What are the geodesic paths on other surfaces? Often these are hard to find, but there is a class of surfaces for which geodesics are easy to find. These surfaces are called "developable" surfaces, surfaces that can be made from a rolled-up sheet of paper. We can make both cylinders and cones by rolling up sheets of paper, for example, so cylinders and cones are both developable.

Suppose an ant crawls between two given point a and b on the surface of a cylinder. How should the ant crawl to minimize the distance it has to move? The answer is easy: All we have to do is to slice the cylinder lengthwise and lay it flat, as shown. With a ruler we draw a straight line from a to b. That straight-line path is a geodesic on the flattened ("developed") cylinder. It is also a geodesic when we roll the cylinder back up, because rolling the cylinder up does not stretch any of the distances involved. The paper can be rolled, but it does not stretch in any direction. So the ant on the cylinder should travel along the line we drew. To find the absolute shortest path, we have to be sure to slice the cylinder in the best way before laying it flat. Unless the endpoints are on exact opposite sides of the cylinder, the angle of b around the cylinder from a will be less in one direction than the other. So we need to slice the cylinder on the side of the cylinder for which the angle of b from a is greater. That way the straight line we draw will be the shortest path possible.

Sample Problem. A rectangular room is 5 m wide, 6 m long, and 4 m high, as shown. An ant starts on the floor at one corner of the room, at point $(x, y, z) = (0, 0, 0)$, and crawls along the room surfaces to the farthest point away within the room. If the ant's top speed is $v = 5$ cm/s, what is the minimum time it will take the ant to reach its destination? ("cm" is the symbol for *centimeters* (1 cm = 10^{-2} m) and "s" is the symbol for *seconds*.)

Solution: The minimum-time path is the same as the minimum-*distance* path, if we fix the speed as the maximum speed possible. The minimum-distance path is easiest to see by slicing edges of the room at strategic places, and laying it flat, as shown. The three points marked "P", the ant's destination, are identical points when the room is reassembled. There are three straight-line paths shown, labeled ①, ②, ③. Path ① is the shortest, with length

$$\sqrt{6^2 + (5 + 4)^2}\ \text{m} = 10.8\ \text{m}.$$

An ant moving at constant speed $v = 5$ cm/s would take the time

$$t = \frac{\text{distance}}{\text{speed}} = \frac{10.8\ \text{m}}{5\ \text{cm/s}}\ [\,100\ \text{cm} / 1\ \text{m}\,] =$$

216 seconds ($= 2 \cdot 10^2$ s to one sig.fig.)

Note: All three paths shown are geodesics, because all three are shorter than any nearby path, i.e., shorter than any path which is a small distortion from the straight line.

Geodesics are closely linked to motion. They are especially important in gravitation physics, in that free particles (particles moving in the absence of nongravitational forces) and light rays move along geodesics in four-dimensional spacetimes curved by massive bodies like the Earth or Sun. The motion of the Earth, for example, is close to that of a geodesic in the curved spacetime created by the Sun.

Special Topic II: Spatial Dimensions

The space around us seems to be 3 dimensional. Tables have height, width, and depth. Starting from the beach at Malibu, a direct trek to the summit of Mt. Whitney requires that one go north, east, and "up".

Sometimes a 1-dimensional or 2-dimensional description is good enough. The location of an ant crawling along a clothesline can be specified quite well by a single number - the ant's distance from one end of the line. The location of a truck traveling on the I10 can be specified quite well by its distance from Santa Monica. Both of these motions are approximately 1-dimensional, since a single number is sufficient to fix the position.

The location of the ant on a flat tabletop requires instead *two* numbers. The location of a car in a supermarket parking lot likewise requires two numbers. Motion in these cases is approximately 2-dimensional.

Specifying the dimensionality of a space does not exhaust its properties, however; we also have the freedom to specify a *topology*. The infinite line is said to have topology R^1, the line extending from $-\infty$ to $+\infty$. An ant crawling in one direction never returns to the same point. A line can also be wrapped into a circle with topology S^1; an ant crawling along this line eventually returns to where it began. A two-dimensional infinite sheet of paper has topology R^2. If a piece of this sheet is rolled into a cylinder and the two edges glued together, the topology is $R^1 \times S^1$. An ant moving in a particular direction on the cylinder can return to its starting point. If a cylinder is sliced to make it finitely long, and if it is made from rubber rather than paper, the two ends of the cylinder can be brought together and glued to form a doughnut or torus, with topology T^2.

Whatever the topology, one number is required to specify location in a 1-dimensional space, and two numbers are required to specify location in a 2 - dimensional

space. A 2 - dimensional space might have topology R^2, $R^1 \times S^1$, T^2, or S^2 (the surface of a sphere). Whatever the topology, a 2-dimensional creature, with no perception of a third dimension, would conclude that its space is 2-dimensional, based upon the fact that two numbers are needed to specify an arbitrary location in it.

We think we are living in a 3-dimensional world because we perceive that three numbers are needed to specify a point within it. The topology may be R^3, infinite in all three directions, but it may not be. It is certainly conceivable that if we traveled far enough in any direction we might return to where we started, with (say) a topology S^3 (a three-sphere), analogous to the surface of a sphere in 2 dimensions.

It is also possible that our space is not really 3-dimensional at all. There are at least two ways we might be fooled. It may be that we are actually living in a universe of more than three spatial dimensions, in which all but three are "hidden" from us in some way. The theory known as *string theory* (or its successor called *M - theory*) suggests, for example, that space is 10- or 11-dimensional. The 6 or 7 "hidden" dimensions might be curled up locally on a very small scale so that we don't perceive them. This would be analogous to the ant walking along the stretched clothes-line; from a distance it looks as though the ant's space is one-dimensional, but on closer inspection the space is two-dimensional, because the string has finite thickness and the bug could walk around the clothesline as well as along it. One of the dimensions is curled up on a small scale, which is hidden from view if you are far away. So we can conceive of the idea that a 6 or 7-dimensional tightly-curled ball may be associated with each point of the 3-dimensional space of our perceptions. If we had a hypothetical microscope of sufficient resolution we could "see" the other dimensions. There is no direct evidence (yet) that this is true, but there are reasons to believe that it might be true, with the other dimensions curled up on the scale of 10^{-35} meters, a much finer scale that we have ever been able to measure.

A second possibility is that we live in an N-dimensional universe with $N > 3$, but that light and the particles we commonly observe (electrons, protons, etc.) are confined to move in a 3-dimensional portion of the full universe. Therefore, so long as our detection apparati only detect light and these common particles, the other dimensions would be unobservable. Of course if *nothing* interacts with the wider N-dimensional realm, the existence of the other dimensions is a question in metaphysics, not physics, because physics is supposed to deal with things that are observable, at least in principle. Some *string theorists* have suggested that gravity alone may be free to escape the three-dimensional portion of our world, and we observe the other dimensions only in their ability to influence our three-dimensional world gravitationally.

The dimensionality of space is currently a lively topic of investigation in string theory, which some hope will be a "Theory of Everything", where the existing fundamental particles and their properties, and everything else in nature, will be determined by the theory. An important feature of this theory is the existence of hidden dimensions.

Questions and Problems from Day One

1Q1. Starting at a wall, an explorer pushes off toward the Bubble interior. How could she come to rest in the interior? If originally at rest in the interior, is there anything she could do to reach the Bubble wall?

1Q2. Explain why the explorers found they were attracted toward the wall, while the Bubble was rotating. [Hint: suppose an explorer is a short distance from the curved wall of the Bubble in the rotating frame. Picture the explorer and Bubble as seen by somebody at rest in an inertial frame, in which the Bubble is seen to rotate. How does the explorer move in the inertial frame, and why does this motion tend to draw the explorer to the wall?]

1Q3. In an inertial frame, a wheel spins freely in space about an axis that stays fixed relative to the frame. In the absence of air resistance, if the wheel turns once around as a free particle moves 1 cm in a straight line, then the wheel turns once *more* around as the particle moves an additional 1 cm, and so on. Use this fact to suggest a simple clock for the inertial frame, so that it isn't necessary to have stopwatches or other manufactured clocks in the Bubble. Such simple clocks show that it is natural to *define time to make motion look simple.*

1Q4. Is it true that if the sum of three vectors is *zero*, the vectors must all lie in the same plane?

1Q5. Saboteurs once tried to derail a train by removing a section of one of the rails, only to find that the train kept moving in a straight line over the ties and rolled right onto the next section of rail. What could the saboteurs have done to derail the train?

1Q6. What are the geodesic paths on the surface of a sphere? How are these related to long-distance airline routes on Earth?

1Q7. The velocity of a ball as a function of time is shown in the graph. At which times or time periods is the ball (a) at rest?
(b) moving to the right?
(c) moving to the left?
(d) not accelerating?
(e) accelerating to the right?
(f) accelerating to the left?

Section 1.1 Inertial motion

1P1. Two trains move in the same direction on parallel tracks. A freight train passes Barstow at 12:00 noon, headed east at v_f = 45 miles/hr. The *Southwest Chief* passes Victorville at 11:30 am the same day, headed east at v_C = 60 miles/hr. Victorville is a distance D = 35 miles west of Barstow. If the trains continue at the same constant speeds, at what time does the *Southwest Chief* overtake the freight train? How far are they from Barstow at this time?

1P2. On a straight stretch of the Indianapolis racetrack, Emerson Fittipaldi's car, moving at speed 120 m/s, is a distance 65 m behind Bobby Unser's car, moving at 98 m/s. Assuming they maintain these speeds, how much time will it take Fittipaldi to overtake Unser? How far will Fittipaldi have moved during this time?

Section 1.2 General one-dimensional motion

1P3. Find both the velocity and acceleration of a ball as functions of time, if the ball's position as a function of time is given by (a) $x = \alpha t^3$, where α is a constant. (b) $x = \dfrac{x_0}{1+\alpha t}$, where x_0 and α are constants.

1P4. Find both the velocity and acceleration of a spaceship as functions of time, if the spaceship's position is given by (a) $x = x_1 \ln(\alpha t + \beta)$, where x_1, α, and β are constants. (b) $x = Ae^{\alpha t}$, where A and α are constants.

1P5. Find both the velocity and acceleration of a golfball as functions of time, if the golfball's position as a function of time is given by (a) $x = x_1 \tan(t/t_0)$, where x_1 and t_0 are constants. (b) $x_1 \operatorname{cotan}(t/t_0)$, where x_1 and t_0 are constants.

1P6. An electron in an antenna has position $x = A\cos(\omega t + \phi)$ as a function of time, where A, ω, and ϕ are constants. Find the electron's (a) velocity (b) acceleration, both as functions of time. (c) From the expression for $x(t)$, one can see that A has units of length and ω has units of inverse time. Verify that your results for (a) and (b) have the correct units for velocity and acceleration.

1P7. The graph as the right shows the position $x(t)$ of a particle as a function of time. Between what times is the particle (a) moving to the right? (b) moving to the left? (c) accelerating to the right? (d) accelerating to the left?

1P8. The position of a car is shown as a function of time. Redraw the graph, and carefully sketch beneath it graphs of the car's (a) velocity (b) acceleration

Section 1.3 Coordinates and the position vector

1P9. A particular chessboard is a square grid containing 64 squares, each one inch on a side. Place an origin of coordinates at one corner of the board, with the x-axis along one edge and the y-axis along the other. The positive z-direction is above the board. (a) What is the position vector $\mathbf{r}$ of a point on the floor 36 inches below the center of the board? (b) What is the distance of this point from the origin?

1P10. Salt Lake City has a rectangular grid of flat north-south and east-west streets. Suppose the blocks are square, each of length L. Pick a particular intersection at ground level as the origin, take east as the positive x-direction, north as the positive y-direction, and "up" as the positive z-direction. (a) Using unit vectors, what is the position vector of a traffic light a height h above an intersection 3 blocks west and two blocks north of the origin? (b) What is the distance of the traffic light from the origin?

Section 1.4 Displacement vectors

1P11. A bee flies from position $\mathbf{r}_1 = (2, 4, 0)$ to $\mathbf{r}_2 = (6, 7, 0)$, where distances are in meters. (a) Sketch $\mathbf{r}_1$ and $\mathbf{r}_2$ in an (x, y) coordinate system. (b) Sketch the displacement vector $\mathbf{D}$ on the same diagram. (c) Find the displacement vector $\mathbf{D}$ numerically, using unit vectors. (d) Find the minimum distance the bee could have flown between the two points.

1P12. A hiker walks from position $\mathbf{r}_1 = (52, 65, 77)$ m to position $\mathbf{r}_2 = (73, 34, 83)$ m. (a) Find the net displacement vector $\mathbf{D}$. (b) Find the magnitude of $\mathbf{D}$. (c) Find the minimum distance the hiker could have walked.

1P13. A golfball on a putting green starts at position $\mathbf{r}_1 = (2, 1, 0)$ and is given successive displacements $\mathbf{D}_1 = (5, -1, 0)$ and $\mathbf{D}_2 = (0, 2, 0)$, where the distances are in meters. (a) Sketch $\mathbf{r}_1$ on an x, y coordinate system and then graphically add $\mathbf{D}_1$ to it. The result is position $\mathbf{r}_2$. (b) Then graphically add $\mathbf{D}_2$ to the sum found in part (a); the result is position $\mathbf{r}_3$. (c) Find $\mathbf{r}_2$ and $\mathbf{r}_3$ numerically, using unit vectors. Do your results appear to be consistent with the graphical results found in parts (a) and (b)? Explain.

Section 1.5 Inertial and noninertial frames of reference

1P14. A ball is thrown from the center of the rotating Bubble toward a target at the rim, a distance $R = 12$ m away. The Bubble turns once around in the time $T = 2.0$ minutes relative to the stars. (a) By how much will the ball miss the target if it is thrown at speed $v = 4.0$ m/s? (b) How fast must it be thrown to miss the target by less than $s = 10$ cm? (c) Sketch the ball's trajectory as seen in an inertial frame at rest relative to the Bubble's center. (d) Sketch the trajectory as seen in the frame of the rotating Bubble.

1P15. The explorers made the Bubble rotate about an axis through its center with a *period* (the time to turn once around) of $T = 2.0$ minutes relative to inertial frames and to the stars. (a) Is there any point within the rotating frame from which a ball can be thrown, such that observers within will see the ball go in a straight line at constant speed? Is there a particular direction in which it must be thrown? (b) How can observers in the rotating frame throw a ball so it will go in a circle, returning to its starting point? Find the speed at which they must throw the ball, in terms of the distance of the starting point from the axis of rotation.

1P16. Children are playing on a merry-go-round. The merry-go-round, of radius $R = 4.0$ m, is turning clockwise as seen from above, once around every 5.0 s. A boy standing at the center throws a ball directly toward a girl standing at the rim who is facing him. (a) To catch the ball, will she have to move to her left or to her right? Explain, with the help of diagrams. (b) If he throws the ball with horizontal velocity 12 m/s, how far will she have to move to catch the ball?

1P17. A hunter standing on the North Pole of Earth throws a spear at a target 100 m away on the ground; it arrives 3.0 s later. Will the spear strike the target at its center, a bit to its left, or a bit to its right, as seen by the hunter? Explain. *Hint:* Consider Earth's rotation. If it does not strike the target's center, by about how much will it miss?

1P18. Two vectors **A** and **B** have magnitudes 4 and 3 units, respectively, and are oriented as shown; **A** makes an angle of 15° with the x-axis, and **B** an angle of 30° with the y-axis.
(a) If $\mathbf{C} = \mathbf{A} + \mathbf{B}$, what are the x and y components of **C** ?
(b) What is the magnitude of **C** ?
(c) What is the angle **C** makes with the x axis?

1P19. A ball is at position $(x, y) = (2.5, 3.6)$ meters. Find (a) the distance r of the ball from the origin, and (b) the angle θ the position vector **r** makes with the y-axis.

1P20. A ball is first displaced by $\mathbf{D}_1 = 3\hat{\mathbf{x}} - 2\hat{\mathbf{y}} + \hat{\mathbf{z}}$, and then by $\mathbf{D}_2 = -2\hat{\mathbf{x}} + \hat{\mathbf{y}} + 4\hat{\mathbf{z}}$, where the distances are in meters. (a) Find the net displacement vector **D**. (b) Find the magnitude of **D**. (c) Is your answer to (b) equal to the total distance the ball moves? Explain.

1P21. A particle is first displaced by $\mathbf{D}_1 = 2\hat{\mathbf{x}} + 2\hat{\mathbf{y}} - \hat{\mathbf{z}}$, then by $\mathbf{D}_2 = -4\hat{\mathbf{x}} + 3\hat{\mathbf{y}} - 4\hat{\mathbf{z}}$, where the distances are in meters. (a) Find the net displacement vector **D**.
(b) Find the magnitude of **D**. (c) Is your answer to (b) equal to the total distance the ball moves? Explain.

1P22. Two vectors **a** and **b** make an angle θ with each other when placed tail to tail. If $\mathbf{c} = \mathbf{b} - \mathbf{a}$, prove that the magnitude of **c** is $c = \sqrt{a^2 + b^2 - 2ab\cos\theta}$. This is the "law of cosines". *Hint:* Show that the triangle constructed from **a**, **b**, and **c** can be composed from two right triangles; then use the Pythagorean theorem.

1P23. (a) Show, by drawing pictures and using trigonometry, that if x and y are the components of an object's position in a Cartesian coordinate system, then

$$x' = x\cos\theta + y\sin\theta \quad \text{and} \quad y' = -x\sin\theta + y\cos\theta.$$

are the coordinates of the object in a *primed* Cartesian coordinate system which is rotated by angle θ in the counterclockwise sense relative to the original axes.
(b) Then show that the *magnitude* of a given position vector is the same in both frames, for *any* rotation angle θ.

1P24. (a) Find the x and y components of a vector of length $\ell = 4.0$ m, oriented at 30° to the x-axis, as shown. (b) Find the x' and y' components of the same vector, in a primed coordinate system rotated by 60° in the counter-clockwise sense, relative to the original axes.

Appendix B: Derivatives and Finite Differences

1P25. Consider the function $y(x) = \sqrt{x}$. (a) Find dy/dx exactly for arbitrary x, and evaluate it at $x = 1.0$. Then find approximate values for dy/dx at $x = 1.0$ using finite differences, if (b) $\Delta x = 0.2$ (c) $\Delta x = 0.1$.

1P26. The table below gives the position x of a car at various times t. (a) Plot x as a function of time. Then using finite differences, find the approximate velocity of the car at times (b) 5 s (c) 10 s. (d) In each of these cases, is the approximation likely to be an *under*estimate or an *over*estimate of the actual velocity?

t(s)	x(m)	t(s)	x(m)
0.0	0.0	7.0	121
1.0	3.5	8.0	134
2.0	9.5	9.0	132
3.0	22.0	10.0	123
4.0	45.9	11.0	109
5.0	79.7	12.0	91
6.0	105		

Appendix C: Velocity and Acceleration Vectors in Three Dimensions

1P27. The vector position of a ball is $\mathbf{r}(t) = (x(t), y(t), z(t)) =$ $(x_0 e^{-\alpha t},\ y_0 e^{-\beta t^2},\ z_0 e^{-\gamma t^3})$ where x_0, y_0, z_0, α, β, and γ are constants. Find (a) the velocity (b) the acceleration of the ball as a function of time.

1P28. A particle is moving counter-clockwise around a circle. Its position vector $\mathbf{r}$ at time t is oriented at angle θ to the positive x axis, as shown. What is the angle of the particle's velocity vector at time t, relative to the positive x axis?

Special Topic I: Geodesics

1P29. Show that if the room is sliced differently in the *Special Topic* Sample Problem, before it is laid flat, one can display three additional geodesic paths for the ant to walk between the same two points. Show also that these three paths have the same lengths as the paths given in the Sample Problem. *Optional*: Are there additional geodesic paths beside the six? How many geodesic paths are there altogether?

1P30. Points on the surface of a right circular cylinder oriented along the z-axis have x and y coordinates given by $x = R\cos\theta$, $y = R\sin\theta$, where R is the cylinder's radius, $R = (4/\pi)$ m, and θ is the angle around it. If an ant crawls along the cylinder from $z_1 = 0$, $\theta_1 = 0$ to $z_2 = 3$ m, $\theta_2 = \pi$ radians,

(a) what is the displacement vector $\mathbf{D}$ of the ant in terms of numbers and unit vectors?

(b) What is the shortest distance the ant can walk between the given points, if it stays on the cylindrical surface?

(c) Draw the displacement vector and the shortest distance path on a sketch of the cylinder.

1P31. A cone has half-angle 30°: That is, every point on the cone is at an angle of 30° to the axis of symmetry, as shown. An ant walks from a to b, two diametrically opposite points on the cone, where each point is a distance D from the cone apex. (a) What is the minimum distance d the ant has to walk, if it stays on the surface? (b) How much less is this than if it walks around the circular rim of the cone, from a to b?

1P32. Hypothetical objects called *cosmic strings* may have formed in the early universe. The space surrounding a straight cosmic string is conical, with the cosmic string passing along the cone's symmetry axis through its apex. Consider a plane perpendicular to the string direction. Particles or light rays moving on this plane behave as though they moved on a cone, whose symmetry axis lies along the string, as shown. Light traveling through space moves on geodesics of the cone. (a) Suppose there is a cosmic string located halfway between Earth and a distant light source such as a QSO (quasi-stellar object). Show that light can travel by two different paths from the QSO to Earth. Therefore a telescope on Earth would see *two* images of the QSO, separated by some angle θ in the sky, as shown.

Hint: A cone can be constructed by cutting a pie-shaped wedge from a cardboard circle, and then gluing the wedge edges together. Place the QSO at a point on the glued edges, and Earth at a point on the opposite side of the string, which runs through the cone apex. Geodesics on the cone have what shape on the flattened cone with the glued edges separated again? (b) If the QSO and Earth are equidistant from the cosmic string, show that $\theta = \delta/2$, where δ is the angle removed in the pie-shaped slice.

Day Two

A Proliferation of Frames

"Too much of a good thing can be wonderful." - Mae West

Staring at the thousands of stars, stationary and unblinking, we notice that one of them slowly grows larger and brighter. It is a softly glowing, transparent sphere coming almost directly towards us, another Bubble similar to our own, with occupants who haven't seen us yet. They are concentrating on the type of experiments we carried out yesterday, pushing balls around, measuring their velocities. They look up and see us, as they drift past at a steady rate.

2.1 Two inertial frames

The same experiment looks different to different observers. We see that people in the other Bubble (we'll call it Bubble *B*) have placed a ball at rest relative to themselves, floating in the middle of their Bubble. From our point of view the ball is moving to the right at a constant velocity, along with the Bubble *B* and everything else inside it. What is at rest to them is moving to us, as shown in Fig. 1(a). And of course a ball at rest to us is moving to them. We are moving to the left from their point of view, so a ball at rest in our Bubble (call it Bubble *A*) moves to the left from their point of view, as shown in Fig. 1(b).

Fig. 1 (a) A ball at rest in Bubble *B* moves to the right as seen by us in Bubble A. (b) A ball at rest in our Bubble *A* moves to the left as seen by people in Bubble *B*.

As Bubble B passes by, someone in it throws a ball toward the *right* at some speed v relative to them. From our point of view the ball moves even *faster* to the right, as shown in Fig. 2(a). The speed of the ball obviously depends upon the observer. Someone else inside B throws a ball "up", in a direction perpendicular to Bubble B's velocity relative to us, as shown in Fig. 2(b). From our point of view that ball moves at an angle, partly "up" and partly to the right.

Fig. 2 (a) A ball thrown to the right in Bubble B moves *faster* to the right as seen by us in Bubble A. (b) A ball thrown "up" in Bubble B moves both "up" and to the right as seen by Bubble A.

Clearly the balls move in straight lines at constant speeds no matter whether *we* are watching them or *they* are watching them. Our frame is inertial, because thrown balls move in straight lines at constant speed from our point of view. And clearly their frame is inertial as well. Although the same ball has a different velocity whether we observe it or they observe it, the ball moves in a straight line at constant speed to them just as it does to us.

Sample Problem 1. An airline traveler needs to get from one terminal to another. She steps on a motorized walkway moving *north* at speed $V = 1.2$ m/s relative to the floor. She notes that a luggage cart is moving *south* at speed $v_o = 3.4$ m/s relative to *her*. How fast is the cart moving relative to the floor?

Solution: She is moving north while the cart is moving south relative to the floor, so the cart must be moving faster relative to her than it is relative to the floor. If the cart moves at speed v_c while she moves at speed V, both relative to the floor, then it makes sense that the cart should move at the larger speed $v_o = v_c + V$ relative to her. Therefore the cart speed is $v_c = v_o - V = 3.4$ m/s $- 1.2$ m/s $= 2.2$ m/s relative to the floor.

2.2 Transformation between inertial frames

There could in fact be an infinite number of inertial frames, all moving at constant velocity relative to our own. If we see a ball moving at constant velocity relative to us, occupants of another inertial frame can look at the same ball and find that it is going at constant velocity relative to them as well. They measure a different velocity for that ball than we do, but whatever that velocity is, it stays constant from their point of view.

We picture two inertial frames as in Fig. 3, moving at constant velocity V relative to one another along (say) their mutual x-axes. The unprimed frame is our rest-frame in Bubble A, while the primed frame is the rest-frame of Bubble B.

Fig. 3. Primed and unprimed inertial frames. The primed frame moves at constant velocity V to the right, relative to the unprimed frame.

We first need to relate the position (x', y', z') of a ball measured in the primed frame to its position (x, y, z) measured in our unprimed frame. That's easy.

Note from Fig. 3 that the y and z coordinates of any point are the same in both frames, but the x coordinates differ by the distance between the two origins. If the origins coincide at time $t = 0$, they will have moved a distance Vt apart by time t. It follows that the *coordinate transformation* between frames is *

$$x' = x - Vt \tag{1a}$$

$$y' = y \tag{1b}$$

$$z' = z. \tag{1c}$$

Equation 1a shows that x' is less than x by the distance Vt, as is clear from Fig. 4.

* This transformation between two inertial reference frames was named the *Galilean Transformation* by Albert Einstein.

Fig. 4. The position x (and x') of an object as observed in the unprimed frame and the primed frame.

Suppose for example that at time t a ball is a distance $x = 5$ m from the origin of the unprimed frame. Its distance from the origin of the primed frame is only $x' = 3$ m, if the primed frame has moved a distance $Vt = 2$ m. Another example: Suppose a ball stays at rest in the primed frame, remaining at the origin $(x', y', z') = (0, 0, 0)$. What is its position in the unprimed frame? From Eqs. 1 we get $(x, y, z) = (Vt, 0, 0)$. This makes sense; the ball's coordinate $x = Vt$ continually increases as the ball moves to the right from the point of view of observers in the unprimed frame.

2.3 The velocity transformation

What if the ball *moves* in the primed frame? How does the velocity seen by observers in the primed frame compare with the velocity of the ball seen by observers in the unprimed frame? All we have to do is differentiate Eqs. 1 because the velocity components in the primed frame are $v'_x = \frac{dx'}{dt}$, $v'_y = \frac{dy'}{dt}$, and $v'_z = \frac{dz'}{dt}$, while in the unprimed frame they are $v_x = \frac{dx}{dt}$, $v_y = \frac{dy}{dt}$, $v_z = \frac{dz}{dt}$. Therefore

$$\frac{dx'}{dt} = \frac{dx}{dt} - V \tag{2a}$$

$$\frac{dy'}{dt} = \frac{dy}{dt} \tag{2b}$$

$$\frac{dz'}{dt} = \frac{dz}{dt}, \tag{2c}$$

or

$$v'_x = v_x - V \tag{3a}$$

$$v'_y = v_y \tag{3b}$$

$$v'_z = v_z \,. \tag{3c}$$

This *velocity transformation* also makes sense*. The ball has the same y and z components of velocity in both frames, but the x-components of velocity differ by V, the relative velocity of the two frames. A ball at rest in the primed frame, for example, has $(v'_x, v'_y, v'_z) = (0, 0, 0)$, so it moves purely in the x direction with velocity V in the unprimed frame, since $(v_x, v_y, v_z) = (V, 0, 0)$ according to Eqs. 3.

* The term "velocity" normally means the *vector*, including both magnitude and direction. The term "speed" normally refers to the magnitude alone.

Sample Problem 2. *Repeat Sample Problem* 1, *using the velocity transformations of Eqs.* 3. The airline traveler again steps on a motorized walkway moving north at speed $V = 1.2$ m/s relative to the floor. She notes that a luggage cart is moving south at speed $v = 3.4$ m/s relative to her. How fast is the cart moving relative to the floor?

Solution: Let her frame be the unprimed frame, and the rest-frame of the floor be the primed frame, moving to the *south* relative to the unprimed frame at speed V. (South is therefore the positive velocity direction according to Fig. 1.) The luggage cart moves south relative to her, so its velocity is positive, $v_x = 3.4$ m/s. In the primed (floor) frame, the velocity of the cart is therefore

$$v'_x = v_x - V = 3.4 \text{ m/s} - 1.2 \text{ m/s} = 2.2 \text{ m/s}.$$

That is, the cart moves south relative to the floor, with speed 2.2 m/s. This is the same result found more intuitively in *Sample Problem* 1. Often the intuitive way is easiest! But in more complicated problems it is good to know that we can always use the formal velocity transformation if necessary.

Sample Problem 3. A hot-air balloon floats at rest relative to the air. A wind blows at speed v_W = 150 mi/hr toward the east relative to the ground. (a) Find the velocity of the balloon relative to the ground. (b) An airplane flies at speed v_A = 300 mi/hr relative to the rest-frame of the air, which is the plane's speed observed by riders in the balloon. If the plane flies toward the south according to its onboard compass, what are the speed and direction of the plane relative to the ground?

Solution: View the airplane and the ground from above and let the wind blow to the right. Then the ground frame is the unprimed frame and the rest-frame of the air is the primed frame.

(a) In the primed (air) frame the velocity of the balloon is $(v'_x, v'_y) = (0, 0)$; relative to the ground the balloon's velocity components are therefore

$$v_x = v'_x + V = 0 + 150 \text{ mi/hr} = 150 \text{ mi/hr}$$

and $v_y = v'_y = 0$ as shown.

(b) In the primed (air) frame the velocity of the plane is $(v'_x, v'_y) = (0, -300 \text{ mi/hr})$; relative to the ground the plane's velocity components are

$$v_x = v'_x + V = 0 + 150 \text{ mi/hr} = 150 \text{ mi/hr}$$

and

$$v_y = v'_y = -300 \text{ mi/hr}$$

as shown. The speed of the plane relative to the ground is therefore

$$v = [(v_x)^2 + (v_y)^2]^{1/2} = [(150)^2 + (-300)^2]^{1/2} \text{ mi/hr} = 335 \text{ mi/hr},$$

and the angle of its velocity relative to due south is

$$\theta = \tan^{-1}(v_x/v_y) = \tan^{-1}(150/300) = \tan^{-1}(1/2) = 26.6^\circ .$$

2.4 Transformations using vectors

The elegance of vector notation is exemplified by rewriting Eqs. 1 and 3 in vector form. That is, the equations

$$x' = x - Vt \tag{1a}$$

$$y' = y \tag{1b}$$

$$z' = z \tag{1c}$$

become simply

$$\mathbf{r}' = \mathbf{r} - \mathbf{V}t \tag{4}$$

in the special case where $\mathbf{V}$ is in the x-direction. In the same case the equations

$$v'_x = v_x - V \tag{3a}$$

$$v'_y = v_y \tag{3b}$$

$$v'_z = v_z \tag{3c}$$

become the single vector equation

$$\mathbf{v}' = \mathbf{v} - \mathbf{V}. \tag{5}$$

When we write out the components of Eq. 4 we get Eqs. 1, and when we write out the components of Eq. 5 we get Eqs. 3, again assuming that $\mathbf{V}$ is in the x direction.*

Note that the velocity transformation $\mathbf{v}' = \mathbf{v} - \mathbf{V}$ follows directly from the position transformation $\mathbf{r}' = \mathbf{r} - \mathbf{V}t$ by differentiating with respect to time, since by definition $\frac{d\mathbf{r}'}{dt} = \mathbf{v}'$ and $\frac{d\mathbf{r}}{dt} = \mathbf{v}$. (The derivative of the constant velocity $\mathbf{V}$ is zero.)

* The vector equations 4 and 5 are actually more general than Eqs. 1 and 3. If $\mathbf{V}$ is *not* in the x-direction, Eqs. 4, 5 are *still true*, but Eqs. 1 and 3 have to be replaced by equations reflecting the correct frame translation direction.

Another advantage of vector notation for velocity is that it makes it easy to relate $\mathbf{v}'$, $\mathbf{v}$, and $\mathbf{V}$ directly by vector addition and subtraction. That is, $\mathbf{v}' = \mathbf{v} - \mathbf{V}$, and so also $\mathbf{v} = \mathbf{v}' + \mathbf{V}$. If we know the velocity $\mathbf{v}$ of a ball in the unprimed frame, we can *subtract* the relative frame velocity $\mathbf{V}$ vector from $\mathbf{v}$ to find $\mathbf{v}'$, as shown in Fig. 5(a). On the other hand, if we know the velocity $\mathbf{v}'$ of a ball in the primed frame, we can find $\mathbf{v}$, by *adding* $\mathbf{V}$ and $\mathbf{v}$ vectorially as shown in Fig. 5(b).

Fig. 5(a) Vector subtraction of $\mathbf{V}$ from $\mathbf{v}$ to find $\mathbf{v}'$ (b) Vector addition of $\mathbf{V}$ and $\mathbf{v}'$ to find $\mathbf{v}$.

Sample Problem 4. A car travels at speed $V = 15$ m/s to the right along a horizontal street, as observed by a pedestrian standing on the sidewalk. Rain is falling in the $-y$ direction, vertically downward in the pedestrian's frame, with speed $v = 20$ m/s. What is the rain's velocity as seen by the driver of the car?

Solution: The car moves to the right as seen by the pedestrian, so the car's frame is the primed frame and the pedestrian's frame is the unprimed frame. In the pedestrian's frame the rain has velocity $\mathbf{v} = (v_x, v_y) = (0, -20 \text{ m/s})$. The rain's velocity in the primed frame can be found using vector subtraction $\mathbf{v}' = \mathbf{v} - \mathbf{V}$, as shown in the diagram at the right. Here $\mathbf{v}' = \mathbf{v} - \mathbf{V}$ is the vector extending from the tip of $\mathbf{V}$ to the tip of $\mathbf{v}$. The result is $\mathbf{v}' = (v_x', v_y') = (-15, -20)$ m/s. The speed of the rain in the primed frame is therefore $v' = \sqrt{(-15)^2 + (-20)^2}$ m/s $= 25$ m/s.

If instead we happen to know that the rain's velocity is $\mathbf{v}' = (-15, -20)$ m/s relative to the (primed-frame) *car*, we can find its velocity in the (unprimed) *ground* frame using vector addition $\mathbf{v} = \mathbf{v}' + \mathbf{V}$, as shown. Here $\mathbf{v}$ is the vector which extends from the tail of $\mathbf{v}'$ to the tip of $\mathbf{V}$. Note that the rain is falling vertically in the ground frame, but that it has angle $\theta = \tan^{-1}(15/20) = 37^{\circ}$ to the vertical in the car's frame.

The results show that rain (or snow) falling vertically appears to approach the windshield from the point of view of the driver, as the car travels forward. The faster the car moves, the more the rain angles toward the windshield.

Sample Problem 5. In the inertial frame of the Bubble a spacerock travels with speed $v = 10$ m/s at angle $\theta = 60^{\circ}$ from the x-axis, as shown. How fast, and in what direction, must another inertial frame move relative to the Bubble, so that the spacerock's velocity will be purely in the y direction in the new frame?

Solution: In the frame of the Bubble, the spacerock's velocity is

$$\mathbf{v} = (v\cos 60^{\circ}\,\hat{\mathbf{x}} + v\sin 60^{\circ}\,\hat{\mathbf{y}})\ \text{m/s}$$

$$= [\,10\,(1/2)\,\hat{\mathbf{x}} + 10\,(\sqrt{3}/2)\,\hat{\mathbf{y}}\,]\ \text{m/s} = (5.00\,\hat{\mathbf{x}} + 8.66\,\hat{\mathbf{y}})\ \text{m/s}.$$

We can eliminate the x-component of the spacerock's velocity by moving at speed $V = 5$ m/s to the right. That is, if we move along with the spacerock's horizontal motion we eliminate that horizontal motion in our new frame. The velocity of the spacerock in the new frame is then $v_y' = 8.66$ m/s in the y-direction. (Our convention is that the primed frame always moves to the right relative to the unprimed frame so the Bubble frame is unprimed and the new frame is primed.) Using vector subtraction the primed and unprimed velocities are $\mathbf{v}' = \mathbf{v} - \mathbf{V}$, as shown at the right. Alternatively, we can solve for V algebraically using Eqs. 3; the equations $v_x' = v_x - V$, $v_y' = v_y$ become

$$0 = 5.00\ \text{m/s} - V, \qquad v_y' = 8.66\ \text{m/s},$$

so $V = 5.00$ m/s and $v_y' = 8.66$ m/s.

2.5 Noninertial frames

So far we have used primed frames that move at *constant velocity* relative to the unprimed frame. We found that if the unprimed frame is inertial, the primed frame is also inertial. Now what if the primed frame *accelerates* relative to the unprimed frame? That is, what if the relative velocity $\mathbf{V} = \mathbf{V}(t)$ depends on time? Is the primed frame still inertial?

The answer is clearly *no*. If a ball is at rest in the unprimed (inertial) frame, for example, its velocity in the primed (accelerating) frame will be $-\mathbf{V}(t)$, which is *not constant*. To observers in the accelerating frame, a ball does not move with constant velocity when nothing touches it, so the accelerating frame is not inertial.

Suppose for example that a supply ship docks at the Bubble. At time $t = 0$ the ship starts to accelerate forward with constant acceleration a_0. Therefore its velocity as a function of time is $V = a_0 t$, since then the acceleration of the ship is $a \equiv \dfrac{dV}{dt} = a_0$ as we know to be true. At this time also, a crew member aboard the ship releases an apple from rest. What is the subsequent motion of the apple according to the crew member?

The ship is at rest relative to the inertial frame at time $t = 0$. The apple, released from rest, *stays* at rest from the point of view of an observer in the Bubble, because the Bubble is an inertial frame. The ship moves with acceleration a_0 to the right relative to the Bubble, so the crew member sees the apple accelerating backward, with acceleration a_0 to the *left*, achieving velocity $V = -a_0 t$ at time t. From the point of view of the crew member, the apple "falls" toward the rear of the ship. It moves in a straight line *but not at constant speed*, so the linearly-accelerating ship frame is *not* inertial.

What about *rotating* frames? Rotating frames are also accelerating frames*. As we saw during *Day One*, in a rotating frame a ball can go in a circular path even when no one touches it. Therefore rotating frames are also noninertial.

*As will become clearer during *Day Five*.

State of the Search

1. We have drastically expanded the number of platforms from which we can view inertial motion. Yesterday we knew of only one inertial frame; now we know there are an infinite number of them. Any frame moving at constant velocity relative to an inertial frame is also an inertial frame.

2. If $\mathbf{V}$ is the velocity of one inertial frame (say the primed frame) relative to another inertial frame (say the unprimed frame), the coordinates of an object in the two frames are related by $\mathbf{r}' = \mathbf{r} - \mathbf{V}t$. In particular, if $\mathbf{V}$ is in the x-direction,

$$x' = x - Vt$$
$$y' = y$$
$$z' = z,$$

called the "Galilean transformation".

3. The velocities of the object in the two frames are related by $\mathbf{v}' = \mathbf{v} - \mathbf{V}$, showing that velocities add (or subtract) vectorially when transforming between two inertial frames. If $\mathbf{V}$ is in the x-direction,

$$v_x' = v_x - V$$
$$v_y' = v_y$$
$$v_z' = v_z .$$

4. A frame *accelerating* relative to an inertial frame is *non*inertial. Accelerating frames include, for example, linearly accelerating frames like an accelerating car or rocket, and rotating frames like Earth or a merry-go-round.

5. Motion is especially simple in inertial frames. In such frames objects stay put if we set at rest, or move in straight lines at constant speed as long as we don't touch them. When later we *do* touch them, we should stand in an inertial frame to watch the action, so we can clearly separate the effects of touching from the effects of the reference frame. If we watch from a *non*inertial frame, objects move in complicated ways even when nothing touches them, so it would be harder to figure out the effects of touching alone.

6. Gravity somehow influences objects without even touching them. So if we can first understand motion here in the Bubble, where it is relatively simple, we can later turn gravity back on and try to understand motion with gravity added.

PICTURE QUIZ 2

Test yourself. Answers are at the end of the text.

1. The Shinkansen bullet train is traveling from Tokyo to Nagoya at steady speed 200 km/hr on a straight section of track.

A crazed passenger seated in the train suddenly throws a teacup directly backwards towards the rear of the train, at speed 60 km/hr relative to the train. As observed by a person standing outside the train on the ground, the speed of the teacup at this instant is

(a) 140 km/hr (b) 200 km/hr (c) 260 km/hr

2. Shortly afterwards the front of the train strikes a falling hailstone, projecting it up and forward at speed $v = 200\sqrt{2}$ km/hr and angled at 45° to the horizontal as seen by the observer standing on the ground. At what angle to the vertical, and with what speed, does the hailstone move as seen by the engineer on the train?

(a) 45°, 200 km/hr (b) 45°, 0 km/hr

(c) 0°, 200 km/hr (d) 0°, $200\sqrt{2}$ km/hr

(e) 90°, $200\sqrt{2}$ km/hr

3. On a windless, snowy day, students on a schoolbus see snowflakes outside their windows (and far from the bus) falling steadily at an angle 30° to the vertical. If the speed of the bus is 30 mi/hr, what is the speed of the snowflakes relative to the bus?

(a) 30 mi/hr

(b) 40 mi/hr

(c) 45 mi/hr

(d) 50 mi/hr

(e) 60 mi/hr

4. A ball is thrown in the Bubble so that nothing touches it until it is caught. In each of the following pictures of the ball's motion as seen by observers within the Bubble, is the Bubble accelerating to the right, to the left, or is it moving at constant velocity relative to an inertial frame?

(a)

(b)

(c)

(d)

PROBLEM QUIZ 2

1. Train A moves at speed $v_A = 30$ m/s along a track to the *right*, while train B moves at speed $v_B = 20$ m/s along a parallel track to the *left*, as shown. Their x-axes are parallel to the tracks, with positive directions to the right, and their origins coincide at time $t = 0$. At time $t = 10$ s, a flashbulb goes off in train A, at a point $x = 40$ m as measured in train A. Observers in train B can also see the flash. What is the x-coordinate of the flash, according to observers in train B?

2. Observers in a spaceship see a swarm of spacerocks moving "downward" at velocity v_r in their reference frame, as shown. Just then another spaceship drifts by to the right, as shown, with velocity v_s. Passengers in this ship will see the spacerocks approaching them at what angle θ to their forward direction? What will be the speed of the spacerocks in their frame?

3. A kayaker on a riverbank at A wants to paddle to a point B on the exact opposite bank of a river, a distance $D = 200$ m away, as shown below. (a) If the kayaker can paddle at speed $v_k = 3$ m/s relative to the water, and the river is flowing at speed $v_r = 1$ m/s relative to the shore, at what angle θ to the straight line from A to B should the kayaker paddle? (b) How long does it take the kayaker to reach the opposite shore?

Special Topic: The Hubble Expansion of the Universe

Edwin Hubble, an astronomer at Mt. Wilson Observatory near Pasadena, California, announced in 1929 his discovery that distant galaxies are receding from us, and that their velocities are linearly proportional to their distances from us. We call this relationship "Hubble's law", expressed as

$$v = H\,\ell\,,$$

where v is the velocity of a galaxy, ℓ its distance from us, and H is the *Hubble constant*, a constant of proportionality. This relationship was the original evidence for a "Big Bang" origin of the universe, in which everything started together at some time in the past, and blew up. According to the Big Bang scenario, the reason why one galaxy is twice as far from us as another is that it has been moving twice as fast for the same length of time.

It is important to find the numerical value of H as accurately as we can, because it can be used to help find the age of the universe. To find H, astronomers have to determine both the distances of galaxies ℓ and their recessional velocities v. The velocities are relatively straightforward to find. Astronomers take spectrograms, in which the light from a galaxy is split up into its constituent colors. Particular wavelengths of light are emitted as fairly sharp lines, characteristic of particular elements (hydrogen, calcium, iron, etc.). Lines are shifted toward longer wavelengths (the red end of the visible spectrum of light) if the galaxy is receding, by what is called the "Doppler effect". The shift is proportional to the galaxy's velocity away from us. Astronomers compare the observed wavelength with the wavelength of spectral lines emitted by the same element in the laboratory, and the difference is used to compute v.

It is considerably harder to find the distance ℓ of a galaxy. A number of methods have been invented, relying on various types of supernovas (large stars which have exploded, so that they can be brighter for a few days than the rest of their galaxy put together), stars called Cepheid variables (stars which oscillate with a period of oscillation which depends upon their brightness), the apparent sizes of entire galaxies or parts of galaxies, and other methods as well. If astronomers can find a "standard candle", objects which shine with known absolute brightness, the observed brightness of the standard candle

in a distant galaxy can be used to determine its distance. In the mid-1990s there were sharp controversies about the distances of galaxies, leading to uncertainties in the Hubble constant of a factor of two or so. The controversies were sometimes so heated that they were called the "Hubble Wars".

More recently two groups of astrophysicists have used "Type 1a Supernovas" as standard candles to measure the distances of very distant galaxies. A Type 1a Supernova occurs in an average galaxy only once every few hundred years. They are thought to be the result of gas from one star falling onto a companion white dwarf star. Beyond a certain size, white dwarfs are known to be unstable. If gas from a companion star is added to a white dwarf, beyond the critical size the white dwarf will collapse and then quickly explode; the explosion emits vast amount of energy, including light energy, which we see as a supernova. Since such supernovas are rare, many distant galaxies must be searched to find them. In practice, images of many distant galaxies are formed and stored; later, pictures of the same group of galaxies are taken again. When the two images are compared, the brightening of a galaxy may indicate the presence of a supernova. The supernova can be as bright or brighter than the rest of the galaxy put together.

When we look at distant galaxies we are also looking back in time. We are seeing the galaxy as it was when the light from it was emitted, a time $t = \ell/c$ ago, where the speed of light $c = 3 \cdot 10^8$ m/s is large but finite. Therefore not only can we measure the Hubble constant, but we can even see if it has been changing with time. We can measure ℓ and v for relatively nearby galaxies to find the present value of H, and we can measure ℓ and v for very distant galaxies to see what its value used to be. Up until quite recently the measurements of ℓ were not sufficiently precise to find H very accurately, say nothing of finding whether H has been changing with time. It has long been believed that we should find that H has been *decreasing* with increasing time, that H should be larger for distant galaxies than for nearby ones. That is because gravity would cause the expansion rate of the universe to decrease with time, for much the same reason that a rock thrown up from Earth's surface slows down as it rises. Galaxies would pull on one another, slowing the expansion rate, and causing H to decrease with time.

It was therefore a great surprise, a shock even, to find that the opposite is true. The new, more precise distance measurements provided by Type 1a Supernovas show that the universe is speeding up its expansion rate. The Hubble constant for the most distant galaxies is *smaller* than the Hubble constant for galaxies closer by. If the data continue to stand up to careful scrutiny, the universe is accelerating in its outward expansion. In that case the universe will never fall back again, and matter in the universe will thin out forever, with distant galaxies growing dimmer and dimmer until they eventually become invisible.

Sample Problem S1. Consider a hypothetical string of galaxies, currently a standard distance ℓ_0 apart from one another, with recessional velocities proportional to their distance from our Milky Way galaxy, as shown. For example, galaxy #1 is ℓ_0 away from us, and has velocity v_0, while galaxy # 3 is $3\ell_0$ away, with velocity $3v_0$.

Hubble's law is obeyed, with $H = v_0/\ell_0$. The law therefore seems to place us in a preferred position in the universe, with our galaxy at rest and everything else streaming away.

Show that if the string of galaxies is infinitely long in both directions, there is in fact no preferred center. Observers on any other galaxy could assume *they* are at rest, since galaxies recede from them also according to Hubble's law.

Solution: Hop onto galaxy #1, for example. Note that galaxy #2 is then a distance ℓ_0 away, and recedes with velocity $2v_0 - v_0 = v_0$, while galaxy #3 is a distance $2\ell_0$ away

and recedes with velocity $3v_0 - v_0 = 2v_0$. Also galaxy # 0 (the Milky Way) and galaxy # -1 are located at $-\ell_0$ and $-2\ell_0$ relative to galaxy #1, respectively, and have velocities $0 - v_0 = -v_0$ and $-v_0 - v_0 = -2v_0$, as seen by galaxy #1. So Hubble's law is also valid from galaxy #1's point of view, with the same value of the Hubble constant. There is therefore no way to tell which galaxy is at the center, if the string is infinitely long.

Questions and Problems from Day Two

2Q1. Is there any way to tell that the frame in which you are at rest is accelerating in a *straight line* relative to the distant stars?

2Q2. Should you run or walk when you have to get from a to b in a rainstorm? That is, how fast should you move to minimize the number of raindrops that hit you? Assume the rain falls vertically in the rest-frame of the ground.

2Q3. Two trains on the same track approach one another, each travelling at $v_t = 30$ mph. When they are a distance $D = 10$ miles apart, a bee, which flies at speed $v_b = 60$ mph, leaves the front of one train headed for the other. When the bee reaches the engine of the other train it immediately turns around and heads back for the first train, and then continues to fly back and forth. By the time the trains crash, what is the total distance the bee has flown?

2Q4. What are the analogs of Eqs. 1 and 3 in case the primed frame moves at speed V in the *z-direction* relative to the unprimed frame?

2Q5. The Sun is at the origin of a Cartesian coordinate system and Earth orbits in the x, y plane. A particular star is very far away along the positive z axis. Accounting for the fact that the speed of light is not infinite, in what direction will the star be seen by people on Earth at a time when Earth is moving in the positive x-direction?

2Q6. A crewmember in a drifting spaceship counts an equal number of stars in the forward and backward directions. The ship then accelerates forward for a long period of time before the engines are cut off. In which direction will the crewmember then see more stars? (Hint: The velocity of light is large but not infinite.)

2Q7. A spaceship is accelerating uniformly in a straight line. A passenger throws a ball sideways, i.e., perpendicular to the acceleration direction. What will the ball's subsequent motion look like to the passenger?

Sections 2.1-2.4 Two inertial frames; Transformation between inertial frames; Velocity transformation; Vector transformations

2P1. A ball moves in the x,y plane at constant speed v_0, at an angle of 45° to the x-axis in the unprimed inertial frame. Find the speed and angle relative to the x' axis in the primed inertial frame, if $V = v_0/(2\sqrt{2})$ is the relative speed of the two frames along their mutual x directions.

2P2. Find the acceleration $\mathbf{a}' = (a'_x, a'_y, a'_z)$ of a ball observed in a primed inertial frame, in terms of its acceleration $\mathbf{a} = (a_x, a_y, a_z)$ in an unprimed inertial frame, using (a) Eqs. 3 (b) Eq. 4(b) .

2P3. A ball moves with speed v at an angle θ to the positive x-axis and angle $90^{\circ} - \theta$ to the positive y-axis in the inertial frame of the Bubble. (a) How fast, and in which direction (toward positive x or negative x), must a primed inertial frame move so the ball's velocity in that frame will be purely in the y' direction?

(b) Find both answers in the special case $v = 5.0$ m/s and $\theta = 45^{\circ}$.

2P4. An airplane flies at speed $v_a = 150$ mi/hr relative to the air. The wind is blowing at speed $v_W = 120$ mi/hr toward the east, relative to the ground.

(a) If the plane flies due north according to its onboard compass, find the speed and direction (relative to north) of the plane as measured by observers on the ground.

(b) If the plane flies due north according to ground observers, how fast does the plane move relative to the ground? In this case, at what angle to north must the pilot fly, according to the onboard compass?

2P5. A stream flows at speed $v_W = 0.50$ m/s between parallel shores a distance $D = 35$ m apart. A swimmer swims at speed $v_S = 1.00$ m/s relative to the water.

(a) If the swimmer swims straight toward the opposite shore, i.e. in a direction perpendicular to the shoreline as seen by the swimmer, how long does it take him to reach the opposite shore, and how far downstream is he swept?

(b) If instead the swimmer wishes to reach the opposite shore at a spot straight across the stream, at what angle should he swim relative to the stream flow direction, so as to arrive in the shortest time? What is this shortest time?

2P6. A river of width D flows at uniform speed V_O. Swimmers A and B, each of whom can swim at speed v_S relative to the water, decide to have a race beginning at the same spot on the shore. A swims downstream a distance D relative to the shore,

and immediately swims back upstream to the starting point. B swims to a point diametrically opposite to the starting point on the opposite shore, and then swims back. Assume $v_S > V_O$. Find the total time for each swimmer. Who wins the race?

2P7. A train is traveling along a straight track at speed $v_t = 26$ m/s. Looking through a side window, a passenger sees snowflakes falling vertically relative to the train, at speed $v_s = 13$ m/s. As seen by someone standing on the ground, how fast are the flakes moving, and at what angle are they falling relative to the vertical?

2P8. Earth orbits the Sun at speed $v_E = 30$ km/s. Light from a distant star (located in a direction perpendicular to the orbital plane) travels at speed $c = 3.0 \cdot 10^5$ km/s.
(a) From the point of view of Earth observers, explain why the star appears to be located in a direction slightly different from its direction in the Sun's frame. How large is the angular discrepancy (b) in radians (c) in seconds of arc? (1 second of arc = (1/3600) degree). (d) How would one observe the discrepancy? (This effect is called the *aberration of light*, and was first noted by the astronomer James Bradley in 1727.)

2P9. Steady rain falls at speed v_r straight down as observed by a pedestrian standing on a sidewalk. A bus travels along the horizontal street at speed v_b.
(a) At what angle θ to the vertical do the raindrops fall, as seen by the bus driver?
(b) Suppose the bus has no windshield, leaving a hole in the flat, vertical front of the bus. The driver wants to travel forward at constant speed in a straight line from point A to point B. To minimize the amount of rain that enters through the hole, should the driver drive the bus very slowly? As quickly as possible? Or how?

2P10. Raindrops are falling at constant speed v_r relative to the ground, at angle α to the vertical, as shown, where α is positive to the right. In the frame of a car moving from left to right at speed v_c along the horizontal street, find expressions for
(a) The speed of the raindrops;
(b) The angle of the raindrops relative to the ground.

2P11. A kayaker wants to cross a river of width $D = 250$ ft, landing at a point on the opposite shore a distance $D = 250$ ft downstream. She paddles the kayak at speed $v_o = 5.0$ ft/s relative to the water. If the river flows at speed $V = 1.0$ ft/s, at what angle should she paddle relative to the flow direction, in the reference frame of the water? Use vector addition of velocities to find the answer.

2P12. An ultralight aircraft is 5 miles due west of the landing field. It can fly 25 miles/hour in stationary air. However, the wind is blowing at 25 miles per hour from the southwest at a 60° angle to the direction toward the landing field. (a) At what angle to the east must the pilot aim her craft to reach the landing field? Explain. (b) How long will it take her to reach the landing field if she flies as described in part (a)?

2P13. A stiff wind is blowing toward the east at speed $v_w = 55$ m/s, carrying a hybrid helium-hot air balloon with it. A plane is flying at the same altitude toward the north relative to the ground at speed $v_p = 75$ m/s. Relative to the plane, find the balloon's (a) speed (b) angle of drift relative to due east. Use vector addition of velocities.

2P14. The atomic nucleus U^{238} decays into an α-particle and a thorium isotope, $U^{238} \rightarrow Th^{234} + \alpha$, with a half-life of $t_{1/2} = 4.5 \cdot 10^9$ yrs. The α-particle is emitted with speed $v_\alpha = 3.4 \cdot 10^5$ m/s in the U^{238} rest-frame. If a particular U^{238} nucleus moving vertically downward in the laboratory at speed $v_U = 1.0 \cdot 10^6$ m/s decays, emitting the α at an angle $\theta = 60^o$ from the downward direction in the U^{238} rest-frame, (a) How fast does the α-particle move in the laboratory? (b) What is the angle of its velocity from the downward direction as observed in the laboratory?

2P15. The unstable beryllium isotope Be^8 decays into two α-particles, $Be^8 \rightarrow 2\alpha$. In the isotope rest-frame, α_1 and α_2 emerge with equal but opposite speeds v_α. Suppose in a particular decay they emerge "horizontally" as shown. Find (a) the speeds v_1 and v_2, and (b) the angles θ_1 and θ_2 relative to the horizontal, for the velocity of each α in the *laboratory* frame in each of the following cases: The Be^8 moves (1) to the *right* in the lab with speed v_α; (2) *upward* in the lab with speed v_α; (3) *upward and to the right* at angle 45° to the horizontal in the lab, with speed v_α. Sketch the velocity vectors of α_1 and α_2 in each of the three cases.

2P16. An allied supply ship in the North Atlantic is heading due east at speed $v_S = 14$ knots. In the evening at 21:00 hours a U-boat captain discovers that he is a distance $d =$ 160 nautical miles due South of the supply ship. The U-boat can achieve speeds up to $v_U = 28$ knots when running on the surface. (1 knot = 1 nautical mile/hour.)

(a) At what angle to the supply ship's path (in the frame of the ocean) should the U-boat move to intercept the supply ship in the least time, assuming the supply ship continues on its course at constant speed? (b) If it starts to get light at 04:00 in the morning, will the U-boat be able to attack under cover of darkness? (c) Upon arrival, what is the speed of the U-Boat relative to the supply ship?

2P17. An allied ship convoy in the North Atlantic is heading due west at speed $v_C = 18$ knots. At 12:00 midnight a U-boat captain discovers that he is a distance $d_x = 35$ nautical miles east and $d_y = 72$ nautical miles north of the convoy. The U-boat can achieve speeds up to $v_U = 32$ knots when running on the surface, and the captain immediately sets course to arrive at a point on the convoy's path 1 hour before the convoy itself arrives (so the U-boat can lie in wait without engines running.)

(a) At what angle to the convoy's path (in the frame of the ocean) should the U-boat set out to arrive on station in the minimum time? (b) At what time will the U-boat engage the convoy if all goes according to plan? (Note: 1 knot = 1 nautical mile/hour.)

2P18. In our reference frame, spaceship A moves at constant velocity $v_A = 55$ km/s along the "x" axis, as shown, starting at time $t = 0$ at $x = -1200$ km. Spaceship B moves at constant velocity $v_B = 85$ km/s along the "y" axis, starting at $t = 0$ at $y = y_O$, as shown. (a) What must y_O be so the spaceships will collide? (b) At what time will they collide? (c) From the point of view of spaceship A, find both the magnitude of B's velocity and the angle B's velocity has relative to the x axis. (d) Find the initial distance of B from A, and then find the time of the collision in A's frame, found by dividing the distance B moves by B's speed.

2P19. A bomber, flying due east at speed $v_b = 550$ mi/hr, is running out of fuel. An airtanker is due south of the bomber a distance $d = 360$ miles at time 9:00 a.m. when it receives an order to refuel the bomber. (a) If the tanker can travel at speeds up to 650 mi/hr and the bomber has sufficient fuel to maintain its course and speed until 10:30 a.m., will the tanker be able to reach the bomber in time to refuel it? (b) If so, at what angle relative to the south will the bomber pilot observe the tanker approaching, in the bomber's reference frame, assuming the tanker flew in the direction to minimize the time to rendezvous? If not, suppose the bomber turns due south and the tanker flies due north to speed up the rendezvous. At what time will they then meet?

2P20. The coordinate transformation of Eqs. 1 between inertial frames, supplemented by the condition $t = t'$ equating times in the two frames, was named the *Galilean transformation* by Albert Einstein. An alternative transformation, named by Einstein the *Lorentz transformation*, is

$$x' = \gamma(x - Vt) \qquad y' = y \qquad z' = z \qquad t' = \gamma(t - Vx/c^2)$$

for the same two frames, where $\gamma = (1 - V^2/c^2)^{-1/2}$ and c is the speed of light.
(a) Show that the Lorentz transformation reduces to the Galilean transformation for $V << c$, if x is also not too large.
(b) Find the velocity transformation analogous to Eqs. 3. Use $v_x = dx/dt$, $v'_x = dx'/dt'$, etc. That is, observers at rest in the unprimed frame use time t to measure velocity, while observers at rest in the primed frame use t'.
(c) Suppose an object moves at the speed of light in the unprimed frame, with $(v_x, v_y, v_z) = (c, 0, 0)$. Find its velocity components in the primed frame moving at speed V in the x-direction. Note that if one uses the Lorentz transformation, velocities do not add vectorially.

2P21. At time $t = 0$, a ball is thrown sideways (i.e., perpendicular to the acceleration direction) by the crew member in the accelerating ship described in Section 2.5. What will be the shape $y(x)$ of its subsequent trajectory, as seen by the crew member? Here the coordinate x is along the acceleration direction, and y is the perpendicular coordinate, with $x = y = 0$ when the ball is thrown.

2P22. The position of a rocket as seen by observers in an inertial frame is $x = \alpha + \beta t + \gamma t^2$ as a function of time, where α, β, and γ are constants. In terms of any or all of these constants, find the rocket's (a) initial position x_0 (its position at time $t = 0$) (b) its initial velocity v_0 (c) its acceleration a. (d) Show that the velocity v of the rocket (at any time) obeys the relation $v^2 = v_0^2 + 2a(x - x_0)$.

2P23. During *Day One* the 12 m-radius Bubble rotated once every 2.0 minutes about an axis through its center. If an explorer pushes a ball from the Bubble center toward the rim at speed $v_b = 0.05$ m/s perpendicular to the axis of rotation, (a) how long does it take the ball to reach the rim? (b) Make a careful sketch of the ball's trajectory as seen by an observer rotating with the Bubble looking perpendicular to the plane of the ball's motion. How many times does the ball spiral around before reaching the rim?

2P24. During *Day One* the 12 m-radius Bubble rotated once every 2.0 minutes about an axis through its center. (a) Find the speed of the Bubble's rim as viewed from an inertial frame at rest relative to the rotation axis. (b) With what speed, and at what angle relative to a line from the rim to the Bubble's center, can a ball be thrown from the rim so that it passes through the Bubble's center and then returns to the thrower? Then sketch the ball's path in (c) the inertial frame (d) the rotating frame.

2P25. Suppose a Bubble frame rotates relative to an inertial frame, about their mutual z axes. The angle θ the x-axis of the rotating frame makes with the x- axis of the inertial frame is $\theta = 2\pi t/T$, as shown, where t is the time and T is the period of rotation, the time to turn once around. (a) If (x, y, z) are the coordinates of a particle in the inertial frame, what are the coordinates (x', y', z') of the particle in the rotating frame of the Bubble, in terms of (x, y, z) and θ? (That is, find the coordinate transformation between an inertial and rotating frame analogous to the coordinate transformation Eqs. 1 between two inertial frames.) Then if $(x, y, z) =$ (1.0, 2.0, 3.0) m, find (x', y', z') at times (b) $t = T/4$ (c) $t = T/2$ and draw a picture of the coordinates in each case to verify that your answers make sense.

2P26. Hubble's law $v = H\ell$ relates the recessional velocities v of distant galaxies to their distances ℓ away from us at the present time. Here H is a constant of proportionality, the same for all galaxies. Suppose Hubble's constant is

$$H = (65 \pm 7)\frac{\text{km/s}}{\text{Mpc}}.$$

Here 1 Mpc = 10^6 parsecs, where 1 pc = 3.26 light-years; one light year is the distance light travels in $1.00\ \text{yr} = 3.16 \cdot 10^7$ s, at speed $c = 3.00 \cdot 10^8$ m/s. If the Doppler redshift of light from a particular galaxy shows that the galaxy is receding from us at speed c/300, what is (a) the minimum (b) the maximum distance the galaxy can be from us, according to the somewhat uncertain Hubble constant given?

2P27. Hubble's law $v = H\ell$ relates the recessional velocities v of distant galaxies to their distances ℓ away from us at the present time. Here H is a constant of proportionality, the same for all galaxies.
(a) Suppose galaxies have retained their original velocities throughout the expansion of the universe. Under that assumption, find the age of the universe in terms of H.
(b) As given in the preceding problem, the current observational bounds on H are $H = (65 \pm 7)\frac{\text{km/s}}{\text{Mpc}}$. Find bounds on the age of the universe (in years), with the assumption made in part (a). (c) Recent observations suggest that the expansion rate is increasing with time, so that the universe is accelerating outward. If this is true, is the universe older or younger than you calculated in part (b)?

2P28. The universe has most likely expanded at different rates at different times. The "scale factor" of the universe, denoted by $R(t)$, is a measure of the distance between two distant galaxies as a function of time. In each of the following eras in the history of the universe, find the relative velocity and relative acceleration between two galaxies, and sketch $R(t)$ *as though it were valid for all* t. (The constant R_0 is the distance between the two galaxies at some particular time $t = t_0$.)

(a) The brief, very early *inflationary era* when $R = R_0 e^{(t - t_0)/t_0}$;
(b) the *radiation-dominated era*, which extended over the first 100,000 years or so, for which $R = R_0(t/t_0)^{1/2}$.
(c) the *matter-dominated era*, which extended from the end of the radiation-dominated era for another few billion years; during this time $R = R_0(t/t_0)^{2/3}$.
(*Note:* The scale factors in (b) and (c) are appropriate if the universe is spatially flat. At present the universe appears to be partly inflationary and partly matter-dominated. In the distant future the universe will become more purely inflationary, i.e., it will expand exponentially, according to our current understanding.)

2P29. According to the scale factor $R(t) = R_0(t/t_0)^{2/3}$, appropriate for the matter-dominated era in the expansion of the universe, the distance $R(t)$ between two galaxies has increased as the two-thirds power of time since the Big Bang. Take R_0 as the present distance between the two galaxies, at the present time t_0. (a) Sketch $R(t)$ from $t = 0$ up to $t = t_0$, assuming the scale factor as given is correct for all times. (b) The Hubble constant H has varied with time in this model. At the present time t_0, $H = \dfrac{dR/dt|_{t_0}}{R_0}$, the present relative velocity of two galaxies divided by their distance apart (the subscript t_0 means that the derivative is to be evaluated at time $t = t_0$). Show that the age of the universe is $t_0 = \dfrac{2}{3H}$.

2P30. During an early inflationary period in the lifetime of the universe, the expansion scale factor R increased with time according to $R = R_0 e^{t/t_0}$, where R_0 and t_0 are constants. The equation means that the distance between two particles increased exponentially in time during this period. At any time, the Hubble "constant" is given by $H = \dfrac{dR/dt}{R}$. (a) Find H as a function of time during inflation. (b) If two protogalaxies (density fluctuations) are a distance R_0 apart at the beginning of inflation, what distance R_f are they apart at time t_f, the end of inflation, if $t_f = 100\ t_0$? Express the answer in the form $R_f = R_0 \cdot 10^n$, for some n.

2P31. In the *Special Topic* it was shown that the Hubble expansion applies equally well from the point of view of any galaxy in a one-dimensional string of galaxies. Consider the two-dimensional array of galaxies shown, with the Milky Way at $(x,y) = (0,0)$. In Milky Way's frame, any galaxy recedes at speed $v = H\ell$, where ℓ is that galaxy's distance from the Milky Way. [For example galaxy $(\ell_0, 0)$ moves to the right at speed $v_0 = H\ell_0$, and the galaxy at (ℓ_0, ℓ_0) moves toward the upper right at speed $v = H\ell = H\sqrt{2}\,\ell_0 = \sqrt{2}\,v_0$.]

(a) Show that if galaxies $(0, \ell_0)$ and (ℓ_0, ℓ_0) obey Hubble's law as seen by the Milky Way, these two galaxies also obey Hubble's law as seen by galaxy $(\ell_0, 0)$.

(b) Show that if *all* galaxies obey Hubble's law as seen by one particular galaxy, then all galaxies also obey Hubble's law as seen by any other galaxy. First consider the two-dimensional array of galaxies shown, and then consider galaxies expanding in three dimensions. Under what circumstances would such a universe have no preferred center?

Day Three

Conservation of Momentum

"Get your facts first, and then you can distort 'em as much as you please" - Mark Twain

During our first two days we investigated the motion of objects when they were left alone. Today we'll see what happens when we make them *collide*.

3.1 Collisions of identical balls

The metal balls we've been using have thin holes drilled through their centers. We thread string through a ball, tie the string taut between opposite sides of the Bubble, and push the ball so it glides almost frictionlessly, straight along the stretched string. When we thread the string through two similar balls, we can make great ringing collisions by pushing the balls along the string toward one another.

We put a "target" ball at rest, and push another ball towards it. The incident ball is nearly stopped by the collision, and the target ball gains most of its velocity. This is like head-on pool shots, where the cue ball nearly stops after hitting its target.

For *arbitrary* initial velocities, the balls roughly *exchange* velocities when they collide. If ball a has velocity v_0 beforehand, then ball b's velocity is *roughly* v_0 afterward. Fig. 1 illustrates a typical experiment.

Fig. 1. A collision of identical balls, before and after. Roughly speaking, the balls exchange velocities with one another: After the collision ball b has *approximately* the same velocity as ball a had before the collision, and *vice versa*.

Table 1 gives the data for initial (i) and final (f) velocities of balls a and b in several experiments. Motion to the right is positive, left is negative, and magnitudes are in centimeters/second.

$(v_a)_i$	$(v_b)_i$	$(v_a)_f$	$(v_b)_f$
100	0	5	95
80	– 20	– 15	75
50	– 50	– 45	45
60	0	3	57
30	– 30	– 27	27

Table 1. Initial and final velocities of balls a and b, in cm/s. Velocities to the right are positive, and velocities to the left are negative.

Even more striking than the *approximate* exchange of velocities between balls a and b is the fact that the *sum* of the velocities of the two balls after the collision is the same as it was before, within experimental error. In our second experiment, for example, the sum of the velocities is 60 cm/s, both before and after. The total velocity stays the same:

$$(v_a)_i + (v_b)_i = (v_a)_f + (v_b)_f . \qquad (1)$$

Note: It is interesting that the first three experiments in Table 1 are in effect the *same* experiment, viewed from three different inertial frames. For example, suppose experiment 1 were viewed by someone moving at speed $V = 20$ cm/s to the right, in the same direction as the velocity of ball a. To that person, ball a would only be moving at velocity 100 cm/s – $V = 80$ cm/s, whereas ball b would be moving at velocity $0 - V = -20$ cm/s, just as in experiment 2. After the collision, ball a would be moving at 5 cm/s – $V = -15$ cm/s, and ball b would be moving at $95 - V = 75$ cm/s, also just as in experiment 2. Similarly, if experiment 1 were viewed by someone moving at speed $V = 50$ cm/s to the right, experiment 1 would look like experiment 3, both before and after the collision. These results illustrate the fact that if velocity is conserved in one reference frame, it is conserved in any other frame as well. We can prove this in general; see Problem 3P5.

Sample Problem 1. Landing on an icy runway, an airliner slows to velocity $v_o = 35$ m/s before running head-on into an identical airliner parked at rest. After the collision the parked plane moves at velocity $v_p = 28$ m/s. What is the landing plane's final velocity v_f?

Solution: According to the rule that the sum of the velocities of identical objects stays the same, the sum of the airliner velocities is the same before and after: $v_o = v_p + v_f$. Therefore the landing plane has the final velocity

$$v_f = v_o - v_p = 35 \text{ m/s} - 28 \text{ m/s} = 7 \text{ m/s}.$$

3.2 Collisions of single balls with multiple balls

Some of the balls can be hooked together. We attach two so they move as a unit when threaded on the string. We set this pair b at rest, and propel a single ball a into it at various speeds, recording the velocities before and after, as given in Table 2.

$(v_a)_i$	$(v_b)_i$	$(v_a)_f$	$(v_b)_f$
60	0	– 18	39
90	0	– 28	59
120	0	– 38	79

Table 2. Initial and final velocities of a single ball with a double ball, in cm/s.

The incident single ball always bounces back off the double-ball, as a tennis ball would bounce off a bowling ball. However, the sum of their velocities no longer stays the same! The sum afterward is always *less* than the initial velocity. In the first experiment, for example, the sum $60 + 0 = 60$ *before* falls to only $-18 + 39 = 21$ *after*.

Is there a rule that works in this case? In our previous experiments with identical single balls, when we projected ball a into ball b at rest, the change in velocity of ball a was equal to the final velocity of ball b: That is,

$$(v_a)_i - (v_a)_f = (v_b)_f, \tag{2a}$$

which is equivalent to the sum of velocities staying constant. If we set $(v_b)_i = 0$ in Eq. 1 we get Eq. 2. Now looking at our latest results, we find that

$$(v_a)_i - (v_a)_f = 2\,(v_b)_f \tag{2b}$$

every time. When we project a single ball into a double-ball at rest, we need to weight the double-ball velocity by the factor "2". In the first experiment in Table 2, for example, the change in ball a's velocity is [60 – (– 18)] cm/s = 78 cm/s, which is also twice the final speed of ball b, 2 x 39 cm/s = 78 cm/s.

If we project a single ball into N balls at rest connected together, we find that

$$(v_a)_i - (v_a)_f = N(v_b)_f \tag{3}$$

in several special cases. We even try projecting N_a balls connected together into N_b balls connected together; we find that

$$N_a[(v_a)_i - (v_a)_f] = N_b(v_b)_f. \tag{4}$$

There is an interesting pattern here!

3.3 A definition of *mass*

Experimenting with some new balls of various sizes and materials, we project one of our standard balls into one of the new balls at rest. For a given new ball b, for various initial velocities of the standard ball a we find that

$$(v_a)_i - (v_a)_f = N\,(v_b)_f\ , \tag{5}$$

just as in Eq. 3, except that now N is not necessarily an integer. One of the balls has N = 1.37, for example, indicating that it is equivalent to 1.37 standard balls. For different balls that seem to be made of the same "stuff", the bigger the ball is, the bigger the value of N we find.

> *We define the mass m of a ball this way.* Let the standard balls have a mass of *one kilogram.* Then we define the mass of the target ball in our latest experiment to be the value of N found, in units of kilograms. That is, $m_{target} = N$ kg. The experiments have shown that mass is *additive*: N standard balls connected together have a mass of N kilograms, for example.

Sample Problem 2. How can we measure the mass of a hockey player, who is standing on frictionless ice, using a 1–kg bowling ball?

Solution: We can slide the ball along the ice toward the hockey player with initial speed v_o. The player catches the ball and holds onto it, so the player and ball have the same speed v_f; that is, $(v_a)_f = (v_b)_f = v_f$ in Eq. 5. Solving for N, we find that $N = (v_o - v_f)/ v_f$. If $v_o = 8.0$ m/s, for example (about 18 miles/hour), and $v_f = 0.10$ m/s, then $N = (8.0 - 0.10)/0.10 = 79$, so the mass of the player is $m = 79$ kg.

3.4 Conservation of momentum

We have found that N need not be an integer in Eq. 5. It is also true that neither N_a nor N_b need be integers in Eq. 4. Using our definition of mass, we can now replace N_a and N_b by masses m_a and m_b,

$$m_a [(v_a)_i - (v_a)_f] = m_b (v_b)_f \tag{6}$$

in which we push a ball of arbitrary mass m_a (which need not be 1 kg) toward a ball of mass m_b at rest. This is a generalization of the experiments leading up to Eq. 4.

Our experiments show also that the mass of a ball is a property of that particular ball, and doesn't depend on which ball it collides with. Once we have found the mass of a ball, that value works in every experiment we try.

Rewriting Eq. 6 in the form

$$m_a(v_a)_i = m_a(v_a)_f + m_b(v_b)_f , \tag{7}$$

we have a general rule, valid in collisions of different-mass balls in which ball b starts at rest.

If ball b is moving initially, a natural generalization of Eq. 7 is

$$m_a(v_a)_i + m_b(v_b)_i = m_a(v_a)_f + m_b(v_b)_f , \tag{8}$$

which we can easily verify in collision experiments.*

* We can also show, using the velocity transformation Eq. 3(a) of *Day Two*, that if in one frame Eq. 7 works with ball b initially at rest, then Eq. 8 must be correct in other frames, in which ball b is initially moving.

We call the experimental law summarized by Eq. 8 the *conservation of momentum*, where the "momentum" of an object is the product of its mass and its velocity:

$$p = m v . \tag{9}$$

By "conservation" we mean that the total momentum

$$(p_a)_i + (p_b)_i = (p_a)_f + (p_b)_f \tag{10}$$

stays constant – it is the same before and after the collision.

Momentum is conserved in all the collisions we have observed. For example, conservation of momentum explains why the sum of $(v_a)_i$ and $(v_b)_i$ in Table 1 is always equal to the sum of $(v_a)_f$ and $(v_b)_f$, since the masses were equal in those experiments.*

* However, a puzzle remains! Conservation of momentum has *not* explained why $(v_a)_f = 5$ and $(v_b)_f = 95$ in the first row of Table 1, instead of $(v_a)_f = 10$ and $(v_b)_f = 90$, for example, or even $(v_a)_f = 0$ and $(v_b)_f = 100$ (with velocities in cm/s). There is an infinite number of choices $(v_a)_f$ and $(v_b)_f$ that conserve momentum. Why does nature make a *particular* choice in our experiments? (We will return to this question during *Day Thirteen.*)

Sample Problem 3. A bowling ball launched with initial velocity v_o collides head-on with a large rubber tire sitting on frictionless ice. The bowling ball bounces back with the velocity $v_f = -(4/5)\, v_o$, and the tire is given the forward velocity $v_x = v_o/10$.

(a) What is the mass of the tire in terms of the mass of the bowling ball?

(b) If instead we set the bowling ball at rest, and the tire, with initial velocity v_o, makes a head-on collision with it, what are the final velocities of the tire and ball?

Solution: (a) Before-and-after pictures of the original collision are shown on the following page. Conservation of momentum of ball b and the tire t gives $(p_b)_i = (p_b)_f + (p_t)_f$; that is,

$$(p_b)_i = m_b \cdot v_o = (p_b)_f + (p_t)_f = m_b \cdot [-(4/5)\, v_o] + m_t \cdot v_o/10.$$

Therefore $m_t = m_b [v_o + (4/5)\, v_o]/(v_o/10) = (9/5) m_b \cdot 10 = 18\, m_b$. (We should *expect* that the tire would have a larger mass than the bowling ball, because the bowling ball bounced *backward* in the collision, like a tennis ball would bounce if thrown at a truck.)

If instead we push the *tire* with velocity v_o toward a stationary bowling ball, the experiment is the same as the first experiment, except that it is now viewed by an observer moving to the right at velocity v_o relative to the original frame. The before-and-after pictures in this new frame are therefore as shown.

In this frame we just add the velocity $-v_o$ (i.e., speed v_o toward the *left*) to each of the velocities in the top set of pictures. Now the bowling ball is initially at rest, the tire has initial velocity $-v_o$, and the final velocities of the bowling ball and tire are $v_f = -9/5\, v_o$ and $-9/10\, v_o$, respectively. It is not surprising that both the ball and tire move to the left; the tire has much more mass than the bowling ball, so it is like throwing a truck at a tennis ball. Both will move off in the direction of the truck's initial velocity.

3.5 Momentum conservation in three dimensions

We remove the strings we have used to guide the balls. We can now throw one ball at another ball set at rest in the middle of the *Bubble* without any string to guide it. Sometimes we connect, although never dead on. The target ball b gets knocked off at some angle ϕ, and the incident ball a bounces off at some other angle θ, as shown in Fig. 2. We can't predict these angles, or the final speeds $(v_a)_f$ and $(v_b)_f$.

Even though we haven't forced the balls to move in a plane, we find that they always do so anyway. We can't predict which plane it will be, however, because our aim isn't good enough. The final velocity vectors $(\mathbf{v}_a)_f$ and $(\mathbf{v}_b)_f$ define a plane, and we notice that the initial velocity vector $(\mathbf{v}_a)_i$ is always in this plane as well.

Fig. 2. An unconstrained collision

It isn't as easy to make measurements as it was for balls on a string, but it is possible to do so using stopwatches, meter sticks, and trig tables. In two or three dimensions the velocities in Eq. 8 have to be *vectors*, rather than the simple positive or negative numbers we have used in one dimension. The more general conservation-of-momentum equation

$$m_a(\mathbf{v}_a)_i + m_b(\mathbf{v}_b)_i = m_a(\mathbf{v}_a)_f + m_b(\mathbf{v}_b)_f \tag{11}$$

is valid in one, two, or three dimensions. It reduces to the correct rule in one dimension (Eq.10) if we say that vectors pointing to the right are positive and those to the left are negative. In terms of the vector momentum

$$\mathbf{p} = m\mathbf{v}\,, \tag{12}$$

the conservation of momentum for the collision of two objects becomes

$$(\mathbf{p}_a)_i + (\mathbf{p}_b)_i = (\mathbf{p}_a)_f + (\mathbf{p}_b)_f\,. \tag{13}$$

Sample Problem 4. A ball of mass m moves toward a ball of mass M, as shown, so they have equal but opposite momenta. If the balls collide so that ball m emerges at angle θ_m to its initial direction, at what angle θ_M does M emerge?

Solution. The total momentum is zero before the collision, so has to be zero afterwards as well. If m emerges at angle θ_m, M must emerge in the exact opposite direction $\theta_M = \pi - \theta_m$, so the balls can have equal but opposite momenta.

In the experiment shown in Fig. 2 the target ball b is initially at rest, so in that case Eq. 12 splits into three equations, corresponding to conservation of momentum along the incident direction (x), the upward direction in the figure (y), and out of the figure (z):

$$x: \qquad (p_a)_i = (p_a)_f \cos\theta + (p_b)_f \cos\phi \tag{14a}$$

$$y: \qquad 0 = (p_a)_f \sin\theta - (p_b)_f \sin\phi \tag{14b}$$

$$z: \qquad 0 = 0, \tag{14c}$$

where a momentum p without bold face is the magnitude of the momentum vector. Here the x-component of $(\mathbf{p}_a)_f$ is $(p_a)_f \cos\theta$, for example, and the y-component of $(\mathbf{p}_b)_f$ is $-(p_b)_f \sin\phi$. Within experimental errors our results agree with predictions. Momentum conservation also explains our observation that all of the velocity vectors lie in the same plane.

There is a useful way to write conservation of momentum with the help of unit vectors. If the target ball is initially at rest, for example, Eq. 12 becomes

$$(p_{ax}\hat{\mathbf{x}} + p_{ay}\hat{\mathbf{y}} + p_{az}\hat{\mathbf{z}})_i = (p_{ax}\hat{\mathbf{x}} + p_{ay}\hat{\mathbf{y}} + p_{az}\hat{\mathbf{z}})_f + (p_{bx}\hat{\mathbf{x}} + p_{by}\hat{\mathbf{y}} + p_{bz}\hat{\mathbf{z}})_f, \tag{15a}$$

which splits up into the three equations

$$(p_{ax})_i = (p_{ax})_f + (p_{bx})_f \tag{15b}$$

$$(p_{ay})_i = (p_{ay})_f + (p_{by})_f \tag{15c}$$

$$(p_{az})_i = (p_{az})_f + (p_{bz})_f \tag{15d}$$

for the three directions. These equations are really the same as Eqs. 14, since for example $(p_{ax})_f = (p_a)_f \cos\theta$ and $(p_{bx})_f = (p_b)_f \cos\phi$, as can be seen from Fig. 2.

Sample Problem 5. A bowling ball of mass M moving with speed v_o strikes a pin of mass m, initially at rest. After the collision, the pin moves at this same speed v_o at right angles to the incident direction of the ball. What is the final speed v_f of the ball, and at what angle θ does it move relative to its incident direction of motion?

Solution: Conserve momentum. Let the bowling ball move initially in the x-direction, and let the pin be knocked aside in the y-direction, as shown above. Conserving momentum in the x and y directions (momentum is a vector, so both components are separately conserved),

P_x $$Mv_o = Mv_f \cos\theta$$

P_y $$0 = mv_o - Mv_f \sin\theta \ ,$$

$$(\text{so } mv_o = Mv_f \sin\theta \ )$$

Dividing this second form of the P_y equation by the P_x equation, we find that $\tan\theta = m/M$. Constructing the corresponding triangle shown, we can use Pythagoras to get $\sin\theta = \dfrac{m}{\sqrt{m^2+M^2}}$.

Therefore

$$v_f = \frac{m\,v_o/M}{\sin\theta} = \frac{\sqrt{m^2+M^2}}{M}\,v_o$$

and

$$\theta = \arctan(m/M).$$

3.6 Addendum on dimensions and units

When an equation involving physical quantities is written, not only must the numerical values agree on both sides, but the *dimensions* (mass M, length L, and time T) must agree for every term in the equation.* In the collision of two equal-mass balls, for example, with ball b initially at rest, we found that

$$(v_a)_i = (v_a)_f + (v_b)_f ;$$

the sum of the final velocities is equal to the initial velocity of the incident ball. Each term in this equation has dimensions $\frac{L}{T}$, so the equation is dimensionally consistent.

Suppose instead we tried to write

$$(m_a)(v_a)_i = (v_a)_f + (v_b)_f .$$

What would be wrong with having dimensions $\frac{ML}{T}$ on the left and $\frac{L}{T}$ for each term on the right? Note that one problem is that the numerical values of each side of the equation could be equal at most in *one* system of units, and for only a special case at that! For example, if in some experiment the mass $m_a = 1$ kg, then the equation above would be valid if we used

SI (meter, kilogram, second) units,

but not if we used

cgs (centimeter, gram, second) units.

That is because in cgs units $m_a = 1000$ g, so the equation would be

$$1000 \cdot (v_a)_i = (v_a)_f + (v_b)_f ,$$

which disagrees with the experimental results. Also if $m_a = 2$ kg, the equation could not agree with experiment even if SI units were used. Therefore equations of physics must be free from human conventions for the unit sizes of M, L, and T.

* This is a different use of the word "dimensions" from that used to define the number of dimensions (1, 2, 3, ..) in a space.

Dimensions can help in solving problems

When solving problems we *must* write out units as well as numbers. For example, if we measure the velocities $(v_a)_f = 5$ cm/s and $(v_b)_f = 95$ cm/s in the collision of two balls, as in our first experiment during *Day Three*, it is important to include the units cm/s; otherwise the numbers are meaningless! Seldom is the meaning clear from the context alone.

Patient: Doctor, I feel *terrible*. How long do you think I have to live?

Doctor: TEN

Patient: Ten *what*?

Doctor: NINE

Patient: I don't understand! What units are you using?

Doctor: EIGHT .....

A famous confusion about units brought about the September 1999 loss of the $125 million Mars Climate Orbiter spacecraft. Engineers at Lockheed-Martin transmitted numerical data on the impulse of the spacecraft in the English units "pound · seconds" (providing the numbers only without the units) to NASA, and the numbers were interpreted by engineers at NASA as having the metric units "Newtons · seconds". The spacecraft was therefore given the wrong impulse, throwing it off course.

The consistent use of units also provides a way to check calculations, because the units of the answer must agree with the units of the quantity we are trying to calculate. For example, if an object moves 10 meters in 2 seconds at constant velocity, its velocity is $v = (10\text{ m}) / (2\text{ s}) = 5$ m/s, and m/s are correct units for velocity. If we had forgotten to divide by 2 s, we would have found $v = 10$ m, and would have noticed that " m " is *not* the correct unit for velocity. This check seems trivial for such a simple problem, but it may not be trivial at all in solving tougher problems.

Checking dimensions throughout a calculation is a great help in preventing and detecting errors. Suppose after solving a long and complicated problem, you find that some acceleration is 2.3 m/s. You realize that the units m/s are wrong. Often it is easiest to find where you made the mistake by going forward or backward through the calculation, and find the first place where the units don't match. That's where a mistake was made! This technique is often easier than checking numbers at each step. Incidentally, you obviously can't use this technique if you plug in numbers alone at the beginning of a problem. It is almost always better to solve a problem algebraically as far as possible, and only plug in numbers (and units) at the end. You can still check the dimensional consistency of equations as you go along.

Dimensional analysis

Dimensional reasoning is a powerful tool for finding out exactly how one quantity depends upon others. The secret is that both sides of the equation must have the same dimensions.

Example 1: A ball of mass m is released at distance h from the "floor" of a spaceship that is uniformly accelerating with acceleration a_0 relative to inertial frames. What is the momentum p of the ball just before it strikes the floor?

It makes sense that the ball's momentum might depend upon its mass, the height from which it is dropped, and the acceleration of the spaceship: That is,

$$p = p(m,\ h,\ a_0).$$

In terms of the fundamental dimensions of mass M, length L, and time T, the dimensions of the relevant quantities are

$$[p] = \frac{\mathrm{ML}}{\mathrm{T}}, \qquad [m] = \mathrm{M}, \qquad [h] = \mathrm{L}, \qquad [a_0] = \frac{\mathrm{L}}{\mathrm{T}^2}.$$

The game now is to find out how m, h, and a_0 can be combined to give the dimensions of momentum. This game is called *dimensional analysis*.

The only way to get the correct M dimension in p is to assume that $p \propto m$. To get the $\frac{1}{\mathrm{T}}$ we have to take the square root of a_0, so that $p \propto m\sqrt{a_0}$. Finally, to get the correct dimensions of length we must multiply by $\sqrt{h}$ (we have to take the square root, because there is already the square root of a length in $\sqrt{a_0}$). Altogether, we have found that $p \propto m\sqrt{a_0 h}$. That is the *only* way to combine m, a_0, and h to get the dimensions of momentum!

The corresponding *equality* is

$$p = k\,m\sqrt{a_0 h}$$

where k is some dimensionless constant we cannot determine by dimensional analysis alone. However, without knowing the numerical value of k (which turns out to be $\sqrt{2}$) we can still answer questions such as:

> "If we drop two similar balls in the same ship from different heights h_1 and $h_2 = h_1/3$, what is the ratio of their momenta when they reach the floor?"

The answer follows from the fact that $p \propto \sqrt{h}$, so that

$$\frac{p_1}{p_2} = \sqrt{\frac{h_1}{h_2}} = \sqrt{3}\,.$$

Example 2. Water flows through a pipe. What is the volume/second of water flowing? That is, what is V/t, the volume of water per second gushing out the end?

First of all, we have to figure out what quantities V/t depends upon. With some experience in physics, one can often make some headway just by thinking about it. This can be easy or not so easy, depending upon the problem and one's experience. But if you are told what these quantities are, a solution is often straightforward.

It turns out that under certain conditions the rate of flow in a pipe depends upon just three quantities. First, it depends on the radius r of the pipe. A fat pipe allows more flow than a skinny pipe. Second, it depends on friction within the water and between the water and sides of the pipe; greater friction tends to decrease the flow rate. Friction in fluids is characterized by a quantity called *viscosity* (which we will discuss in more detail at the end of *Day Nine*), designated by the Greek letter η ("eta"). The viscosity of a particular fluid (which generally depends upon temperature) can be looked up in tables. The greater the viscosity, the lower the flow rate; molasses (with high viscosity) flows more slowly than a light oil (with low viscosity).

Finally, the flow rate depends upon the pressure difference between the ends of the pipe. The greater the pressure difference, the greater the flow rate. Fire fighters need lots of water pressure to project water quickly out the end of their hoses. More precisely, because of friction with the walls of the pipe, the flow rate depends upon the pressure difference *per unit length* of the pipe. That is, for given pressure difference a long pipe has a lower flow rate than a short pipe, because friction is more important in a long pipe (there is more pipe surface tending to hold back the flow). We can eliminate this effect by using the pressure difference per unit length (or pressure *gradient*) $\Delta P/\ell$, where ℓ is the pipe length. If the length of the pipe is doubled, the pressure difference must be doubled as well to maintain the same flow rate.

Now we can figure out how the flow rate V/t depends upon pipe radius r, viscosity η, and pressure gradient $\Delta P/\ell$. The dimensions of V/t, the quantity we are searching for, are $[V/t] = \left[\frac{\text{volume}}{\text{time}}\right] = \frac{L^3}{T}$. We need to find some combination of r, η, and $\Delta P/\ell$ which has the same dimensions.

We know that the dimension of r (denoted by $[r]$) is simply the length dimension L. It turns out that the dimensions of viscosity are $[\eta] = \frac{M}{LT}$ and those of the pressure gradient are $[\Delta P/\ell] = \frac{M}{L^2T^2}$. Now what is the formula for V/t in terms of r, η, and $\Delta P/\ell$?

(i) *Quick, informal solution.*

The quantity V/t contains no dimension of mass, so the two quantities with dimensions of mass on the right must be divided one into the other to eliminate M. Since V/t is expected to grow with increased pressure and shrink with increased viscosity, we take the ratio $\frac{\Delta P/\ell}{\eta}$, with dimensions

$$\left[\frac{\Delta P/\ell}{\eta}\right] = \frac{M/L^2T^2}{M/LT} = \frac{1}{LT}.$$

With this ratio alone, the time dimension is already correct. The length dimension can be made right by multiplying the ratio $\frac{\Delta P/\ell}{\eta}$ by the fourth power of the pipe radius, with dimensions L^4. Therefore we have found by dimensional reasoning that

$$\frac{V}{t} = k\frac{\Delta P/\ell}{\eta} r^4$$

where k is a dimensionless constant of proportionality (which of course we can't find by dimensional reasoning alone.) *The flow rate is proportional to the pressure gradient, to the fourth power of the pipe radius, and inversely proportional to the viscosity.* There is no other way to combine r, η, and $\Delta P/\ell$ to get the dimensions of volume/second. It is remarkable how much can be learned by dimensional reasoning! We have learned that if the radius of the pipe is doubled, for example, the volume of water pushed through per second is sixteen times as large, all else (the pressure gradient and viscosity) being kept the same.

(ii) *Formal solution.*

Assuming we have the correct variables, the formula for the flow rate takes the form

$$\frac{V}{t} = k(\Delta P/\ell)^{\alpha}(\eta)^{\beta}(r)^{\gamma}$$

for some set of exponents α, β, γ, where k is a dimensionless constant. In terms of the dimensions of each quantity,

$$\frac{L^3}{T} = k\left(\frac{\mathrm{M}}{\mathrm{L}^2\mathrm{T}^2}\right)^{\alpha}\left(\frac{\mathrm{M}}{\mathrm{LT}}\right)^{\beta}(\mathrm{L})^{\gamma}.$$

We now make consistent the powers on both sides for L, T, and M.

L: $$3 = -2\alpha - \beta + \gamma$$

T: $$-1 = -2\alpha - \beta$$

M: $$0 = \alpha + \beta$$

The second and third of these equations can be solved together for α and β, with the result $\alpha = -\beta = 1$; then substituting these values into the first equation gives $\gamma = 4$. Therefore

$$\frac{V}{t} = k\frac{\Delta P/\ell}{\eta} r^4 .$$

The formal approach is straightforward and relatively foolproof, but more mechanical and not as much "fun". (The flow through a pipe as described here is called *Pouseille flow*.)

Conversion of units

Often we have to convert from one set of units to another. For example, suppose the magnitude of a particle's momentum is $p = 2.5$ g cm/s , and we want the result in the Standard International units kg m/s instead. A convenient way to convert grams to kilograms and centimeters to meters is to multiply the result by factors of unity, so that grams and centimeters cancel out. We know that 1 kg = 1000 g and 1 m = 100 cm, so

$$p = 2.5 \frac{\text{g cm}}{\text{s}} \left(\frac{1 \text{ kg}}{1000 \text{ g}}\right) \left(\frac{1 \text{ m}}{100 \text{ cm}}\right) = 2.5 \cdot 10^{-5} \frac{\text{kg m}}{\text{s}} .$$

State of the Search

1. We have determined the mass of a ball from a collision experiment. We place the ball at rest, and let a standard 1-kg ball hit it head-on. If v_i and v_f are the initial and final velocities of the 1-kg ball (positive to the right, negative to the left), and if v is the recoil velocity of the target ball, we have defined the mass of the target ball to be

$$m = \left(\frac{v_i - v_f}{v} \right) \text{kg} .$$

2. We have defined the momentum

$$\mathbf{p} = m\mathbf{v}$$

of an object to be the product of the object's mass and velocity. Momentum is useful because it is *conserved* in collisions. That is, the vector sum of the momenta of two balls before a collision is the same as the vector sum of their momenta afterward:

$$(\mathbf{p}_a)_i + (\mathbf{p}_b)_i = (\mathbf{p}_a)_f + (\mathbf{p}_b)_f$$

Momentum is a *vector*, so *each component* of the total momentum is separately conserved. That is, the sum of the x-components of momentum is the same before and after, as are the sum of the y-components and the z-components.

Momentum conservation is the first of three conservation laws we will find, all experimental laws. Conservation laws are profoundly significant in physics.

3. Keeping track of the fundamental dimensions (mass, length, and time) is important in solving problems and avoiding errors. The dimensions on one side of an equation must be the same as they are on the other side. The corresponding units (kg, m, s, for example) are as important in a solution as the numerical values. The answer to a problem is *meaningless* if only the numerical result is given without the corresponding units.

4. Dimensions can used as a tool for solving problems, using the techniques of *dimensional analysis*. Sometimes it is possible to find the dependence of one quantity on other quantities, simply by making the dimensions consistent on both sides of an equation.

PICTURE QUIZ 3

Take it to test your understanding. Answers are given at the end of the text.

1. A cannon is mounted on a railway flatcar; both are moving to the right at speed v_0 relative to the tracks, with the cannon pointed horizontally in the direction of motion, as shown. When the cannon fires a cannonball the cannon and flatcar recoil, so they move

to the *left* at speed v_0, also as shown. If M is the total mass of the cannon and railway car without the cannonball, and m is the mass of the cannonball, what is the speed of the cannonball relative to the tracks just after it has been fired?

(A) $2\left(\frac{M}{m}+1\right)v_0$ (B) $\left(\frac{2M}{m}+1\right)v_0$ (C) $\left(\frac{M}{m}+2\right)v_0$

(D) $(M+2m)v_0$ (E) $2v_0$ (F) $\frac{M}{2m}v_0$

2. Ball A, initially at rest, is struck by ball B. Is it possible for ball A to have a larger final momentum than the initial momentum of ball B?

(A) Yes.
(B) No, because this would violate conservation of momentum.

3. Which of the following equations *cannot* be correct in all systems of units? The m's are masses, v's are velocities, and p's are momenta.

(a) $p_1 + p_2 = m_3 v_3 / (m_1 + m_2)$ (b) $m_1 v_1 = m_2 v_2 + v_3$

(c) $m_1 + m_2 = p_3 / v_1$ (d) $v_1 + v_2 = (p_3 / m_1) + v_4$

4. Which of the following sets of before-and-after pictures might correspond to an actual collision between two equal-mass pool balls? (The length of the vector corresponds to the magnitude of the velocity in each case.)

(a) before

after

(b) before

after

(c) before

after

(d) before

after

PROBLEM QUIZ 3

Answers are given at the end of the text.

1. A car of mass m moving with some unknown initial velocity has a head-on collision with pickup truck of mass $M = 1.5\,m$ initially at rest on an icy road.

(a) After the collision the car's velocity is v_f and the truck's velocity is $2v_f$, both in the forward direction. What was the car's initial velocity v_o, expressed as a multiple of v_f?
(b) Suppose instead that the vehicles become entangled together in the collision, moving off together at the same velocity v_f. What was the car's initial velocity v_o?

2. A pool ball of mass m moving horizontally with speed v_o strikes a lead ball of mass M which is initially at rest on the pool table. The pool ball bounces off at right angles to its original direction of motion, moving horizontally with speed v_f. What is the final speed v' of the lead ball, and at what angle θ does it move relative to the original velocity direction of the pool ball?

top view

3. The escape velocity from Earth is 11.2 km/s. What is this velocity expressed in miles/hour?

4. A simple pendulum is a ball of mass m hung on a string of length ℓ in gravity g. If the pendulum swings back and forth only slightly away from equilibrium, the period P (the time for one complete oscillation), depends only on m, ℓ, and g. By what factor will the period change (a) if the length ℓ is doubled (keeping m and g fixed); (b) if the mass m is doubled (keeping ℓ and g fixed)? Given: g has the same dimensions as acceleration.

5. (a) The natural logarithm of 2, denoted by $\ell n\,2$, is 0.693 to three significant figures. What then is $\ell n\,8$, to three significant figures?
(b) The quantity $e^{0.3}$ is 1.35 to three significant figures. What then is $e^{0.6}$, to three significant figures?

Special Topic: Alpha and Beta Decays of Atomic Nuclei

Momentum is conserved in the collisions of individual atoms or atomic nuclei as well as in the collisions of large objects. Here we will describe atomic nuclei and their so-called α and β decays, in which momentum is also conserved.

An atom consists of a small, positively-charged nucleus surrounded by negatively-charged electrons. A typical nucleus is only about 10^{-14} m in diameter, while the atom as a whole is some 10,000 times larger. A nucleus is composed of positively-charged protons and electrically neutral neutrons. Neutrons and protons have roughly equal masses, while electrons are some 1800 times lighter.

A typical nucleus is a spherical assembly of Z protons and N neutrons. Here Z is the *atomic number*, equal also to the number of electrons in an electrically neutral atom. The element and most of the atom's chemistry is characterized by Z. The total number of *nucleons* (i.e., neutrons plus protons) in a nucleus is $A = Z + N$, called the *atomic weight.*

Each nucleus is represented by a symbol for the element (H, He, Li, . .), with a subscript indicating (redundantly) the atomic number (${}_1H$, ${}_2He$, ${}_3Li$, . .), and a superscript indicating the atomic weight (e.g, ${}_1H^1$, ${}_1H^2$, ${}_1H^3$, ${}_2He^3$, ${}_2He^4$, etc.) Here ${}_1H^1$ is a single proton, for example, the nucleus of an ordinary hydrogen atom. Also ${}_1H^2$, which consists of a proton and a neutron, is called a *deuteron*, the nucleus of a deuterium atom, and ${}_1H^3$, which consists of a proton and two neutrons, is called a *triton*, the nucleus of a tritium atom. These nuclei are all different *isotopes* of hydrogen, each with a single proton but differing numbers of neutrons. The isotopes of helium include ${}_2He^2$, ${}_2He^3$, ${}_2He^4$, etc., all with two protons, but with differing numbers of neutrons.

Some isotopes are stable and some are unstable, flying apart or decaying into various other particles. The isotope ${}_2He^2$ disintegrates immediat ly into two protons, for example; it is violently unstable. There are two stable isotopes c ' hydrogen, ${}_1H^1$ and ${}_1H^2$. The isotope ${}_1H^3$ is unstable, as are all more massive isotopes of hydrogen. There are also two stable isotopes of helium, ${}_2He^3$ and ${}_2He^4$. Such light stable isotopes tend to have roughly equal numbers of protons and neutrons.

Heavy stable isotopes, including for example the lead isotope ${}_{82}Pb^{208}$, with 82 protons and 126 neutrons, always contain more neutrons than protons. A plot of the number of protons *vs.* the number of neutrons in stable nuclei shows a so-called "valley of stability", as indicated in the figure.

What happens to an isotope which happens to be *near* but not *in* the valley of stability? Most of them undergo either alpha (α) decay or beta (β) decay. The most common uranium isotope ${}_{92}U^{238}$ decays into an α -particle and the thorium isotope ${}_{90}Th^{234}$, with a *half-life* $t_{1/2} = 4.5 \cdot 10^9$ years:

$$ {}_{92}U^{238} \rightarrow {}_{90}Th^{234} + \alpha \, . $$

The α -particle is in fact a helium nucleus ${}_{2}He^{4}$; it was called an α -particle before the identification was discovered, and the name stuck. The half-life means that the ${}_{92}U^{238}$ isotope has a 50% chance of decaying within the time $t_{1/2} = 4.5 \cdot 10^9$ years. It is not possible to know when a particular ${}_{92}U^{238}$ nucleus will decay; some decay quickly and some last much longer than $t_{1/2}$. In a large sample, however, very close to half will be gone by $t_{1/2}$. Of those that remain, half will have decayed by the time another $4.5 \cdot 10^9$ years has passed, and so on. There are no stable isotopes of uranium; the reason why there is still a good deal of uranium within Earth is that U^{238} has a long half-life, coincidentially about the same as Earth's age. Therefore about half of Earth's ${}_{92}U^{238}$ still exists.

Experiments confirm that momentum is conserved in α decays. If the uranium nucleus is at rest, for example, the thorium nucleus and α -particle emerge in opposite directions with velocities which satisfy $m_\alpha v_\alpha = m_{Th} v_{Th}$.

Not all alpha decays have billion-year half-lifes.

Parent Nucleus	$T_{1/2}$
Th^{232}	$1.39 \times 10^{10} y$
Ra^{226}	$1.62 \times 10^{3} y$
Th^{228}	$1.9 y$
Em^{222}(Rn)	$3.83 d$
Po^{218}	$3.05 m$
Po^{216}	0.16 sec
Po^{214}	1.5×10^{-4} sec
Po^{212}	3.0×10^{-7} sec

*Data from I. Kaplan, *Nuclear Physics*, Addison-Wesley, Reading, Mass., 1955.

Other radioactive isotopes undergo *beta decay* instead. The triton, for example, is observed to decay into an electron and the helium isotope ${}_2He^3$:

$$ {}_1H^3 \rightarrow {}_2He^3 + e^- $$

with a half-life of 12.3 years. The emitted electron e^- (minus for the negative electric charge, to distinguish it from the electron's antiparticle e^+ which is emitted in some other decays) was originally called a *beta particle* before it was recognized as an electron. The name beta particle is no longer used, but the term *beta decay* is standard. Other beta decays include the famous ${}_6C^{14}$ decay ${}_6C^{14} \rightarrow {}_7N^{14} + e^-$ with half-life 5730 years, used to date ancient manuscripts, shrouds, and mummies, anything formed of organic material within the past several thousand years.

Observers were surprised to find that the final particles did not usually emerge in opposite directions in β-decays; the ${}_2He^3$ isotope and electron from a ${}_1H^3$ decay might appear as shown, for example, even though the ${}_1H^3$ nucleus was initially at rest. This led some people to suggest that momentum is not conserved in beta decays. The Swiss physicist Wolfgang Pauli didn't agree, however; he proposed that a light, neutral particle is also emitted in the decay, which is not observed because the detectors register only charged particles. The hypothetical extra particle, subsequently named the *neutrino* by the Italian physicist Enrico Fermi, would emerge as shown in the example already mentioned, so as to conserve momentum. In this decay the emerging particle is actually the *anti*neutrino $\bar{\nu}$, and the β-decay is written

$$ {}_1H^3 \rightarrow {}_2He^3 + e^- + \bar{\nu}. $$

The neutrino and antineutrino have since been discovered experimentally, and overall momentum is conserved in β-decays.

Sample Problem S1. The tritium nucleus ${}_1H^3$, a hydrogen isotope with one proton and two neutrons, decays into the helium isotope ${}_2He^3$ and an electron, two charged particles which can readily be detected, and an antineutrino $\bar{\nu}$, which is harder to observe. The decay is written ${}_1H^3 \rightarrow {}_2He^3 + e^- + \bar{\nu}$.

In a *particular* decay of a tritium nucleus at rest, suppose the ${}_2He^3$ isotope and electron emerge at right angles to one another as shown, the ${}_2He^3$ with speed $v_{He} = 1.42 \cdot 10^3$ m/s and the electron with speed $v_e = 7.79 \cdot 10^6$ m/s. In what direction is the antineutrino emitted, and what is the magnitude of its momentum? The masses of the ${}_2He^3$ nucleus and the electron are, respectively, $5.00 \cdot 10^{-27}$ kg and $9.11 \cdot 10^{-31}$ kg.)

Solution: The momentum vectors of the three final particles are shown. The antineutrino momentum has been drawn at an angle θ below the horizontal axis; we have to find this angle along with the magnitude of p_n to make the three vectors sum to zero. The horizontal components and vertical components of momentum must each sum to zero, so

$$p_n \cos\theta = p_{He^3} = M_{He^3} v_{He^3} = 5.00 \cdot 10^{-27} \text{ kg} \cdot 1.42 \cdot 10^3 \text{ m/s} = 7.1 \cdot 10^{-24} \text{ kg m/s}$$

$$p_\nu \sin\theta = p_e = M_e v_e = 9.11 \cdot 10^{-31} \text{ kg} \cdot 7.79 \cdot 10^6 \text{ m/s} = 7.1 \cdot 10^{-24} \text{ kg m/s.}$$

Therefore $\tan\theta = (p_n \sin\theta)/(p_n \cos\theta) = 1.0$, so $\theta = 45^\circ$.

Then

$$p_n = \sqrt{p_\nu^2 \sin^2\theta + p_\nu^2 \cos^2\theta} = 7.1\sqrt{2} \cdot 10^{-24} \text{ kg m/s} = 1.0 \cdot 10^{-23} \text{ kg m/s.}$$

Questions and Problems from Day Three

3Q1. How can a Bubble explorer measure her own mass using a 1-kg ball, a meter-stick, and a stopwatch?

3Q2. A canoeist returning from a trip tries to step onto the dock. Why does he have to take a larger step than he might think?

3Q3. A Ford driver finds herself stalled at rest on the frictionless frozen surface of a fjord in February. What can she do to rescue herself and her Ford?

3Q4. Assuming momentum conservation, (a) if a ball is thrown at another ball at rest, is it true that the collision necessarily takes place in a single plane? That is, are all velocity vectors, before and after, in the same plane? (b) Suppose the target ball is also moving initially. Will the collision necessarily be confined to some plane in this case?
(c) Suppose the thrown ball hits two balls, both initially at rest. Is the collision confined to a plane in this case?

3Q5. A very heavy pendulum is suspended within a closed truck, as shown. The pendulum is pivoted at the upper end. What would happen to the pendulum as the driver steps on the brakes? What is then likely to happen after the truck has stopped?

3Q6. While watching the *fourth* experiment in Table 1, how fast would you have to move relative to the stretched string so that it would look like the *fifth* experiment?

3Q7. If the total momentum of a system of particles is conserved in one inertial frame, is the total momentum of the system also conserved in all other inertial frames?

Section 3.1 Collisions of identical balls

3P1. Using the data given in Table 1, predict the final velocities of two standard single balls a and b in the Bubble if they collide head-on with initial velocities
(a) $(v_a)_i = 75$ cm/s, $(v_b)_i = -25$ cm/s,
(b) $(v_a)_i = 45$ cm/s, $(v_b)_i = -15$ cm/s,

3P2. A cue ball with initial velocity $v_{ci} = 50$ cm/s has a head-on collision with an identical target ball at rest. After the collision the cue ball moves forward at $v_{cf} = 2$ cm/s. (a) What is the velocity v_{tf} of the target ball after the collision? (b) If instead the cue ball and target ball have initial velocities $v_{ci} = 25$ cm/s and $v_{ti} = -25$ cm/s, respectively, what would be the final velocity of each ball, assuming they collide head-on?

3P3. The cue ball, moving with initial velocity $v_{ci} = 3.0$ m/s on a pool table, has a head-on collision with the identical 8-ball, initially at rest. Afterwards the cue ball has the final velocity $v_{cf} = 0.1$ m/s (a) What is the final velocity v_{8f} of the 8-ball? (b) If instead the cue ball moves initially at $v_{ci} = 2.0$ m/s toward the 8-ball, while the 8-ball moves toward the cue ball at $v_{8i} = 1.0$ m/s, how fast will each ball be moving after the collision?

3P4. An airtrack on Earth is an aluminum track through which many tiny holes have been drilled, so that small jets of air escape when a high-pressure hose pumps air into the track. Aluminum carts on the track are suspended by the small jets, so they can glide nearly frictionlessly along the track. We observe that when identical carts A and B are projected along the track toward one another, cart A with velocity $v_{Ai} = 25$ cm/s and cart B with velocity $v_{Bi} = -15$ cm/s, after they collide cart A has velocity $v_{Af} = -14$ cm/s. (a) What is cart B's final velocity? (b) If instead cart A were projected with velocity $v_{Ai} = 40$ cm/s at cart B at rest, what would be the velocities of A and B after the collision?

3P5. Two identical balls collide in one dimension. Show that if the sum of their velocities is conserved when viewed from one frame of reference, then the sum of their velocities is also conserved when viewed from any other frame of reference. *Hint:* Use Eq. 3(a), *Day Two*.

3P6. One Ford Taurus collides head-on with a similar Taurus parked on an icy street. Immediately after the collision one car has velocity 17 mi/hr and the other has velocity 4.0 mi/hr. (a) What was the velocity of the original moving car? (b) A passing motorist, driving parallel to the original moving car, sees that after the collision one of the cars has velocity 8.0 mi/hr from her point of view. How fast was the original parked car moving from her point of view?

Section 3.2 Collisions of single balls with multiple balls

3P7. A single ball of velocity v_0 makes a head-on collision with a three-ball combination, initially at rest. Afterwards the three-ball combination has the forward velocity $v_0/2$. (a) What is the final velocity of the single ball? Which direction is it moving? (b) The three-ball combination, still with velocity $v_0/2$, strikes another single ball, initially at rest. Afterwards the single ball has the forward velocity v_0. What is the final velocity of the three-ball combination? Is it moving forward or backward?

3P8. Two identical freight cars attached together slide along an icy rail with initial velocity v_0 into a combination of three identical freight cars, also attached together, which are initially at rest. After the collision the three-car combination moves forward with velocity $v_0/2$. (a) What is the final velocity of the two-car combination? In which direction is it moving? (b) Consider the same collision from the point of view of passengers on a train moving on a parallel track at velocity $v_0/2$ in the forward direction. What are the initial and final velocities of each combination of cars in this frame?

3P9. Three parked Honda Accords, chained together on an icy street, are struck by a single Accord moving at 22.0 miles/hour. After the collision the three chained cars move at 10.0 miles/hour. How fast, and in what direction, does the single Accord move after the collision?

3P10. Four spaceships, bolted together, are moving toward the Orion Nebula at one-tenth the speed of light. Two spaceships, bolted together and moving in the same direction as the first four at one-fifth the speed of light, strike the first four from behind. All six spaceships are identical to one another. If the group of two is moving at one-tenth the speed of light after the collision, how fast is the group of four then moving?

3P11. (a) Ball a of unknown mass and initial velocity $(v_a)_i$ = 100 cm/s collides head-on with ball b, a 1 kg ball, initially at rest. Ball a has final velocity $(v_a)_f = 50$ cm/s, and ball b has final velocity $(v_b)_f = 75$ cm/s. Find the mass of ball a.
(b) Now view the experiment described in part (a) from the standpoint of a primed frame in which ball a is initially at rest. Find the velocity of ball a after the collision in this frame, and the velocities of ball b before and after.
(c) Using the results of (a) and (b), show that momentum is conserved in this collision, in the primed frame.

3P12. Consider the collision between a single ball moving initially at velocity v_o, and an N-ball initially at rest, such that they attach together at the moment of contact.
(a) Assuming momentum conservation, find the velocity of the final $(N+1)$-ball.
(b) Another single ball, also with velocity v_o in the same direction, now collides with the moving $(N+1)$-ball, and sticks. Find the speed of the resulting $(N+2)$-ball. (Collisions in which objects stick together are said to be *totally inelastic*.)

3P13. A particular automobile has mass $m_a = 1500$ kg, and a particular baseball has mass m_b = 150 g. (a) How fast would the baseball have to move to have the same momentum as the automobile moving at 2 cm/s? (b) Could a pitcher throw the ball that fast?

3P14. Using balls threaded on strings, we projected a standard 1-kg ball into a stationary ball of unknown mass, and used measured velocities to define the mass of the stationary ball.
(a) Describe an alternative procedure for defining mass in which the ball of unknown mass is projected into standard 1-kg balls, initially at rest.
(b) Develop a general formula from which the unknown mass can be calculated.
(c) Use the velocity transformation $v_x' = v_x - V$ to show that the two methods for defining mass are really equivalent. That is, show that the masses found are the same in one method as they are in the other.

3P15. Using the velocity transformation $v_x' = v_x - V$ (Eq. 3(a) of *Day Two*), show that if Eq. 7 is valid in the inertial frame in which ball b is initially at rest, then Eq. 8 is valid in other inertial frames, in which ball b is initially moving.

3P16. A hockey player of mass M is skating toward the goal at speed V. The puck, of mass m, has just been hit by the goalie directly toward him at speed v. The player hits the puck back toward the goalie at speed v'. With what speed v_f is the player moving just after he hits the puck, (a) in terms of M, m, V, v, and v'? (b) in meters/sec, if $M = 75.0$ kg, $m = 0.16$ kg, $V = 15.0$ m/s, $v = 22$ m/s, and $v' = 25$ m/s. (All speeds are relative to the ice.)

3P17. A pig of mass M_p floating in gravity-free space holds a gun of mass M_g containing a bullet of mass M_b. The pig fires, holding onto the gun. Afterwards, if the velocity of the bullet is (a) v_1 relative to the *original* rest-frame of the pig, what has been the pig's change of speed, in terms of the masses and v_1? (b) v_2 relative to the *final* rest-frame of the pig, what has been the pig's change of speed, in terms of the masses and v_2?

3P18. In 1980, L. Alvarez, W. Alvarez, F. Asaro, and H. Michel* suggested that an asteroid struck Earth sixty-five million years ago, leading to the mass extinction of the dinosaurs and many other species, and bringing the Cretaceous period to a close. Suppose an asteroid of uniform density $\rho = 2.2$ g/cm^3 and diameter $d = 10$ km struck head-on with speed $v_0 = 25$ km/s relative to the surface. (These values are consistent with those given by Alvarez *et al.*) (a) Find Earth's recoil velocity, assuming the asteroid was absorbed and there was no splashback of material into space. (b) There would probably be some splashback. Would Earth's recoil velocity likely be less or more than that calculated in (a)? Note: Earth's mass is $M = 5.97 \cdot 10^{24}$ kg. **Science* **208**, 1095 (1980)

3P19. A space traveler pushes off from her coasting spaceship to give herself some space. She slowly drifts away from the ship at speed v_0. Her mass, together with that of the spacesuit, is M; she is also carrying a pair of binoculars of mass m. Twenty minutes later she decides to return, but her thruster doesn't work. In another forty minutes her oxygen supply will be used up, so she immediately throws the binoculars with velocity v_b relative to the ship, directly away from the ship.
(a) After the throw, what is her speed relative to the ship?
(b) Suppose $v_0 = 0.25$ m/s, $M = 120$ kg, and $m = 1.2$ kg. What is the minimum binocular speed v_b required so she will make it back in time?

3P20. In some supernova explosions, the core of the exploding star implodes to form a neutron star, which is kicked away from the exploding star with a very high velocity. Suppose a particular neutron star has the mass of our Sun and a velocity $v = 1200$ km/s. (a) What is its velocity expressed as a fraction of the speed of light? (b) What is its momentum, in SI units?

Section 3.5 Collisions in three dimensions

3P21. The cue ball strikes the 8-ball, which is initially at rest on a pool table. After the collision, the cue ball moves at 2.00 m/s at an angle of 30.0° to its initial direction of motion, while the 8-ball moves at an angle of 60.0° to its initial direction. Find (a) the final speed of the 8-ball; (b) the original speed of the cue ball. (The balls have the same mass.)

3P22. A bowling ball of mass m and velocity v_0 strikes a pin of mass $m/3$, initially at rest. After the collision the pin moves with speed $v_0/2$ at an angle of 60° to the forward direction. Find (a) the final speed of the ball as a multiple of v_0 (b) the final angle the ball makes with the forward direction.

3P23. In the middle of an icy intersection, a 1981 Honda Civic moving north at some unknown speed crashes into a 1989 Lincoln Towncar moving west at $v = 30$ mi/hr. The police investigation shows that the two cars became locked together upon impact and slid in the northwest direction. (The Civic and Towncar weigh 2050 lbs and 4300 lbs, respectively, including the weight of their drivers. The weight of an object is proportional to its mass.)
(a) How fast was the Civic travelling before the collision?
(b) How fast do they both travel after the collision?

3P24. The starship *Enterprise*, of mass M, moving to the "right" at speed v_E, is struck by a Klingon vessel of mass $M/2$, moving "down" at some unknown speed, as shown. The ships become locked together in the collision, moving off at an angle of 30° to the original *Enterprise* direction of motion. As a multiple of v_E, what is (a) the initial speed of the Klingon ship? (b) the final speed of both ships?

3P25. A bowler makes a strike, knocking down all ten pins. Just before hitting any pins the 2.0 kg ball moves at 8.0 m/s. After it has stopped hitting pins, it moves at 7.2 m/s at an angle of 12° to its original direction. At this time what is the magnitude of the total momentum of the pins in the horizontal plane, if none of them has yet hit the backstop? Neglect friction with the alley or air.

3P26. In spherical coordinates, the direction of a momentum vector $\mathbf{p}$ is specified by two angles θ and ϕ. The polar angle θ is the angle $\mathbf{p}$ makes with the z-axis of a Cartesian coordinate frame, and the azimuthal angle ϕ is the angle the projection of $\mathbf{p}$ onto the x, y plane makes with the x axis, as shown. Suppose a particle has a momentum of magnitude p directed with $\theta = 45^o$ and $\phi = 30^o$. Find the x, y, and z components of the momentum and express $\mathbf{p}$ in terms of these components and unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$.

3P27. Given the masses of two balls and the initial velocity of the incident ball (the target ball is at rest), there are four unknowns in a collision in three dimensions, $(v_a)_f$, $(v_b)_f$, θ, and ϕ. (See Fig.2) Assuming momentum conservation, what is the minimum number of quantities which have to be measured so that all four unknowns are determined?

3P28. Demonstrate from the velocity transformation $\mathbf{v}' = \mathbf{v} - \mathbf{V}$ (Eq. 4b of *Day Two*) that in a three-dimensional collision of two balls of (possibly) different mass, if in an unprimed frame

$$(\mathbf{p}_a)_i + (\mathbf{p}_b)_i = (\mathbf{p}_a)_f + (\mathbf{p}_b)_f ,$$

then in any primed frame,

$$(\mathbf{p}_a')_i + (\mathbf{p}_b')_i = (\mathbf{p}_a')_f + (\mathbf{p}_b')_f .$$

3P29. A cosmic-ray proton of unknown initial velocity collides with the stationary nucleus of an atmospheric atom, and scatters off at $\theta = 90.0^o$ with speed $v_p = 2.30 \cdot 10^7$ m/s. The nucleus recoils with speed $v_N = 2.10 \cdot 10^6$ m/s at angle $\phi = 43.2^o$, measured from the momentum direction of the incident proton, as shown.

(a) Express the initial and final momentum vectors of both particles in terms of their masses and speeds, using Cartesian components and unit vectors. Assume the proton moves initially in the x-direction, and the scattering takes place in the x-y plane.
(b) Find the mass of the nucleus, in terms of the proton mass. (c) What nucleus is this likely to be? (d) Find the initial velocity of the proton, in m/s.

3P30. A neutron of mass m and velocity v_0 makes a head-on collision with a $_{92}U^{235}$ isotope of mass M at rest in a nuclear reactor. The neutron is absorbed to form $_{92}U^{236}$. (a) Find the velocity v_A of the $_{92}U^{236}$ isotope in terms of m, M, and v_0. (b) The $_{92}U^{236}$ isotope subsequently fissions into two $_{46}Pd^{118}$ isotopes of equal mass, each emerging at angle θ to the forward direction, as shown below. Find the speed v_B of each $_{46}Pd^{118}$ isotope in terms of m, M, v_0, and θ. (c) Suppose instead that one of the $_{46}Pd^{118}$ isotopes emerges at angle 60° and the other at angle 30° to the forward direction. Find the speeds of each of these particles in terms of v_0.

3P31. In a collision in which the target ball is initially at rest, the final momentum vectors $(\mathbf{p}_a)_f$ and $(\mathbf{p}_b)_f$ in a three-dimensional collision must sum to $(\mathbf{p}_a)_i$. By vector addition, the picture is as shown.

(a) Show that the scattering angles θ and ϕ are two of the interior angles in this triangle. (See Fig. 2)
(b) Use the *law of sines* to find an equation identical to the conservation of momentum in the y-direction, as given by Eq.13b. (c) Use the *law of cosines* to find an equation for $(p_b)_f$ in terms of $(p_a)_i$, $(p_a)_f$, and θ. (d) Show that your result in (c) may also be derived by eliminating ϕ between Eqs.13a and 13b for the conservation of momentum in the x- and y-directions.

3.6 Addendum on dimensions and units

3P32. (a) What is the numerical size of the quantity $\frac{1}{2}mv^2 = 7.2\ \frac{\text{kg m}^2}{\text{s}^2}$, in units of $\frac{\text{g cm}^2}{\text{s}^2}$? (b) What is the density of water, $\rho = 1$ g/cm^3, in kg/m^3 ?

3P33. (a) The mass of an electron is $9.1 \cdot 10^{-31}$ kg. What is its mass in grams?
(b) Planck's constant $h = 6.627 \cdot 10^{-34}$ kg m^2/s. How large is Planck's constant in units g cm^2/s?

3P34. The velocity of waves on the surface of a lake depends only upon gravity g and the depth h of the lake, as long as the wavelength (the distance between two successive wave crests) is large compared with h. If the depth is increased by a factor of two, by what factor will the wave velocity be changed for such surface-water waves? Given: g has the same dimensions as acceleration. Solve using (i) the quick, informal method, and also (ii) the formal method introduced in the addendum.

3P35. The velocity of waves on the surface of a lake depends only upon gravity g and the wavelength λ (the distance between two successive wave crests), as long as the depth of the lake is large compared with λ. If λ is increased by a factor of two, by what factor will the wave velocity be changed for such surface-water waves? Given: g has the same dimensions as acceleration. Solve using (i) the quick, informal method, and also (ii) the formal method introduced in the addendum.

3P36. Capillary waves on the surface of a liquid are related to the liquid's surface tension σ, which has the dimensions M/T^2. The velocity of capillary waves depends upon σ and also upon the wavelength λ (the distance between two successive wave crests) and the density ρ (the mass per unit volume) of the liquid. Two waves on the same liquid have wavelengths λ_1 and $\lambda_2 = 2\,\lambda_1$. What is the ratio of their velocities? Solve using (i) the quick, informal method, and also (ii) the formal method introduced in the addendum.

3P37. (a) The "Planck length" ℓ_p depends only upon the speed of light c, Newton's constant of gravity G (with units $m^3\,kg^{-1}\,s^{-2}$) and Planck's constant (divided by 2π) $\hbar$ (with units $kg\,m^2\,s^{-1}$.) If Planck's constant were twice as large as it is, how would ℓ_p be affected? (b) Do the same for the "Planck time" t_p and "Planck mass" m_p; each of these quantities also depends only upon c, G, and $\hbar$.

3P38. A ball of mass m is tossed straight upward from the ground with velocity v_0. The time it takes for it to rise and fall back to the ground might depend upon m, v_0, and the gravitational field g (which has the dimensions of acceleration.) If v_0 were doubled, by what factor would the time above ground increase? What would happen if m were doubled, keeping v_0 and g the same?

3P39. Two astronauts are instantaneously at rest above a spherical asteroid of mass m and radius R. One astronaut is at distance r and the other at distance $2r$ from the asteroid's center, where $r >> R$. If it takes time T for the first astronaut to fall to the asteroid, about how long does it take the second astronaut to fall? *Hint*: The time to fall depends upon m, r, and Newton's gravitational constant G, with dimensions $L^3\,M^{-1}\,T^{-2}$

Special Topic: Alpha and Beta Decays

3P40. In the α-decay of a uranium nucleus $U^{238} \rightarrow Th^{234} + \alpha$, the α-particle emerges with speed $3.4 \cdot 10^5$ m/s in the U^{238} rest-frame. The masses of Th^{234} and of the α-particle are 234.0 u and 4.0 u, respectively. Here "u" is the symbol for *atomic mass unit,* a system in which the mass of the C^{12} nucleus is set equal to 12, *exactly.* Momentum is conserved in the decay. What is the speed of the Th^{234} nucleus (a) in the U^{238} rest-frame? (b) in the rest-frame of the α-particle?

3P41. A $_4Be^8$ nucleus moving to the right in the lab at speed V decays quickly into two α-particles, one of which moves to the left at speed v in the lab.

(a) With what speed, and in what direction, does the other α-particle move in the lab?

(b) In the rest frame of the $_4Be^8$ nucleus, with what speeds and in which directions do the two α-particles move?

(c) The two α-particles from the decay of a second $_4Be^8$ nucleus move along the positive and negative y axis in the rest-frame of the nucleus, each with speed v_0. In the lab frame in which the nucleus moves in the positive x-direction with speed V, how fast does each α-particle move? What is the angle each makes with the x axis?

3P42. In a particular $_2H^3$ (tritium) decay, the $_2He^3$ nucleus and the electron emerge at right angles to one another; the antineutrino emerges at the angle θ shown. (See *Sample Pbm S1*). The electron is found to have speed $v_e = 9.31 \cdot 10^6$ m/s, and the $_2He^3$ nucleus has speed $v_{He} = 9.80 \cdot 10^2$ m/s. Find the antineutrino's (a) angle θ; (b) momentum magnitude p_ν. [The masses of $_2He^3$ and electron are $5.00 \cdot 10^{-27}$ kg and $9.11 \cdot 10^{-31}$ kg, respectively.]

3P43. In a carbon-14 beta-decay $_6C^{14} \rightarrow {}_7N^{14} + e^- + \bar{\nu}$, with the carbon nucleus at rest, the nitrogen nucleus emerges along the positive y-axis with momentum p_N, and the electron emerges along the positive x-axis with momentum $p_e = p_N/2$. At what angle to the positive x-axis does the antineutrino emerge, and what is the magnitude of its momentum as a multiple of p_N?

3P44. (a) A free neutron (i.e., a neutron not bound up inside an atomic nucleus) decays into a proton, electron, and antineutrino $n \rightarrow p + e^- + \bar{\nu}$ with a half-life of about 15 minutes. In a particular decay of a neutron at rest, the proton emerges in the positive x-direction at speed $5.00 \cdot 10^5$ m/s, and the electron emerges in the positive y-direction with the speed $9.14 \cdot 10^6$ m/s, as shown. What is the magnitude of the antineutrino's momentum, and at what angle does it emerge relative to the negative x-axis? The masses of the proton and electron are $1.67 \cdot 10^{-27}$ kg and $9.11 \cdot 10^{-31}$ kg, respectively.

(b) Now view the same decay from a frame of reference moving in the positive x-direction at speed $5.00 \cdot 10^5$ m/s relative to the reference frame of part (a). Find the magnitude and direction of the momentum of (1) the proton (2) the electron in this new reference frame.

Open-ended problem:

Consider an experimental determination of mass different from that presented in Section 3.3. A small spring is threaded on a stretched string, between two identical balls. The balls are squeezed together, compressing the spring, and then released simultaneously from rest. The velocities of the two balls are observed to be equal but opposite. Then consider a single standard "1-kg" ball on one side, and N such balls hooked together on the other side. What relative velocities would be observed? Finally, consider a "1-kg" ball on one side, and an arbitrary ball on the other. Is the new definition of mass equivalent to the one used in the text? If so, is any additional assumption necessary? If not, which is a better definition?

Day Four

Center of Mass and Impulse

"Ninety percent of this game is half-mental." - Yogi Berra

4.1 The center of mass

We toss a banana. It tumbles end over end as it moves from one side of the Bubble to the other. Each end of the banana moves in a complicated way - certainly not in a straight line at constant speed. A point near the banana's center, however, *does* seem to move in a straight line at constant speed, while the rest of the banana circles around it. We call this

point the *banana's center of mass*, and the velocity of this point we call V_{CM}, the *velocity of the center of mass*. Where exactly *is* this center of mass, and how do we find V_{CM}?

A banana is a complicated object, and tumbles in a complicated way. Simpler is a system of two balls, the colliding balls we studied yesterday. Take two balls of equal mass, as in the sequence of pictures shown in Fig. 1. Ball 1, initially moving, collides with ball 2, initially at rest. What is the point that moves at constant velocity before, during, and after the collision? It is the point halfway between the balls. Fig. 1 shows the position of this point at equal time intervals. Note that the point advances the same distance each time, so moves at constant velocity.

Fig. 1. A collision of equal-mass balls, shown at equal time intervals. The point halfway between them, the center of mass of the system, moves at constant velocity.

Why is there a point that moves at constant velocity during the collision of two balls? Yesterday we found that the total momentum P of balls a and b is conserved when they collide head-on. Their total mass M also stays the same, so the ratio

$$\frac{P}{M} = \frac{m_a v_a + m_b v_b}{m_a + m_b} \tag{1}$$

has to stay constant as well. This ratio has the dimensions of velocity, and is conserved. It must be the constant velocity V_{CM} we sought. That is, the center-of-mass velocity for two balls a and b is

$$V_{CM} = \frac{m_a v_a + m_b v_b}{m_a + m_b}, \tag{2}$$

which stays the same before, during, and after a collision. The individual balls change velocities as they collide, but the center of mass keeps moving at constant velocity.

Sample Problem 1. A Volkswagen Beetle of mass m moves with initial velocity v_o along an icy street and collides head-on with an initially parked Cadillac Eldorado of mass $2m$. After the collision the Cadillac moves forward with speed $\frac{1}{3}v_o$.

(a) What is v_f, the final velocity of the Volkswagen?

(b) What is V_{CM}, the center-of-mass velocity of the two cars?

Solution: (a) From conservation of momentum it follows that

$$mv_o = mv_f + 2m\,\frac{1}{3}v_o\,,$$

which gives $v_f = \frac{1}{3}v_o$. The VW moves ahead with one-third its initial speed, which is the same speed as the Cadillac.

(b) Before the collision, the center-of-mass velocity is

$$V_{CM} = \frac{m_a v_a + m_b v_b}{m_a + m_b} = \frac{mv_o + 2m\,(0)}{m + 2m} = \frac{1}{3}\,v_o\,,$$

one-third the velocity of the VW. After the collision both cars are moving together with velocity $\frac{1}{3}v_o$. Their center-of-mass velocity afterwards is therefore

$$V_{CM} = \frac{m_a v_a + m_b v_b}{m_a + m_b} = \frac{m\left(\frac{1}{3}v_o\right) + 2m\,(\frac{1}{3}v_o)}{m + 2m} = \frac{1}{3}\,v_o,$$

the same as it was before the collision. The velocity of each car changes, but the center-of-mass velocity stays the same. (It is not surprising that the CM velocity is $\frac{1}{3}v_o$ after the collision, because the cars are moving together at this same speed.)

What is the location of the center of mass of two balls (or of two cars)? Where is the point that moves at the same velocity before, during, and after the collision? We note that the velocity VCM can be represented as a time derivative,

$$V_{CM} = \frac{m_a v_a + m_b v_b}{m_a + m_b} = \frac{d}{dt}\left(\frac{m_a x_a + m_b x_b}{m_a + m_b}\right), \tag{3}$$

since the masses are constant, and the individual velocities are the derivatives $v_a = \frac{dx_a}{dt}$, $v_b = \frac{dx_b}{dt}$. Here x_a and x_b are the positions of the two balls, measured from some arbitrary origin, as shown in Fig. 2. The quantity in parenthesis in Eq. 3 has the dimensions of length. We call it *the position* X_{CM} *of the center of mass* of the two balls. That is, $V_{CM} = \frac{dX_{CM}}{dt}$, where

$$X_{CM} = \frac{m_a x_a + m_b x_b}{m_a + m_b}. \tag{4}$$

Fig. 2. The positions of *a* and *b*.

If the two masses are equal, Eq. 4 gives $X_{CM} = \frac{1}{2}(x_a + x_b)$, their average position, i.e., halfway between them. This makes sense for the center of mass of two equal balls.

We can rearrange Eq. 4 to give

$$X_{CM} = \frac{(m_a + m_b - m_b)x_a + m_b x_b}{m_a + m_b} = x_a + \left(\frac{m_b}{m_a + m_b}\right)(x_b - x_a), \tag{5}$$

the position of the left-hand ball plus a term proportional to $x_b - x_a$, which is the distance between the two balls. If the two masses are equal, the second term is $\frac{1}{2}(x_b - x_a)$, so X_{CM} is halfway between the balls. If $m_b = 2m_a$, the second term is $\frac{2}{3}(x_b - x_a)$,.

indicating that X_{CM} is two-thirds of the way to x_b, twice as far from a as from b. Figure 3 shows a sequence of equal-time pictures of a collision between two balls, where the target ball has twice the mass of the incident ball. (Shown is an "elastic" collision between the balls, unlike the "inelastic" collision between the VW and Cadillac of *Sample Problem* 1, where the objects stuck together after the collision. The differences will be explored during *Day Thirteen.*)

Fig. 3. Sequence of equal-time pictures in the collision of two balls, where the ball at the right has twice the mass of the ball at the left. The center of mass of the system moves forward at constant velocity.

Sample Problem 2. (a) Take two balls a and b with masses m_a and m_b. Show that if x'_a is the distance of ball a from their center of mass and x'_b is the distance of ball b from their center of mass, then $m_a x'_a = m_b x'_b$.

Solution: The distance of ball a from the center of mass is $x'_a = \left(\frac{m_b}{m_a + m_b}\right)(x_b - x_a)$, as given by Eq. 5. The sum of the distances of the two balls from their center of mass must equal the total distance between the two balls; that is, $x'_b + x'_a = (x_b - x_a)$, so

$$x'_b = (x_b - x_a) - x'_a = [1 - \frac{m_b}{m_a + m_b}](x_b - x_a) = \left(\frac{m_a}{m_a + m_b}\right)(x_b - x_a).$$

Comparing these formulas for x'_a and x'_b, clearly $m_a x'_a = m_b x'_b$, as we had to prove. The mass of one ball times its distance from the center of mass *is* equal to the other mass times its distance from the center of mass. If ball a has three times the mass of ball b, for example, then ball b is three times as far as ball a from their mutual center of mass.

What about an arbitrary number of balls? What is the center-of-mass velocity of N balls, and where is their center of mass located? For a collection of balls threaded on a string in a straight line, their total momentum is

$$P = \sum_i p_i = \sum_i m_i v_i ,$$

the sum of the momenta of all the balls. The total momentum is conserved as they collide with one another, and their total mass $M = \sum_i m_i$ also stays constant. Therefore the ratio P/M stays constant as well, which we identify as the center-of-mass velocity

$$V_{\mathrm{CM}} = \frac{P}{M} = \frac{m_{\mathrm{a}} v_{\mathrm{a}} + m_{\mathrm{b}} v_{\mathrm{b}} + \cdots}{m_{\mathrm{a}} + m_{\mathrm{b}} + \cdots} = \frac{\sum m_i v_i}{\sum m_i} . \tag{6}$$

The center of mass position of N balls is the generalization of Eq. 4 for two balls,

$$X_{\mathrm{CM}} = \frac{m_{\mathrm{a}} x_{\mathrm{a}} + m_{\mathrm{b}} x_{\mathrm{b}} + \cdots}{m_{\mathrm{a}} + m_{\mathrm{b}} + \cdots} = \frac{\sum m_i x_i}{\sum m_i} . \tag{7}$$

As for two balls, the center-of-mass velocity is $V_{\mathrm{CM}} = \dfrac{dX_{\mathrm{CM}}}{dt}$.

Sample Problem 3. Four balls are threaded on a string oriented along the x-axis, at intervals of 50 cm, as shown. In order, the balls have masses 1 kg, 2 kg, 3 kg, and 4 kg.

(a) How far is the center of mass from the 1-kg ball?

(b) If, at time $t = 0$, the four balls are given the velocities $v_1 = 1$ m/s, $v_2 = -2$ m/s, $v_3 = 3$ m/s, and $v_4 = -4$ m/s, describe the subsequent motion of the center of mass.

Solution: (a) With the origin located at the 1-kg ball, the position of the center of mass is

$$X_{CM} = \frac{\sum m_i x_i}{\sum m_i} = \frac{(1\text{ kg} \cdot 0) + (2\text{ kg} \cdot .5\text{ m}) + (3\text{kg} \cdot 1.0\text{ m}) + (4\text{ kg} \cdot 1.5\text{ m})}{1\text{ kg} + 2\text{ kg} + 3\text{ kg} + 4\text{ kg}}$$

$$= \frac{10\text{ kg m}}{10\text{ kg}} = 1.0\text{ m}.$$

The CM turns out to be located at the position of the 3-kg ball. (Note that it is *not* true that half the mass of the system is on each side of the CM!)

(b) The center-of-mass velocity is

$$V_{CM} = \frac{\sum m_i v_i}{\sum m_i} = \frac{(1\text{ kg})(1\text{ m/s}) + (2\text{ kg})(-2\text{ m/s}) + (3\text{ kg})(3\text{ m/s}) + (4\text{ kg})(-4\text{ m/s})}{10\text{ kg}}$$

$$= -1\text{ m/s}.$$

As the balls move, including collisions between the first and second balls and between the third and fourth balls, the center of mass of the four balls moves at a steady rate of 1 m/s to the left relative to the fixed string.

Sample Problem 4. Find the CM position of a thin stick of mass M and length L, lying along the positive x-axis with one end at $x = 0$, if the stick has (a) uniform mass density λ ; (b) linearly increasing mass density $\lambda = k\,x$, where k is a constant.

Solution: (a) By symmetry, we *expect* that the CM for the uniform stick is in the middle of the stick, at $x = L/2$. How do we calculate it from the definition $X_{CM} = \dfrac{\sum m_i x_i}{M}$?

We can think of the stick as composed of many thin crosswise slices, with total mass

$$M = \sum m_i = \Delta m_1 + \Delta m_2 + \ldots ,$$

the sum of the slice masses, and

$$\sum m_i x_i = \Delta m_1 x_1 + \Delta m_2 x_2 + \ldots ,$$

where for example x_1 is the position of the first slice, as shown.

Now we take the limit of an infinite number of infinitely-thin slices, so the sum signs are replaced by integrals, and m_i *is replaced by* dm. *That is,* $X_{CM} = \dfrac{\int x\,dm}{\int dm}$ *for a continuous mass distribution.*

If $\lambda(x)$ is the linear mass density, then the mass dm of a short length of the stick dx is $dm = \lambda(x)\,dx$, as shown below, and the total mass of the stick is $M = \int \lambda(x)\,dx$.

In part (a) the mass density is constant, so $M = \int_0^L \lambda\,dx = \lambda\,L$, as expected, and

$$X_{CM} = \frac{\int_0^L \lambda\,x dx}{M} = \frac{\lambda \int_0^L x\,dx}{M} = \frac{\lambda L^2/2}{M} = \frac{L}{2}$$, also as expected.

(b) If $\lambda = kx$, where k is a constant, then the mass of the stick is

$$M = \int_0^L \lambda \, dx = k\int_0^L x \, dx = \frac{1}{2}kx^2\Big|_0^L = \frac{1}{2}kL^2$$

and the center-of-mass position is

$$X_{CM} = \frac{\int_0^L \lambda \, x dx}{M} = \frac{k\int_0^L x^2 \, dx}{M} = \frac{k\,L^3/3}{M} = \frac{k\,L^3/3}{\frac{1}{2}kL^2} = \frac{2L}{3}.$$

The center of mass is now shifted over toward the more massive end of the stick. If we threw this stick in the gravity-free Bubble, it would turn end over end while this center of mass position on the stick moved in a straight line at constant speed, as shown below.

4.2 The center of mass in three dimensions

So far we have only found the center of mass of two or more objects spread out along a straight line. How do we find the center of mass for a three-dimensional object, like a banana or a human being?

We can generalize the definition to three dimensions using vectors. For an arbitrary number of balls $m_1, m_2, m_3 \cdots$, it is natural to define the CM position as

$$\boxed{\mathbf{R}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots}{m_1 + m_2 + \ldots} = \frac{\Sigma_i \, m_i\mathbf{r}_i}{\Sigma_i \, m_i} = \frac{\Sigma_i \, m_i\mathbf{r}_i}{M}} \qquad (8)$$

where the Cartesian components of $\mathbf{R}_{CM}$ are (X_{CM}, Y_{CM}, Z_{CM}), and the Cartesian components of the individual balls are $\mathbf{r}_i = (x_i, y_i, z_i)$.

Sample Problem 5. Find the center of mass of three balls of masses $m_i = 2, 4$, and 6 kg, arranged at the vertices of an equilateral triangle of side $d = 1$ m, as shown.

Solution: Choose an origin of Cartesian axes at the 2-kg ball, with x-axis extending toward the 4-kg ball. Then the balls are located at

$$\mathbf{r}_2 = (0\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}}), \quad \mathbf{r}_4 = (1\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}})\text{ m}, \quad \text{and } \mathbf{r}_6 = (0.5\,\hat{\mathbf{x}} + \sqrt{3}/2\,\hat{\mathbf{y}})\text{ m}.$$

Therefore

$$\Sigma_i\, m_i\mathbf{r}_i = [4\,(1\,\hat{\mathbf{x}}) + 6\,(0.5\,\hat{\mathbf{x}} + \sqrt{3}/2\,\hat{\mathbf{y}})]\text{ kg m} = (7\,\hat{\mathbf{x}} + 3\sqrt{3}\,\hat{\mathbf{y}})\text{ kg m},$$

so

$$\mathbf{R}_{CM} = \frac{\left(7\hat{\mathbf{x}} + 3\sqrt{3}\,\hat{\mathbf{y}}\right)\text{ kg m}}{(2+4+6)\text{ kg}} = \left(\frac{7}{12}\hat{\mathbf{x}} + \frac{\sqrt{3}}{4}\hat{\mathbf{y}}\right)\text{ m}$$

as shown. Note that $\mathbf{R}_{CM}$ is closer to the more massive balls.

The velocity of the center of mass for three-dimensional motion is $\mathbf{V}_{CM}$, where

$$\boxed{\mathbf{V}_{CM} = \frac{d\mathbf{R}_{CM}}{dt} = \frac{\Sigma_i\, m_i\, \mathbf{v}_i}{M} = \frac{\mathbf{P}}{M}.} \tag{9}$$

As for one-dimensional motion, the center-of-mass velocity is conserved in collisions; the CM drifts along in a straight line at constant speed throughout the course of the collision. If for example the three balls shown in *Sample Problem* 5 are not stationary, but are moving toward one another in such a way that their center of mass remains at rest before they collide with one another, then their center of mass remains at rest during and after the collisions as well.

Sample Problem 6. An isoceles triangle whose smallest angle is α is cut from a piece of cardboard. How far is the center of mass of the triangle from the vertex at α?

Solution: For a continuous mass distribution, the CM is located at

$$\mathbf{R}_{\text{CM}} = \frac{\int dm\,\mathbf{r}}{M},$$

the generalization of the discrete-mass definition of Eq. 8. Here dm is an infinitesimal part of the triangle's mass, $\mathbf{r}$ is its position vector, and M is the total mass of the object. From symmetry it is clear that the CM must lie somewhere along the horizontal symmetry axis (the x-axis) shown above, so all we need to calculate is

$$X_{\text{CM}} = \frac{\int dm\,x}{M}.$$

We will place the origin at the left-hand vertex, and let the mass per unit area of the cardboard be denoted by σ. Then the mass dm of an infinitesimal slice of the triangle of width dx located at horizontal distance x from the vertex is $dm = \sigma\,dA = \sigma\,(2y\,dx)$, as shown, where the area of the slice is dA = height ($2y$) times the width (dx) of the slice. Here $y = x\tan(\alpha/2)$ is the height of the upper leg of the triangle at x. Therefore

$$X_{\text{CM}} = \frac{\int dm\,x}{M} = \frac{\int dm\,x}{\int dm} = \frac{2\sigma\tan(\alpha/2)\int_0^L dx\,x^2}{2\sigma\tan(\alpha/2)\int_0^L dx\,x}$$

$$= \frac{x^3/3\Big|_0^L}{x^2/2\Big|_0^L} = \frac{L^3/3}{L^2/2} = \frac{2}{3}L.$$

The center of mass is located *two-thirds* of the way from the vertex to the right-hand side of the triangle. The result does not depend upon the angle α!

(Note that this is the same answer we got for the one-dimensional stick in *Sample Problem* 4. Why do the results come out the same?)

4.3 "Particles" and the center-of-mass frame

The equations of momentum conservation and the definitions of the center-of-mass position and momentum can be used for general objects, including bananas and humans, as well as the metal balls, if we think of general objects as composed of a huge number of very small pieces called *particles*. A particle is supposed to be small enough that it can be characterized by a single position and velocity. No single position can exactly characterize a ball, banana, or human being. No single velocity can characterize them either, because these complicated objects may wiggle or rotate. However, the equations

$$\mathbf{V}_{\mathrm{CM}} = \frac{d\mathbf{R}_{\mathrm{CM}}}{dt} \quad \text{and} \quad \mathbf{V}_{\mathrm{CM}} = \frac{\mathbf{P}}{M}, \tag{10a}$$

valid for a collection of particles, are similar to the equations

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \quad \text{and} \quad \mathbf{v} = \frac{\mathbf{p}}{m} \tag{10b}$$

for a single particle. *We can therefore think of a composite object as a particle, if we follow the center of mass.* If the total momentum $\mathbf{P}$ of the particles comprising a banana is conserved, for example, then the CM of the banana will move in a straight line at constant speed in the Bubble, even though the banana may rotate as it moves. This is exactly what we observed when we threw one. A drifting bomb may even blow up in midflight; its CM continues to move in the same straight line at the same constant speed.

The velocity $\mathbf{V}_{\mathrm{CM}}$ of a system of particles stays constant during collisions. If we ourselves drift along at this velocity, we will find in our new frame that $\mathbf{V}_{\mathrm{CM}} = 0$. We will call this frame the *center of mass frame* or *CM frame*. It is the frame in which the total momentum $\mathbf{P} = M\mathbf{V}_{\mathrm{CM}}$ of the system is zero. If two equal-mass balls collide, for example, in the CM frame the balls approach one another with equal but opposite velocities v, and later depart with equal but opposite velocities v', as shown.

If the masses of two balls aren't the same, their velocities are different, but their momenta are always equal but opposite in the CM frame.

Sample Problem 7. A ball of mass $m_1 = 1$ kg is given a velocity $v_1 = 12$ m/s and a ball of mass $m_2 = 2$ kg is given a velocity $v_2 = -3$ m/s in the rest frame of the Bubble, as shown.

(a) How fast is the CM frame moving relative to the Bubble?
(b) How fast is each ball moving in the CM frame?

Solution: (a) The velocity of the center of mass of the two balls is

$$V_{CM} = \frac{P}{M} = \frac{1\text{ kg }(12\text{ m/s}) + 2\text{ kg }(-3\text{ m/s})}{3\text{ kg}} = 2\text{ m/s},$$

which by definition is also the velocity of the CM *frame* relative to the Bubble frame.

(b) Using the velocity transformation Eq. 3(a) of *Day Two*, the velocity of ball a in the CM frame is

$$v_a' = v_a - V_{CM} = 12\text{ m/s} - 2\text{ m/s} = 10\text{ m/s},$$

and the velocity of ball b in the CM frame is

$$v_b' = v_b - V_{CM} = -3\text{ m/s} - 2\text{ m/s} = -5\text{ m/s},$$

as shown. Note that the total momentum of the balls in the CM frame is

$$P_{CM} = (1\text{ kg})(10\text{ m/s}) + (2\text{ kg})(-5\text{ m/s}) = 0$$

as expected.

The total momentum of an isolated system in the CM frame is always zero.

4.4 Impulse

Is the momentum of an object always conserved? Obviously not! We can change the momentum of a single ball just by hitting it or catching it. And in the collision of two balls, it is only the *sum* of their momenta that is conserved. The momentum of each ball individually is changed by the collision.

What is the explanation of momentum change? We need to understand so we can decide, in a new situation, whether momentum should be conserved or not.

It seems clear that the momentum of a ball changes if it *touches* something else, like our hand when we throw it or another ball in a collision. If it doesn't touch anything, it coasts along with constant velocity, and so has constant momentum.

What is it about *touching* something that causes a ball's momentum to change? Suppose that when ball a collides with ball b, ball a exerts a quantity we will call impulse $\mathbf{I}(t)$ on b, causing it to change its momentum. That is,

$$\mathbf{I}(t) = \Delta\mathbf{p} = \mathbf{p}(t) - \mathbf{p}_o . \tag{11}$$

where $\mathbf{p}_o$ is ball b's momentum before the collision, $\mathbf{p}(t)$ is its momentum at a time t later, and $\Delta\mathbf{p}$ is the change in its momentum. If the impulse $\mathbf{I}(t) = 0$ at some time t, ball b's momentum hasn't changed; the collision hasn't happened yet. If $\mathbf{I}(t) \neq 0$, the momentum has changed, indicating that the collision is either under way or has already happened. The impulse $\mathbf{I}$ is the cause of ball b's momentum change. It is not simply a *name* for the momentum change; it is to *blame* for the momentum change.

We can derive an interesting property of $\mathbf{I}$ from the equation $\mathbf{I} = \Delta\mathbf{p}$. In the collision of two balls, if $\mathbf{I}_{ab}$ is the impulse ball b exerts upon ball a, and $\mathbf{I}_{ba}$ is the impulse a exerts upon b, then

$$\mathbf{I}_{ab} = (\mathbf{p}_a)_f - (\mathbf{p}_a)_i \tag{12a}$$

and

$$\mathbf{I}_{ba} = (\mathbf{p}_b)_f - (\mathbf{p}_b)_i \tag{12b}$$

Therefore their *sum*

$$\mathbf{I}_{ab} + \mathbf{I}_{ba} = [(\mathbf{p}_a)_f + (\mathbf{p}_b)_f] - [(\mathbf{p}_a)_i + (\mathbf{p}_b)_i] = \mathbf{P}_f - \mathbf{P}_i = 0, \tag{13}$$

since the total momentum of the balls is conserved. It follows that

$$\mathbf{I}_{ba} = -\mathbf{I}_{ab};$$

the impulse ball b exerts upon ball a is equal but opposite to the impulse a exerts on ball b. In other words, *for every impulse there is an equal but opposite counterimpulse.* This result follows directly from conservation of momentum and the equation $\mathbf{I} = \Delta\mathbf{p}$.

Sample Problem 8. A cannon of mass m_c sits in empty space. It contains a cannonball of mass m_b. A charge in the cannon explodes, causing the cannonball to be shot out with velocity v_b in the original rest-frame of the cannon.

(a) What is the final velocity of the cannon?
(b) What impulse did the cannon exert on the cannonball?
(c) What impulse did the cannonball exert on the cannon?

Solution: (a) Momentum is conserved throughout the firing. Beforehand, the momentum was zero in the rest-frame of the cannon. Therefore

$$0 = p_{\text{ball}} + p_{\text{cannon}} = m_b v_b + m_c v_c.$$

Solving for the final velocity of the cannon, $v_c = -(m_b v_b / m_c)$. The minus sign means that the cannon recoil; i. e., it moves off in the opposite direction from the cannonball.
(b) The impulse on the cannonball by the cannon is $I_{bc} = \Delta p_{\text{ball}} = m_b v_b$
(c) The impulse on the cannon is $I_{cb} = \Delta p_{\text{cannon}} = m_c v_c$. It follows from part (a) that $m_c v_c = -m_b v_b$, so $I_{cb} = -I_{bc}$, which we know must be true.

4.5 Impulsive *changes*

The impulse given to a ball as we throw it is sketched in Fig. 5(a). The ball starts from rest, so the equation $\mathbf{I}(t) = \Delta\mathbf{p}(t) = m\mathbf{v}(t)$ shows that $\mathbf{I}$ increases as the ball speeds up, going from zero to its maximum value $\mathbf{I}_f$ when the ball is released. Our hand causes $\mathbf{I}$ to increase, and when we release the ball, $\mathbf{I}$ levels off to its final value. So it is really a *change* in $\mathbf{I}$ that is due to contact with our hand, and not $\mathbf{I}$ itself. Similarly, in the collision of two balls the impulse given to one ball by the other is illustrated in Fig. 5(b). The impulse I changes only duidring the split-second of contact; when $\mathbf{I}$ become $\mathbf{I}_f$ and levels off, nothing more is transferred from one ball to the other. After that, each ball just coasts along.

Fig. 5. (a) Impulse on a ball as we throw it (b) Impulse in the collision of two balls

The final impulse given to a ball in a particular collision may be the same as the impulse our hand gives the ball in a particular pitch, but the collision is strong and quick while the pitch is weaker and more gradual. The rate $\frac{d\mathbf{I}}{dt}$ of the transfer of momentum therefore has a lot of physical meaning. It can tell the difference between the strong shock of the collision and the steadier push of the hand while throwing the pitch.

The time derivative of $\mathbf{I}(t)$ has another interesting property. Since in general $\mathbf{I}(t) = \mathbf{p}(t) - \mathbf{p}_O$, where $\mathbf{p}_O$ is the original momentum, taking the derivative $\frac{d\mathbf{I}}{dt}$ wipes out the constant vector $\mathbf{p}_O$. Only the strength of the collision itself is kept: the past, as represented by $\mathbf{p}_O$, doesn't matter when we take the derivative.

The magnitude $\frac{dI}{dt}$ of the impulse derivative is sketched in Fig. 6 for two cases, throwing a ball and colliding two balls together.

Fig. 6. *dI/dt* as a function of time (a) while throwing a ball (b) while two balls collide

We plotted each of the graphs by finding the slope of the graph for I(t) in the corresponding case. For example, the graph of Fig. 6(a) is the slope of the graph in Fig. 5(a). The slope is nonzero only during the actual contact of the ball with our hand or with another ball.

The slope of I seems to describe the actual collision better than I itself, and so deserves a name of its own. We call $\frac{d\mathbf{I}}{dt}$ the *force* $\mathbf{F}$. The momentum of an object changes when we push or pull on it. It is when we exert a force on one of the balls, by pushing it, or by crashing another ball into it, that its momentum changes. That's exactly what the derivative $\frac{d\mathbf{I}}{dt} = \mathbf{F}$ does: It measures the rate of momentum change.

The impulse is $\mathbf{I}(t) = \mathbf{p}(t) - \mathbf{p}_0$, so its derivative is $\frac{d\mathbf{I}}{dt} = \frac{d\mathbf{p}}{dt}$. That is,

$$\boxed{\mathbf{F} = \frac{d\mathbf{I}}{dt} = \frac{d\mathbf{p}}{dt}}$$

The force on an object is equal to the time rate of change of its momentum. We hot that force has the dimensions $[F] = ML/T^2$, and the SI units kg m/s^2. [Eq. 15 is one form of *Newton's second law of motion.* Force is such an important quantity that it is given its own unit. In the SI system, the unit is *Newtons*: 1 Newton (1 N for short) is 1 kg m/s^2.]

Sample Problem 9. A pitched baseball of mass $m = 147$ g approaches the batter at velocity $v_0 = 42$ m/s (94 mi/hr). The bat connects for a time $\Delta t = 1.0$ ms, sending the ball straight back toward the pitcher at speed $v_f = 60$ m/s.

(a) Find the impulse exerted on the ball by the bat.

(b) Find the average force exerted on the ball during the period of contact.

Solution: The momentum change of the ball is $\Delta\mathbf{p} = m\mathbf{v}_f - m\mathbf{v}_0$; its magnitude is

$$\Delta p = m(60 \text{ m/s}) - m(-42 \text{ m/s}) = 147 \text{ g } (60 \text{ m/s} + 42 \text{ m/s}) = 15 \text{ kg m/s}$$

(Note that the initial and final velocities are in opposite directions, so they have opposite signs.) The impulse exerted on the ball is therefore $I = 15$ kg m/s directed toward the pitcher.

(b) The average force exerted by the bat on the ball while the two are in contact is directed toward the pitcher, with magnitude

$$\langle F \rangle = \frac{\Delta I}{\Delta t} = \frac{\Delta p}{\Delta t} = \frac{15 \text{ kg} \cdot \text{m/s}}{1.0 \cdot 10^{-3} \text{s}} = 15 \cdot 10^3 \text{ N}$$

According to Eq. 15 the force $\mathbf{F}(t)$ on an object is the time rate of change of the impulse $\mathbf{I}(t)$ on the object, and it is also the time rate of change of the object's momentum $\mathbf{p}(t)$. So if we know $\mathbf{I}(t)$ or $\mathbf{p}(t)$ it is straightforward to find $\mathbf{F}(t)$ by differentiation.

We showed in Section 4.4 (Eq. 14) that in the collision of two balls, the impulses on the balls were equal but opposite: i. e.,

$$\mathbf{I}_{ab} = -\mathbf{I}_{ba}$$

if the total momentum of the two balls is conserved. We had another way of saying it: *For every impulse there is an equal but opposite counterimpulse.*

The force on a due to b at each moment during the collision is $\mathbf{F}_{ab} = \dfrac{d\mathbf{I}_{ab}}{dt}$, and the force on b due to a is $\mathbf{F}_{ba} = \dfrac{d\mathbf{I}_{ba}}{dt}$. From $\mathbf{I}_{ab} = -\mathbf{I}_{ba}$ it then follows that

$$\boxed{\mathbf{F}_{ab} = -\mathbf{F}_{ba}.} \tag{17}$$

At each moment the force on a due to b is equal but opposite to the force on b due to a. That is, *for every force there is an equal but opposite counterforce.**

If the only force on ball a is caused by ball b, and the only force on b is caused by a, as in a collision, then $\mathbf{F}_{ab} = \dfrac{d\mathbf{p}_a}{dt}$ and $\mathbf{F}_{ba} = \dfrac{d\mathbf{p}_b}{dt}$. Therefore their sum is

$$\mathbf{F}_{ab} + \mathbf{F}_{ba} = \frac{d(\mathbf{p}_a + \mathbf{p}_b)}{dt}, \tag{18}$$

so $\mathbf{F}_{ab} = -\mathbf{F}_{ba}$ if and only if the total momentum of the two balls is conserved, that is, if and only if the total momentum $\mathbf{p}_a + \mathbf{p}_b$ is constant.

Conservation of momentum is equivalent to the force-counterforce law according to Eq. 18. If either rule fails, so must the other. If a collision is ever discovered in which the forces are not equal but opposite, then momentum is not conserved in collision, if Eq. 18 continues to hold. And if a collision is ever discovered in which momentum is not conserved, then the forces are not equal but opposite.

* This is *Newton's third law*, often expressed as "For every action there is an equal but opposite reaction."

State of the Search

1. The center of mass (CM) of a collection of objects with masses m_i is located at $\mathbf{R}_{CM} = \frac{\sum m_i \mathbf{r}_i}{M}$, where M is the total mass. Here $\mathbf{r}_i$ is the position vector of the ith object measured from the origin of coordinates. If the mass distribution is continuous, then $\mathbf{R}_{CM} = \frac{\int \mathbf{r}\, dm}{M}$. It follows from these vector expressions for the center-of-mass position that the x-coordinate of the CM is $X_{CM} = \frac{\sum m_i x_i}{M}$, with similar expressions for the y and z coordinates.

2. The center of mass moves at constant velocity throughout collisions. This center-of-mass velocity is $\mathbf{V}_{CM} = \frac{d\mathbf{R}_{CM}}{dt} = \frac{\sum m_i \mathbf{v}_i}{M} = \frac{\mathbf{P}}{M}$, where $\mathbf{P}$ is the total momentum of the colliding objects.

3. The net external impulse $\mathbf{I} = \Delta\mathbf{p}$ is the cause of a change of momentum $\Delta\mathbf{p}$ of a system of objects. *For every impulse, there is an equal but opposite counter-impulse.* That is, if object a exerts an impulse $\mathbf{I}_{ba}$ on object b, then object b exerts an impulse $\mathbf{I}_{ab}$ back on object a, where $\mathbf{I}_{ab} = -\mathbf{I}_{ba}$. Rocket motion is an example of this law. A rocket exerts an impulse on a bit of newly-burned fuel, ejecting it as exhaust; the burned fuel exerts an equal but opposite counter-impulse on the rocket, causing it to accelerate forward.

4. Just as the net external impulse $\mathbf{I} = \Delta\mathbf{p}$ is the cause of a change of momentum $\Delta\mathbf{p}$ of a system, the net external force $\mathbf{F} = \frac{d\mathbf{I}}{dt}$, the time derivative of the net impulse, causes the momentum of the system to change with time according to $\mathbf{F} = \frac{d\mathbf{p}}{dt}$, one form of *Newton's second law.*

5. For every impulse there is an equal but opposite counter-impulse, so it follows that for every force there is an equal but opposite counterforce. If object a exerts a force $\mathbf{F}$ on object b, then b always exerts an equal but opposite counterforce $-\mathbf{F}$ back on a. This is sometimes stated "for every action there is an equal but opposite reaction", known as *Newton's third law*.

PICTURE QUIZ 4

1. The object shown is cut from a piece of uniform plywood. Which of the points shown is closest to the object's center of mass?

(A) a (B) b (C) c (D) d

2. (a) If a six-foot tall high jumper jumps over a bar six feet above the ground, about how far must her center of mass rise?

(A) 0 ft (b) 3 ft (c) 6 ft

(b) Is it possible for a high jumper's center of mass to pass *beneath* the bar during a successful jump?

(A) yes (b) no

3. A penny of mass m is glued to one end of a uniform stick of mass $2m$ and length L. How far from the other end of the stick is the center of mass of the stick + penny?

(A) $L/2$ (B) $2L/3$ (C) $3L/4$ (D) $5L/6$ (E) $7L/8$

4. A rocket of mass 25000 kg is at rest in outer space. The rocket's engine is turned on for 15 seconds, burning 25 kg of fuel and causing the rocket to gain a velocity 50 m/s.

(a) What impulse was exerted on the rocket by the exhaust (all in units kg m/s) ?

(A) $1.25 \cdot 10^3$ (B) $1.25 \cdot 10^4$ (C) $1.25 \cdot 10^5$ (D) $1.25 \cdot 10^6$ (E) none of these

(b) What average force was exerted on the rocket during the 15 second burn?

(A) $8 \cdot 10^1$ N (B) $2 \cdot 10^2$ N (C) $6 \cdot 10^3$ N (D) $8 \cdot 10^4$ N (E) $2 \cdot 10^5$ N

(c) What was the approximate average speed of the exhaust during the 15 second burn?

(A) 2 m/s (B) $25 \cdot 10^2$ m/s (C) $5 \cdot 10^4$ m/s (D) $7 \cdot 10^5$ m/s (E) $3 \cdot 10^8$ m/s

5. A railway train collides head-on with small car stuck on the tracks.

During the collision,

(A) The train exerts a greater amount of force on the car than the car exerts on the train.

(B) The car exerts a greater amount of force on the train than the train exerts on the car.

(C) Neither exerts a force on the other; the car gets crushed simply because it is at rest in a different reference frame than the rest-frame of the train.

(D) The train exerts a force on the car but the car does not exert a force on the train.

(E) The train exerts the same amount of force on the car as the car exerts on the train.

PROBLEM QUIZ 4

1. Three beads, with masses 2 g, 3 g, and 4 g, are strung along a straight string at 10-cm intervals, as shown.

(a) Measured from the 2-gram bead, where is the location X_{CM} of the center of mass of the system? (b) Is there the same total mass on each side of X_{CM} ? (c) The string is then cut and the beads are given the velocities 1 cm/s, 3 cm/s, and 4 cm/s, respectively, all to the right. What is the center-of-mass velocity V_{CM}? (d) Find the velocity of each bead in the CM frame, and verify that the total momentum in the CM frame is *zero*.

2. A car of mass m and initial speed v_0 is traveling directly toward a truck of mass $3m$ and initial speed $v_0/2$ traveling in the opposite direction on an icy highway. (a) What is the speed of the truck according to passengers in the car? (b) What is the speed of the car+truck center-of-mass frame relative to the road? (c) If the car and truck collide and become entangled together in the collision, how fast are they moving relative to the road? How fast are they moving in the CM frame?

3. A pool ball of mass M and velocity v_0 collides head-on with a ping-pong ball of mass m, initially at rest. The collision gives the ping-pong ball a speed $v = 2v_0$. (a) What is the magnitude of the impulse given to the ping-pong ball by the pool ball in the collision? (b) What is the magnitude of the impulse given to the pool ball by the ping-pong ball in the collision? (c) If the collision acts for a time Δt, what is the magnitude of the average force $<F>$ exerted on the pool ball by the ping-pong ball during this time?

4. A laser beam is aimed directly at the surface of a mirror. The beam consists of N photons/second, each with momentum p_γ, all moving in the same direction. Half the photons are absorbed by the mirror, and half are reflected back to the laser. What average force $<F>$ does the beam exert on the mirror? [This problem uses ideas from *Special Topic* I]

Special Topic I: The Average Force Due to Many Small Collisions

A tiny grain of sand, drifting through space, strikes the outside of the Bubble and bounces off. The force it exerts lasts for only the brief time interval of the collision, as illustrated in Figure (a) below. If many such sand grains strike about the same place on the Bubble at random times and with random velocities, the force on the Bubble might look as shown in Figure (b). What *average* force would these strikes exert on the Bubble?

The force exerted by (a) a single collision of a sand grain (b) collisions of many sand grains. (c) The average force exerted by many sand grains.

To find their average force, we can average over many collisions, by replacing the definition $F = \dfrac{dI}{dt}$, the time rate of change of the impulse, by $F_{\text{av}} \equiv \langle F \rangle = \dfrac{\Delta I}{\Delta t}$, where ΔI is the finite change of impulse caused by a substantial number of striking grains, and Δt is the time interval during which those grains strike. The average force is then the smoothed-out version shown in Figure (c).

[Note that the force due to an individual grain can be much larger or smaller than the average force. The local large force due to a fast-moving grain might leave a pit in the surface for example, so on a small scale the average force wouldn't mean much. The average force *would* be useful in finding the subsequent motion of the Bubble as a whole, however, if it is struck by a huge number of sand grains.]

How do we find the change in impulse ΔI ? We know that the impulse exerted upon the Bubble by a colliding sand grain is equal but opposite to the impulse exerted upon the sand grain by the Bubble. We know also that the impulse exerted upon the sand grain by the Bubble is equal to the momentum change of the sand grain, Δp , which in turn is the mass of the grain multiplied by the grain's velocity change Δv . So to find the average force exerted upon the Bubble, we just add up the momentum change of all the grains striking the Bubble over some period of time Δt , and divide by Δt .

Sample Problem S1. A swarm of sand grains, all with mass m and speed v, are incident normally upon a surface of area A. There are n grains per unit volume, and each grain bounces back with the same speed v after it collides with the surface. Find the average pressure (the force/area) exerted upon the surface by the grains.

Solution: The average force $< F >$ on the surface has the same magnitude as the average force on the grains, which is

$$< F > \;=\; \frac{\Delta\,\text{Impulse}}{\Delta t} \;=\; \frac{\text{Impulse}}{\text{grain}} \cdot \frac{\text{no. of grains}}{\Delta t}\,.$$

The number of grains which strike per unit time Δt is equal to the number of grains per unit volume in the swarm, multiplied by the volume of grains striking per time interval Δt; that is,

$$< F > \;=\; \frac{\text{Impulse}}{\text{grain}} \cdot \frac{\text{no. of grains}}{\text{volume}} \cdot \frac{\text{volume}}{\Delta t}\,.$$

The volume of grains striking per time Δt is equal to the area of the surface times the distance moved by the grains per time Δt toward the surface, which is the velocity of the grains. That is,

$$< F > \;=\; \frac{\text{Impulse}}{\text{grain}} \cdot \frac{\text{no. of grains}}{\text{volume}} \cdot \text{Area} \cdot \text{velocity}.$$

$$= \quad 2mv \cdot n \cdot A \cdot v\,.$$

Here the impulse per grain is the grain's momentum change $\Delta p = 2mv$, the factor "2" arising from the fact that the final momentum of the grain is equal but opposite to its initial momentum. The pressure on the surface is therefore

$$\text{Pressure} = < F >/A = 2mnv^2$$

proportional to the square of the grain velocity v, because both the impulse exerted per grain and the number of grains per unit time striking the surface are proportional to v. Again, the local pressure can be greater or smaller than the average pressure.

Special Topic II: Project Orion

Project Orion was a secret project in the late 1950's to study the feasibility of propelling long-range spaceships with nuclear bombs. The bombs would be carried on board, and ejected one at a time toward the rear. When a bomb had drifted a few kilometers behind the ship, it would be detonated, and a huge hemispherical bowl, called the *pusher*, attached to

shock absorbers at the rear of the ship, would absorb the blast and propel the ship forward. Freeman Dyson, a project member, has described a particular design.* The ship is loaded with 30 million bombs, each of mass m = one metric ton (1000 kg). Bombs are detonated a distance d = 10 km behind the ship, in the center of a hemisphere of radius R = 10 km, attached to the ship. Dyson estimates the ejected particles from a bomb to have a typical speed U between 1% and 10% the speed of light, i.e, 3000 km/s $\leq U \leq$ 30,000 km/s. He also assumes the explosions are spherically symmetric. What impulse does each explosion give to the ship? That is, how much momentum does the ship gain from each explosion?

If an explosion were somehow focussed so that half the ejected particles moved directly toward the ship and the other half directly away from it, the impulse imparted to the ship would be the momentum of the half moving toward the ship, $\frac{1}{2}mU$, where m is the total mass of the bomb. (This assumes that as the ejected particles strike the hemisphere they are not reflected backward. If they are reflected, the impulse is greater.) We are supposing that the explosion is spherically symmetric, however. Therefore only one component of the momentum of each particle is directed toward the ship; the perpendicular

*See "Interstellar Transport" by Freeman J. Dyson in *Physics Today,* October, 1968

component does not push the ship forward. The net impulse to the ship will therefore be $I < \frac{1}{2}mU$. As shown in the Appendix, the right answer is $I = \frac{1}{4}mU$. Each spherically symmetric explosion delivers an impulse to the hemispherical "pusher" which is half what it would be if half the ejected particles moved straight toward the ship. In the particular design of the Orion team, the fully-loaded ship has a mass $M = 40$ million tonnes. What is the boost Δv in ship velocity caused by the impulse from a single explosion, for a fully-loaded ship and for an empty ship?

The impulse $I = mU/4$ causes a momentum change $\Delta p = M\Delta v$ of the ship, so the ship's velocity is increased by $\Delta v = I / M$, where M is the ship's mass. For a fully loaded ship, and making the conservative choice $U = 3000$ km/s,

$$\Delta v = \frac{1000\,\text{kg} \cdot 3000\,\text{km/s} / 4}{4 \cdot 10^{10}\,\text{kg}} = 0.02\ \text{m/s}.$$

The ship is so massive that a nuclear explosion only boosts its speed by two centimeters per second! When the last few bombs are used the ship's mass will have dropped by 75%, so Δv will increase by a factor of 4, to 0.08 m/s.

We can estimate the final speed of the ship, after all 30 million bombs have been used. In particular, lower and upper limits to the ship's final velocity can be found by multiplying the number of explosions (30 million) times the lower and upper limits of Δv per explosion, found by assuming the ship is fully loaded or empty, respectively. Fully loaded, $v_{\text{final}} = 3 \cdot 10^7 \cdot .02$ m/s = 600 km/s; empty, $v_{\text{final}} = 3 \cdot 10^7 \cdot .08$ m/s = 2400 km/s. So

$$600\ \text{km/s} < v_{\text{final}} < 2400\ \text{km/s}.$$

For a more precise result, see Problem 4P44. The upper limit $v_{\text{final}} = 2400$ km/s is a bit less than 1% of the speed of light. That speed would get the ship to the nearest star in about 500 years, if the speed were gained comparatively early in the trip, followed by a long period of drift. It is technologically feasible to build the ship now, but it would be much too expensive. In one or two hundred years it might be affordable if world economies advance sufficiently.

[Readers who wish to build such a spaceship now are encouraged not to turn on the propulsion system described until the ship is a considerable distance from Earth. They should also be aware that they will have to seek an exemption from an international treaty prohibiting nuclear explosions in space.]

Questions and Problems from Day Four

4Q1. Is the center of mass of an object necessarily within the object itself?

4Q2. A batter hits a home run over the center-field fence to the north of home plate. Using the equal impulse and counter-impulse rule, is it fair to say that just after hitting the ball, the bat's CM must be traveling south?

4Q3. A batter hits the baseball with a bat. Argue for or against conservation of momentum of the following systems: (a) the ball; (b) the ball and bat; (c) the ball, bat, and batter; (d) the ball, bat, batter, and Earth.

4Q4. The graphs show the impulse given to the space shuttle during launch by four possible first-stage rocket engines.
(a) Which will "jerk" the astronauts most? (Take "jerk" to be the time rate of change of acceleration.)
(b) Which gives the greatest final speed to the shuttle?

4Q5. Sitting at rest in gravity-free space, a rocket starts to burn fuel and so moves away toward the right. In which direction does the CM of the system move, where the system is the ship together with all of its exhaust ?

4Q6. At a particular instant, two positive electric charges q_1 and q_2, with masses m_1 and m_2, are moving as shown. Charge 1 is moving at velocity v_1 directly toward charge 2, while charge 2 is moving at velocity v_2 at right angles to v_1. The two charges exert equal but opposite repulsive electrical forces F_E on each other. Charge q_2 also exerts a magnetic force F_B on q_1 as shown, but q_1 does not exert a magnetic force on q_2. No other forces are exerted on the charges.
(a) Is the force-counterforce rule (Newton's third law) valid in this situation?
(b) Is the total momentum of q_1 and q_2 conserved?
(c) Does the center of mass of the two charges move at constant velocity?

4Q7. Starting from rest, the distance x a spacecraft moves depends upon its mass M, the impulse I imparted to it, and the time interval t. Using dimensional reasoning, how far will it move in the same time (as a multiple of x) if *twice* the impulse is imparted?

4Q8. Suppose that we bat a baseball, imparting an impulse $I(t)$ to it in one inertial frame. What impulse have we given it as viewed by observers in an inertial frame moving at velocity v relative to the first one? There is a corresponding force $F(t)$ we have exerted on the baseball in the original frame. What is $F(t)$ in the other inertial frame?

4P1. Given two particles of masses m_1 and m_2, located on the x-axis at $x_1 = 10$ cm and $x_2 = 30$ cm. Find the CM location for (a) $m_2/m_1 = 1/2$ and (b) $m_2/m_1 = 3$. In each case sketch the positions of m_1, m_2, and the CM.

4P2. Find the CM location for a CO molecule, measured from the carbon atom. The ratio of the masses of the carbon and oxygen atoms is $m_C/m_O = 12/16$, and the distance between the two atoms is d.

4P3. Four rock climbers, with masses 50 kg, 60 kg, 70 kg, and 80 kg, are attached to a vertical rope at 5.0 m intervals, as shown, with the 50 kg climber at the top. (a) Measured from the 80 kg climber, where is the location Y_{CM} of the center of mass of the system of climbers? (Neglect the mass of the rope.) (b) Is there the same total mass below as above Y_{CM} ? (c) At some instant the top two climbers are stationary, while the 70 kg and 80 kg climbers are falling with velocities 1.0 m/s and 2.0 m/s, respectively. What is the system's center-of-mass velocity V_{CM} ?

4P4. Find the distance of the CM of the Earth-Moon system from the center of Earth, expressed as a fraction of the radius of Earth. Data: $M_{moon} / M_{earth} = 0.012$; Earth-Moon distance (between their centers) = 380,000 km ; radius of Earth = 6400 km. (b) Jupiter has more mass than all the other planets combined, so crudely speaking, the solar system is the Sun-Jupiter system. Find the CM of the Sun-Jupiter system expressed as a fraction of the radius of the Sun. Data: $M_{Jupiter} / M_{sun} = 0.00095$; Jupiter-Sun distance = $7.8 \cdot 10^8$ km ; radius of the Sun = $7.0 \cdot 10^5$ km.

4P5. One of the Bubble explorers has mass $m_1 = 72$ kg and the other has mass $m_2 = 53$ kg. Each is initially at rest, a distance $d = 8.2$ m apart, and they hold a cord of negligible mass between them. If they then pull themselves together until they meet, how far would the 53-kg explorer have moved?

4P6. A pig of mass $m = 12$ kg lives in the left side of a rectangular box of mass $M = 120$ kg and length $\ell = 22$ m, sitting in gravity-free empty space, as shown. Through a porthole, the pig spots a meteor approaching, and calculates it will strike the box a distance $d = 3.0$ m from the right-hand end. Is there anything the pig can do to prevent the collision, if it can only move along the length of the box?

4P7. Two friends of mass m_A and $m_B = 2m_A$ hold onto opposite ends of a uniform pole of mass $M = 3m_A$ and length L, which sits horizontally at rest on a frictionless sheet of ice. Each pulls herself at speed u (relative to the pole) toward the opposite end; they cross each other at the middle.

(a) By the time they have exchanged places, how far has the center of the pole moved?
(b) Now suppose the stick and friends are all initially sliding to the right at speed v_0 with respect to the ice. By the time they have exchanged places, how far has the center of the pole moved?

4P8. A gas of N molecules, each of mass m, is compressed at the far left end of a box of mass M and length L by a membrane, as shown. The box is in gravity-free empty space, initially at rest in some inertial frame. If the gas breaks through the membrane and distributes itself throughout the box, when the system comes to equilibrium, (a) how far has the box moved? (b) what is its velocity?

4P9. Find the CM locations of the following thin sticks, each of mass M and length L, whose left ends are at $x = 0$.

(a) A stick with uniform mass density $\lambda_1 = 2M/3L$ along the left half, and with uniform mass density $\lambda_2 = 4M/3L$ along the right half.

(b) A stick whose mass-density λ increases quadratically with distance from the origin: i.e., $\lambda = k(x/L)^2$, where k is a constant.

Section 4.2 The center of mass in three dimensions

4P10. Four balls of masses $m_1 = 1$ kg, $m_2 = 2$ kg, $m_3 = 3$ kg, and $m_4 = 4$ kg are located at the corners of a square of side $d = 2$ meters, as shown. If m_1 is chosen as the origin of Cartesian coordinates, find the CM of the system.

4P11. Three stars have equal mass. Two of them are a distance D apart, and the third is a distance d ($d > D/2$) from each of the first two. How far from the third star is the CM of the system?

4P12. Find the distance of the CM of a water molecule from the oxygen atom, in terms of the distance d between the O atom and each H atom. The O has 16 times the mass of each H, and the angle between the two OH bonds is $\theta = 108°$, as shown.

4P13. The ammonia (NH_3) molecule has the shape of a pyramid, with the nuclei of the three hydrogen atoms forming a triangular base of side $d = 1.63$ Å, and the nucleus of the nitrogen atom at the apex, a distance $\ell = 1.01$ Å from each H atom. [1 Ångstrom (Å) = 10^{-10} m] (a) Find the distance (in Å) of the CM of the three H atoms, measured from each of them.

(b) Find how far the CM of the entire molecule is above the triangular base. The nitrogen/hydrogen mass ratio is $M_N/M_H = 13.9$.

4P14. Find the CM position of a uniform-density semicircular hoop of mass M and radius R.

4P15. A semicircle of radius R and mass M is cut from sheet aluminum. Find the location of its center of mass.

4P16. A coathanger wire is bent to form an equilateral triangle of side ℓ, as shown in diagram A below. Find its CM, measured from each vertex, in terms of ℓ.

4P17. A solid equilateral triangle of side L is cut from a piece of cardboard, as shown in diagram B. Find its CM, measured from each vertex.

4P18. A uniform-density right circular solid cone has half-angle α with respect to its symmetry axis, and height h as shown. How far above its apex is its CM?

Section 4.3 "Particles" and the center of mass frame

4P19. Star α, of mass m, is headed directly toward Star β, of mass $3m$, with velocity v_0 as measured in β's reference frame. (a) What is the velocity of their mutual CM, measured in β's frame? (b) How fast is each star moving in the CM frame? (c) If the two stars merge upon collision, how fast is the new star moving in the CM frame?

4P20. The cue ball, moving with velocity $v = 1.20$ m/s in the positive x direction relative to the pool table, makes a head-on collision with the "six" ball, initially at rest. Both balls have the same mass. (a) What is the initial velocity of each ball in their mutual center-of-mass frame? (b) If, after the collision, the cue ball is moving with speed $v = 0.58$ m/s in the CM frame, how fast is the "six" ball moving relative to the pool table?

4P21. Spaceship A moves with velocity $v_A = 1.0 \cdot 10^4$ km/s in the x-direction, while Spaceship B moves with velocity $v_B = 2.0 \cdot 10^4$ km/s in the y-direction. Spaceship A has twice the mass of Spaceship B. Find (a) the velocity (b) the speed of each ship in their mutual center-of-mass frame.

Section 4.4 Sudden impulses

4P22. The first-baseman catches a $v = 75$ m/s line drive. The baseball's mass is $m = 145$ g. (a) What impulse does the first-baseman exert upon the ball? (b) What impulse does the ball exert upon the first-baseman?

4P23. Floating at rest in the Bubble, a pitcher of mass $m_p = 75$ kg throws a ball of mass $m_b = 5.0$ kg in the forward direction, imparting to it the impulse I shown, ending with a final impulse $I_f = 15$ kg m/s. (a) What is the pitcher's speed v_p after the ball has been thrown? (b) A catcher of mass $m_c = 55$ kg, originally floating at rest, catches the ball. What impulse is imparted to the catcher by the ball? (c) What is the velocity of the CM of the system *pitcher plus ball* before the ball is caught? (d) What is the velocity of the CM of the same system, after the ball is caught? (e) What is the total impulse imparted to this same system by the time the ball is caught? (f) What is the velocity of the CM of the larger system *pitcher plus ball plus catcher* after the ball is caught? (g) What is the total impulse imparted to this larger system by the end of the game?

4P24. A soccer player of mass $m_1 = 55$ kg holds a soccer ball of mass $m_b = 1.5$ kg; both are at rest on a frictionless sheet of ice. The player kicks the ball horizontally, exerting an impulse $I_1 = 20$ kg m/s. A second player, of mass $m_2 = 45$ kg, also initially at rest relative to the ice, intercepts the ball and kicks it horizontally back toward the first player, giving it a speed $v' = 10$ m/s relative to the ice. (a) What is the ball's speed v_0 after the first kick? (b)What is the first player's speed v_1 after kicking the ball? (c) What is the impulse I_2 exerted *on* the second player by the ball? (d) What is the speed v_2 of the second player after kicking the ball? (e) What is the total impulse I_T exerted on the ball, including both kicks?

4P25. A two-stage rocket is coasting in space with velocity $v_0 = 6.30$ km/s. The first stage, which has already used up its fuel, has mass $m_1 = 21.0 \cdot 10^3$ kg; the fueled second stage has mass $m_2 = 42.0 \cdot 10^3$ kg kg. Explosive bolts blow the two stages apart, so that the first stage now has velocity $v_1 = 6.18$ km/s. (a) What impulse I_1 does the first stage exert on the second? (b) What impulse I_2 does the second stage exert on the first? (c) What is the velocity v_2 of the second stage after the explosion? (d) What is the velocity v_{CM} of the center of mass of the system after the explosion?

4P26. A ball of mass m is thrown in such a way that the impulse delivered to it as a function of time t is $I = (I_f t^2)/(t^2 + \tau^2)$, where I_f and τ are constants.
(a) Sketch $I(t)$ for $t \geq 0$.
(b) Find the velocity of the ball as a function of time, if it starts from rest.
(c) Find the acceleration a of the ball as a function of time, and sketch $a(t)$.

4P27. A set of pool balls is confined to the two-dimensional surface of a pool table. Each ball can collide with other balls, with a pool cue, and with the table boundary.

(a) If $\mathbf{P} = \Sigma\, \mathbf{p}_i$ is the total momentum of the balls, where $\mathbf{p}_i$ is the momentum of the ith ball and $\mathbf{P} = 0$ initially, show that at any later time $\mathbf{P} = \mathbf{I}_{cue} + \mathbf{I}_{table}$, where $\mathbf{I}_{cue}$ is the sum of the impulses given to the balls by the pool cue, and $\mathbf{I}_{table}$ is the sum of the impulses given to the balls by the table boundaries as they bounce off, up to that time. (b) Show in general for a system of particles that $\mathbf{I}_{ext} = \mathbf{P}_f - \mathbf{P}_i$ where $\mathbf{P}_i$ and $\mathbf{P}_f$ are the initial and final total momenta of the system, and $\mathbf{I}_{ext}$ is the impulse exerted on the system by external influences only. That is, for the system as a whole, show that impulses between the particles themselves don't count in the impulse equation.

4.5 Impulsive changes: the *force*

4P28. A rifle bullet of mass $m = 9.0$ g is fired at velocity $v = 5.0 \cdot 10^2$ m/s into a large stationary block of wood. If the bullet is stopped in the time interval $\Delta t = 2.5$ ms $= 2.5 \cdot 10^{-3}$ s, what is the average force exerted on (a) the bullet by the block, and (b) the block by the bullet, during the time the bullet is slowing down?

4P29. A tennis ball moves toward a racquet (negative velocity) and is then struck by the racquet so that it moves away again (positive velocity). The velocity of the ball as a function of time is shown in the graph. The mass of the ball is $m = 75$ g. Estimate the maximum force (in Newtons) exerted on the ball during this period of time.

4P30. A baseball of mass $m = 150$ g is caught by the first baseman. If the magnitude of the force on the ball by the player's glove as the ball is brought to rest is as shown, estimate the initial speed of the ball.

4P31. An impulse I acting in the x-direction is exerted on a ball of mass m. The force dI/dt on the ball is shown in the graph. (a) Find the impulse $I(t)$ exerted on the ball as a function of time, in terms of t_0, t_1, and I_f, if $I = 0$ at $t = 0$ and $I = I_f$ at $t = t_1$. (b) Draw a graph of $I(t)$. (c) At $t = t_0$, the ball has velocity v_0. Find its velocity at $t = t_1$, in terms of defined parameters.

4P32. A baseball of mass m is struck by a bat. Approximate the force given to the ball by the bat by the Gaussian function $F = dI/dt = A\,e^{-b(t-t_0)^2}$ shown in the figure. (This cannot be the exact expression, because the Gaussian function only becomes zero as $t \to \infty$. However, the function is very small in the "wings" if b is large.) (a) Find the constant A in terms of the maximum force F_{max} of the bat on the ball. (b) The Gaussian $A\,e^{-b(t-t_0)^2}$ is shown. Sketch the Gaussian if b were doubled. (c) Find a condition relating the constants A and b if the batter hits head-on the ball pitched with speed v_0, and gives it the speed v_f back towards the pitcher. (Hint: the integral $\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}$.) (d) Find the constant b in terms of F_{max}, m, v_0, and v_f.

4P33. (a) The force acting on an automobile at rest at time $t = 0$ is $F = At^2$, where A is a constant. Find the momentum of the car at any later time. (b) If instead $F = A\,e^{-\alpha t}$, where A and α are constants, what then is the car's momentum at any later time?

4P34.(a) During some time interval we want the momentum of a proton to be $p = p_0 \cos\omega t$ where p_0 and ω are constants. What force $F(t)$ should we apply to the proton? (b) If instead we want $p = p_0 e^{-\omega t}$, what then is the force acting on the proton?

Special Topic I: Average Force Due to Many Small Collisions

4P35. Hailstones of diameter $D = 1.50$ cm are falling vertically with speed $v = 21$ m/s. There are 55 of them per cubic meter of air. (a) What is the pressure p (average force/area) in Newtons/m^2 on the flat roof of a building due to the impact of hailstones, assuming they don't bounce? (The density of ice is $\rho = 0.92$ g/cm^3. Ignore the weight of hailstones piling up on the roof.) (b) What is the pressure p if they bounce back elastically with the same speed v ? (Ignore the effect of repeat bounces.) (c) If in part (b) each hailstone is in contact with the roof for $1.0 \cdot 10^{-3}$ s, what is the average force exerted on each hailstone by the roof during the time it is in contact?

4P36. A terrorist fires a 30-round burst from an Uzi automatic weapon. The gun fires bullets of mass $m = 7.5$ g each with muzzle velocity $v = 380$ m/s, at the rate of 1800 rounds/minute. What average force F must the terrorist exert on the gun to keep it in place while firing? *Note:* One "round" is one bullet.

4P37. Sandblasters shoot high-velocity airstreams containing sand grains to clean surfaces. A particular manufacturer advertises a blaster whose nozzle shoots out air at 7 cfm@80 psi (*i.e.*, 7.0 cubic feet/minute@80 pounds/square inch). The tank holds 1.5 cubic feet of sand, and the sand lasts for 1.5 hours of blasting. (a) What is the velocity of the airstream (in cm/s) if the emitting area of the hose nozzle is 1.0 cm^2? (b) What is the volume/second of sand (in cm^3/s) shot out? (c) The density of the sand is 1.5 g/cm^3. If the sand strikes a wall head-on without bouncing back, what force does it exert on the wall, in Newtons? (d) Is the force exerted by the sand likely to be greater than or less than the force exerted by the airstream itself? Explain.

4P38. The solar wind is a stream of protons and electrons emitted from the Sun's corona. When the stream reaches Earth the particles can influence radio communication, cause auroras, and affect the weather. The velocity and density of the stream varies. Let the average stream velocity be v and average number density of protons be n. Each proton has mass m; the mass of the electrons can be neglected. (a) Find an expression for the force on a spacecraft due to the solar wind, if the craft has cross-sectional area A normal to the stream direction. Neglect any recoil of protons when they strike the spacecraft. (b) Find the force numerically (in Newtons) if $A = 2.5$ m^2, using the observed values $v = 450$ km/s, $n = 8.0$ protons/cm^3, $m = 1.67 \cdot 10^{-27}$ kg. (c) About how far will the craft be deflected by the solar wind during one week?

4P39. Consider the following crude model of a gas inside a cubical box of side ℓ. The gas consists of N atoms, each with the same mass m and speed v. At any instant, 1/6*th* of the atoms are moving in each of six directions, exactly toward $+x$, $-x$, $+y$, $-y$, $+z$, $-z$. When an atom collides with a wall, it bounces directly back with the same speed v it had before. (a) Pick one of the six walls of the box. In terms of N, v, and ℓ, show that the number of collisions per second made by atoms with that wall is $Nv/6\ell$. (b) Find an expression for the pressure p (Force/Area) on each wall due to the gas described, in terms of ℓ, m, v, and N. (c) The temperature T of the gas obeys $\frac{1}{2}mv^2 = \frac{3}{2}kT$, where k is *Boltzmann's constant.* Find the pressure p in terms of N, k, T, and the box volume V.

4P40. A police water cannon directs a horizontal stream of water at speed $v = 22$ m/s from a gun of nozzle diameter $D = 5.4$ cm, at a demonstrator standing at rest.
(a) What horizontal force $< F >$ (in Newtons) is exerted on the demonstrator, assuming the water flows downward afterwards?
(b) What is the initial acceleration a of the demonstrator, whose mass $m = 65$ kg, assuming other forces on the demonstrator are negligible in this case?

4P41. The standard firehose in the U.S. delivers 250 gallons/minute of water through a hose of diameter 2 1/2 inches, out a nozzle of diameter 1 1/8 inches. (a) What is the velocity of the water as it emerges from the nozzle, in meters/second? (b) If a fire-fighter directs the water horizontally at a wall, and the water flows directly downward after striking the wall, what force does the water exert on the wall, in Newtons? (Note: One gallon of water has a volume 3.79 liters = 3,790 cm^3, and 1 inch = 2.54 cm.)

4P42. A proposed interstellar ram-jet operates on the following principle: It sweeps up deuterons (heavy-hydrogen nuclei) in space, burns them in a fusion reactor, and shoots the reaction products out the tail of the ship. In a reference frame instantaneously at rest relative to the ship, many deuterons, each of mass m, approach the ship at velocity v. They are burned, and the burn products, with essentially the same total mass, are ejected from the rear of the ship at velocity $v + u$. The ship mass M stays constant, the cross-sectional area of the ship is A, and the number of deuterons/unit volume of space is n.

(a) Find dN/dt, the number of deuterons swept up per unit time, in terms of n, A, and v. (b) Find dP/dt, the change in total momentum of the ship per unit time. (*Hint:* conserve overall momentum of deuterons + ship.) (c) Show that the velocity of the ship increases exponentially in time in the frame in which the deuterons are at rest, i.e., that $v = v_0 e^{\alpha t}$, where v_0 is the ship's velocity at time $t = 0$, and α is a constant. Find α in terms of given parameters. Assume that u is constant.

4P43. A spaceship with cross-sectional area A is traveling through interstellar space at speed v. Space is filled with n hydrogen atoms per cubic meter, each with mass m. (a) Find the drag force F_D acting on the ship in terms of given parameters, assuming that on the average atoms struck by the ship gain the ship's forward velocity. (b) Find F_D in Newtons, given that $A = 1000\ \text{m}^2$, $v = 3 \cdot 10^6$ m/s, $n = 10\ /\text{m}^3$, and $m = 1.7 \cdot 10^{-27}$ kg. (c) Suppose the ship has mass $M = 5 \cdot 10^6$ kg. Estimate the time T (in years) it would take the ship, assuming the data in part (b), to lose 1 m/s of velocity due to the drag force.

Special Topic II: Project Orion

4P44. The boost given an Orion spaceship by a single explosion is $\Delta v = I / M$, where M is the mass of the ship and $I = mU/4$ is the impulse given the ship by the explosion. Here m is the mass of a single bomb, and U is the speed of the debris. To find the final velocity of the ship v_f, we have to sum all the Δv's. The change in the ship's mass is $\Delta M = -m$ from each bomb released, so $\Delta v = -\dfrac{\Delta M}{M}\dfrac{U}{4}$ from a single explosion, and $v_f = \Sigma \Delta v = \Sigma\,[\,(-\Delta M / M)(U/4)]$ from all of them together. The ship mass M keeps changing as bombs are ejected. Since $m = -\Delta M$ is very small compared with M, we can approximate the mass loss as continuous, so

$$v_f = \int_0^{v_f} dv = -\frac{U}{4}\int_{M_0}^{M_f} \frac{dM}{M}$$

where M_0 and M_f are the masses of the original fully loaded ship and of the final empty ship, respectively. (a) Find v_f for a ship as described in the *Special Topic,* with debris velocity $U = 3000$ km/s. (b) Assuming one bomb is exploded every three seconds, for how many years does the ship accelerate? (c) About how long would it take for the ship to reach α-Centauri, a star 4.3 light-years away? (A light-year is the distance light moves in one year.) Find roughly how far the ship goes during its acceleration phase, and then find the time for the entire trip. (*Hint:* An approximate answer can be found by noting that the bombs are all used up quite early in the trip, so the ship coasts at speed v_f for nearly all of its voyage.)

Day Five

Force

"Don't fight forces; use them" – Buckminster Fuller

Yesterday we found that *force* causes the momentum of an object to change with time, according to $\mathbf{F} = \frac{d\mathbf{p}}{dt}$. What good is this equation? What can it really tell us?

We originally *defined* force as the time rate of change of the impulse $\mathbf{I}$, and then used the equation $\mathbf{I} = \Delta\mathbf{p} = \mathbf{p} - \mathbf{p}_0$ to show that

$$\mathbf{F} = \frac{d\mathbf{I}}{dt} = \frac{d\mathbf{p}}{dt}.$$

Therefore the equation is little more than a definition of force. If the only way we can measure the force on a horse (say) is to find the time rate of change of the horse's momentum, the equation is useless. It is then just a definition, which gets us nowhere. We could sit around defining all kinds of things, and afterwards understand nothing more than we did before.

Physics can't just be definitions! Physics is supposed to be about "understanding" nature, being able to predict what will happen next in various circumstances. So we need some independent definition of force, which we can use to predict the time rate of change of the momentum of an object in a particular situation from $\mathbf{F} = \frac{d\mathbf{p}}{dt}$. Then we can carry out the experiment and see if our prediction matches reality. If it does, our confidence in the equation will grow, and we can use it to predict $\frac{d\mathbf{p}}{dt}$ in other experiments as well. If it continues to work in these other experiments, our confidence will grow still greater.

5.1 Springtime in the Bubble

We find some springs we might use to define force. Many of them appear identical. We grab opposite ends of one of them, and pull gently. The spring stretches. If we pull harder, it stretches more. We feel it pull back on us as well, more and more as the spring is stretched. These spring-pulls seem likely candidates for our definition of force.

We check that the springs have equal strength by attaching them in pairs to opposite sides of one of our metal balls, using hooks as shown in Fig. 1. We pull the far ends so

Fig. 1. Two springs with a ball in equilibrium

the ball doesn't move; it then makes sense to say that the *total* force on the ball is *zero*. The forces due to the two springs have cancelled one another, so the ball remains in equilibrium. In this case the two forces have the same magnitude, but are in opposite directions.

From the experiment we find that the two springs stretch by an equal amount – they are equally stiff. If we attach *two* springs on the left and only *one* on the right, and again pull the springs very slowly apart so that the ball doesn't move, the spring on the right stretches twice as much as either of those on the left, as shown in Fig. 2. The force exerted by each of our springs therefore seems to be directly proportional to its stretch.*

We gather further evidence for this rule by attaching three springs on the left; then the

Fig. 2. Three springs with a ball in equilibrium

* Springs whose restoring force is proportional to their stretch are called *Hooke's-law* springs, after Robert Hooke, a contemporary of Newton. Springs can be specially made to produce linear restoring forces over an appreciable range of stretch, but of course they all depart from linearity if the stretch is too large. Hooke's law springs, assumed to have exact linearity, are sometimes called *ideal springs*.

single spring on the right stretches three times as much as each of those on the left. Additional experiments confirm the relationship. We now have a measurable source of force: Each of our springs exerts a force whose magnitude is

$$F_{\text{spring}} = k\Delta x \tag{1}$$

where k is some constant and Δx is the stretch. We cannot find the actual numerical value of the constant k by experiment until we have decided upon a *scale* for F.

The spring stretch Δx is the stretch away from the natural length of the spring. That is, suppose we set the left-hand end of a spring at $x = 0$ and keep it there, and then find that the right-hand end of the spring is located at $x = x_0$ if there is no force on it. The "natural" length of the spring is therefore x_0. Now if we stretch the right-hand end of the spring to some new position x, the spring stretch is $\Delta x = x - x_0$.

5.2 Is it true that $\mathbf{F}_{\text{spring}} = \dfrac{d\mathbf{p}}{dt}$?

Next we stretch a string from one side of the Bubble to the other and make marks on it every twenty-five centimeters. We then attach a single spring to one of the "1 kg" balls. Pulling on the other end of the spring, we start the ball moving alongside the stretched string. The ball starts up slowly, so it is not hard to keep the spring stretched by $\Delta x = 5$ cm for half a minute or so.

One of us pulls while the other notes the time the ball passes each mark on the string. We repeat the experiment several times, and average the results. We plot this averaged data on a graph of x vs. t, as shown in Fig. 3a . The slope of the graph, which is the velocity $v = dx/dt$, is shown in Fig. 3b. The slope of v *vs* t is constant within the scatter in our data; therefore the acceleration $a = dv/dt$ seems to be constant. Measuring from the slope of Fig. 3b, the acceleration is $a = 0.25$ cm/s^2.

Fig. 3. The position (a) and velocity (b), of a ball pulled by a spring with constant stretch

We assume that the spring with constant stretch still exerts a constant force on the ball as it did when they were both at rest. Then we have found that a constant force produces a constant acceleration. If we keep the spring stretched by a constant 10 cm instead, we discover in the same way that the acceleration is *twice* what it was before. After a number of additional experiments with different spring stretches, we satisfy ourselves that $a \propto F$. The acceleration is directly proportional to the applied force.

When we attach *two* 1-kg balls together and accelerate them both with a spring stretched by 5 cm, the acceleration is only half what it was in the same experiment with only a single ball. The same pattern is evident when we accelerate any one of the other balls whose mass we had measured a couple of days back: For the same force (that is, for the same spring-stretch), the acceleration is *inversely* proportional to the mass being pulled. From the two types of experiments, varying F and varying m, we conclude that the acceleration is directly proportional to F but inversely proportional to m: That is, $a \propto F / m$, or what is the same thing,

$$a = K \frac{F}{m} , \tag{2}$$

where K is some constant of proportionality.

Now we need to find this constant K. The momentum of a ball of mass m moving in one dimension is $p = mv$, so it follows that

$$\frac{dp}{dt} = m\frac{dv}{dt} = ma\,. \tag{3}$$

It *is* true that $F_{spring} = \dfrac{dp}{dt}$ *if we choose* $K = 1$ *in Eq. 2!* Then in effect we have chosen a scale for the force, so that

$$\boxed{F_{\text{spring}} = \frac{dp}{dt} = ma\,.} \tag{4}$$

If a certain constant force causes a mass $m = 1$ kg to have the acceleration $a = 1$ m/s^2, then that force has the magnitude $F = 1$ kg m/s^2 = 1 N (i.e., 1 *Newton*).

Furthermore, we have found that our new, independent definition of force in terms of spring-stretch is consistent with yesterday's definition in terms of momentum change. That is, defining force by spring-stretch turns out to be equivalent to defining force by $\mathbf{F} = \dfrac{d\mathbf{p}}{dt}$, at least for constant-force, straight-line motion.

We can't be sure the same equation will work for *varying* spring forces, for example, unless we test it experimentally. It seems reasonable that the spring force is still proportional to spring stretch, whether the stretch is constant or not. If at some instant the spring is stretched by a certain amount, it should exert a force at that instant corresponding to that particular stretch. As the stretch changes, so should the force. It is then natural to suppose that the acceleration is $a = F/m$ even for varying F, although the idea needs to be tested experimentally. Then knowing "a" we can figure out everything about the ball's motion, including its velocity and position.

How do we find the ball's motion if we know its acceleration? The acceleration and velocity of the ball are related by $a = dv/dt$, so if a is known, we can find the velocity by integration:

$$\int_0^t a\,dt = \int_{v_0}^{v} dv = v - v_0\,,$$

or

$$v = v_0 + \int_0^t a\,dt \tag{5}$$

where v is the velocity at time t, and v_0 is the initial velocity, at $t = 0$. Then knowing v, we can calculate the position of the ball using $v = dx/dt$. That is,

$$\int_0^t v\,dt = \int_{x_0}^{x} dx = x - x_0\,,$$

so

$$x = x_0 + \int_0^t v\,dt \tag{6}$$

where x is the position at time t, and x_0 is the position at $t = 0$.

Sample Problem 1. A ball of mass m experiences the time-dependent force $F = 0$ (for time $t < 0$) $F = \alpha\sqrt{t}$ $(t \geq 0)$. Find the ball's (a) acceleration (b) velocity (c) position as functions of time, if x_0 and v_0 are the ball's position and velocity at $t = 0$.

Solution: After time $t = 0$, (a) the acceleration is $a = F/m = (\alpha/\mathrm{m})\sqrt{t}$ $(t \geq 0)$ using Newton's second law. (b) The velocity of the ball (using Eq. 5) is

$$v = v_0 + \int_0^t a\,dt = v_0 + (\alpha/\mathrm{m})\int_0^t \sqrt{t}\,dt = v_0 + \frac{2\alpha}{3m}\,t^{3/2}.$$

(c) The position of the ball (using Eq. 6) is

$$x = x_0 + \int_0^t v\,dt = x_0 + \int_0^t \left(v_0 + \frac{2\alpha}{3m}\,t^{3/2}\right) dt = x_0 + v_0 t + \frac{4\alpha}{15m}\,t^{5/2}.$$

We can check these results by differentiating x to get v, and then differentiating v to get a.

5.3 Uniformly accelerated motion

Suppose the force is *constant* , caused by a spring whose stretch is kept constant. Then the acceleration $a = F/m$ is also constant, with $a = a_0$ always. The acceleration can then be brought outside the integral in Eq. 5, so

$$v = v_0 + a_0 \int_0^t dt = v_0 + a_0 t . \tag{7}$$

The velocity grows linearly with time if $a_0 > 0$, as shown in Fig. 4. The slope of $v(t)$

Fig. 4. Velocity as a function of time for constant positive acceleration

in the figure is the acceleration a, and the intercept on the v-axis is the initial velocity v_0. The only difference between this result and our experimentally derived graph of Fig. 3(b) is that our experiments applied to the special case $v_0 = 0$. Knowing v from Eq. 7, we can then calculate x from Eq. 6:

$$x = x_0 + \int_0^t v\, dt = x_0 + \int_0^t (v_0 + a_0 t)\, dt$$

$$= x_0 + v_0 t + \frac{1}{2} a_0 t^2 . \tag{8}$$

A typical graph is shown in Fig. 5.

Fig. 5. Position as a function of time for constant positive acceleration

The slope of this graph is $v(t)$, starting with v_0 at $t = 0$, and the intercept on the t-axis is the initial position x_0. Again the difference between this graph and our experimental graph of position in Fig. 3a is that our experiments were carried out for the particular initial conditions $x_0 = 0$ and $v_0 = 0$.

For the <u>special case of constant acceleration</u>, we have derived the equations

$$\boxed{\begin{aligned} v &= v_0 + a_0 t \qquad (9a) \\ x &= x_0 + v_0 t + \frac{1}{2} a_0 t^2 \qquad (9b) \end{aligned}}$$

for the particle velocity and position, given a_0 and the initial values v_0 and x_0. We can use these to derive two other equations

$$\boxed{\begin{aligned} x - x_0 &= \frac{(v + v_0)}{2} t \qquad (9c) \\ v^2 &= v_0^2 + 2a_0(x - x_0). \qquad (9d) \end{aligned}}$$

Equation 9c gives the displacement $x - x_0$ as the product of an "average" velocity $(v + v_0)/2$ and time; we get it by eliminating the acceleration a between (9a) and (9b). Equation 9d gives us a way to find the velocity of a particle for known acceleration and displacement, without needing to know the time t. We get this equation by eliminating the time between (9a) and (9b).

All of these equations were derived *assuming the acceleration is constant, and they are valid only for that case.*

Sample Problem 2. A world record time for a Top-Fuel Dragster was acheived by Gary Scelzi in Houston, Texas, on October 31, 1999. The car covered 1/4 mile from a standing start in 4.48 seconds. In the following, pretend that the car's acceleration a was constant. (a) Find a (b) Find the car's velocity v when it crossed the finish line.

Solution: (a) Equation 9b, $x = x_0 + v_0 t + \frac{1}{2} a_0 t^2$, relates position, acceleration, and time. Solving for a_0 (with x_0 and v_0 both zero),

$$a_0 = \frac{2x}{t^2} = \frac{1/2 \text{ mile}}{(4.48 \text{ s})^2} \cdot \frac{5280 \text{ ft}}{1 \text{ mile}} = 132 \text{ ft/s}^2$$

(This is over four times the acceleration of gravity (32 ft/s^2) on Earth's surface.)

(b) Equation 9a, $v = v_0 + a_0 t$, relates velocity, acceleration, and time. With $v_0 = 0$,

$$v = a_0 t = 132 \text{ ft/s}^2 \cdot 4.48 \text{ s} = 591 \text{ ft/s} \ (= 403 \text{ mi/hr}),$$

about half the speed of sound! [Alternatively, we can solve Eq. 9c for v, giving $v = 2x/t$, or solve Eq. 9d for v, giving $v = \sqrt{2a_0 x}$. Substitution of numbers into each equation gives the same value for v.]

"Top-Fuel" dragsters use four-stroke engines with large displacement (large cylinder volume) and superchargers to force more air into the cylinders. Most particularly the "top fuel" engines are "nitro-burning" engines, burning about 85% nitromethane (CH_3NO_2) and 15% methanol (CH_3OH). Nitromethane comes with its own oxygen atoms to increase burning, helping to give nitromethane about 2 1/2 times the power of gasoline (C_8H_{18}). A typical drag-racing engine produces about 6000 horsepower and uses about a gallon of fuel per second!

Sample Problem 3. Equations 9 (relating the position, velocity, and acceleration of a particle to time) are only correct if the force on the particle (and therefore its acceleration) is constant. Suppose instead that F increases linearly with time according to $F = \alpha t$, where α is a constant. Find the particle's (a) velocity $v(t)$ (b) position $x(t)$. That is, find the analogs of Eqs. 9a and 9b for the force $F = \alpha t$.

Solution: (a) The acceleration of a particle of mass m is $a = \dfrac{dv}{dt} = \dfrac{F}{m} = \dfrac{\alpha t}{m}$, so

$$\int_{v_0}^{v} dv = (\alpha/m) \int_0^t t\, dt \quad \text{or} \quad v - v_0 = \frac{\alpha}{2m} t^2 .$$

The analog of Eq. 9a is therefore

$$v = v_0 + \frac{\alpha}{2m} t^2 .$$

(We could insert the acceleration $a = F/m = \alpha t/m$ into the equation by eliminating α, but this would not be helpful, because a is not a constant.)

(b) We can integrate again, using $v = \dfrac{dx}{dt} = v_0 + \dfrac{\alpha}{2m} t^2$. Then

$$\int_{x_0}^{x} dx = \int_0^t [v_0 + \frac{\alpha}{2m} t^2]\, dt \text{ , giving } x - x_0 = v_0 t + \frac{\alpha}{2m}\left(\frac{t^3}{3}\right).$$

The analog of Eq. 9b is therefore

$$x = x_0 + v_0 t + \frac{\alpha}{6m} t^3 .$$

The diagrams show the acceleration a, velocity v, and position x of a particle that begins with velocity v_0 and position x_0 at $t = 0$, and is subjected to the linearly increasing force $F = \alpha t$. Note that the velocity curve plots the slope of the position curve, and the acceleration curve plots the slope of the velocity curve.

5.4 Circular motion

We have so far tested $F = ma$ only for *one-dimensional* motion. Force and acceleration are both vectors, however, with directions as well as magnitudes. Therefore we expect that more generally the equation should be $\mathbf{F} = m\mathbf{a}$, with a vector force and a vector acceleration, which would apply to motion that is two or three dimensional as well as one dimensional. According to the full vector law $\mathbf{F} = m\mathbf{a}$, if we pull on a ball in some arbitrary direction the ball will accelerate in that same direction.

The full three-dimensional nature of the equation $\mathbf{F} = m\mathbf{a}$ would be needed to predict the motion of a ball attached to one end of a spring that is whirled around in a circle. The other end of the spring is held in place, while the ball circles around the fixed point. The circular orbit would have some radius R and period of rotation T (the time to complete one orbit), both of which could be measured.

In each orbit the ball travels a distance $2\pi R$, say at constant speed $v = 2\pi R/T$. The spring is stretched along the radius of the circle, so it exerts an inward, radial force $k\Delta x$, where Δx is the stretch in the spring. Nothing else touches the ball, so this is the only force on it. From $\mathbf{F} = m\mathbf{a}$ the acceleration of the ball is therefore also directed toward the center of the circle, with magnitude $a = k\Delta x/m$. The velocity is tangent to the circle, perpendicular to the radial direction.

How is it possible that the ball accelerates but has constant speed? Isn't that a contradiction? It is certainly impossible for straight-line motion. How can it happen for circular motion?

It can happen because velocity is a *vector*. A vector has both magnitude and direction, so there are two ways the velocity can change with time, when we calculate $\mathbf{a} = d\mathbf{v}/dt$. The *magnitude* of the velocity can change, which is what happens when a ball moves in a straight line while changing speed. Or the *direction* of the velocity can change, which is what happens when the ball moves in a circle.

What is the acceleration of a ball moving at constant speed v in a circle of radius r? By the definition of derivatives as a limiting process, the acceleration

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ell im_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \ell im_{\Delta t \to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t}, \tag{10}$$

where the velocity vectors at the two nearby times t and $t + \Delta t$ are shown in Fig. 6a. Both have the same length, but point in slightly different directions.

Fig. 6. Velocity vectors for a ball traveling in a circle at constant speed.
(a) Velocity vectors at t and $t + \Delta t$ (b) Subtraction of the vectors.

To subtract these vectors, we bring their tails together, as in Fig. 6(b). The difference vector is $\Delta\mathbf{v} = \mathbf{v}(t + \Delta t) - \mathbf{v}(t)$, extending from the tip of $\mathbf{v}(t)$ to the tip of $\mathbf{v}(t + \Delta t)$. Note that since the vector $\mathbf{v}(t)$ is tangent to the circle, it is perpendicular to the radius vector $\mathbf{r}(t)$. In the limit $\Delta t \to 0$, then also $\Delta\theta \to 0$, and the vector $\Delta\mathbf{v}$ likewise becomes perpendicular to $\mathbf{v}(t)$. Therefore in the limit, $\Delta\mathbf{v}$ is (anti)parallel to $\mathbf{r}(t)$, which means that it points *radially inward.*

For the narrow triangle the magnitude of $\Delta\mathbf{v}$ is $\Delta v = v \sin(\Delta\theta) \to v\,\Delta\theta$ in the limit of small angle $\Delta\theta$, where also $\Delta\theta$ is the angle the ball moves around the circle during time Δt. But this angle is *by definition* $\Delta\theta = \Delta s/r$, where Δs is the small arc-length around the circle.

That is, angles in radians are defined to be the arc-length subtended on a circle, divided by the radius of the circle.

For *any* angle θ in radians, $\theta = s/r$, where s is the arc-length subtended on the circle, as shown in Fig.7. That's why there are 2π radians in an entire circle, for example.

Fig.7. The definition of angle in radians: $\theta \equiv s / r$

At constant speed v, the ball moves around the circle for a distance $\Delta s = v\Delta t$, so

$$\Delta v = v\,\Delta\theta = v\,\frac{\Delta s}{r} = v^2\frac{\Delta t}{r}. \tag{11}$$

The magnitude of the inward (or *centripetal*) acceleration is therefore

$$\boxed{\mathbf{a}_{\text{centripetal}} = \left| \ell im_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} \right| = \frac{v^2}{r}.} \tag{12}$$

An object moving at constant speed v around a circle of radius r accelerates toward the center of the circle with $a = \dfrac{v^2}{r}$. It *is* possible for an object to accelerate even if its speed stays the same!

Now finally we can see whether or not $\mathbf{F} = m\mathbf{a}$ works in this experiment by comparing the inward force $F = k\Delta x$ with mass times acceleration, $ma = mv^2/r$.

> *We note that mv^2/r is mass times acceleration; it is* not *a force.*
> *The only force acting on the ball is the spring force, $k\Delta x$.*

We have measured k, Δx, m, r, and the period T. Using our data and the formula $v = 2\pi r/T$, we *do* find (within experimental error) that

$$k\Delta x = ma = \frac{mv^2}{r}. \tag{13}$$

Now we have evidence, at least from a single experiment, that the vector law $\mathbf{F} = m\mathbf{a}$ works beyond the one-dimensional experiments we used to discover it.

Sample Problem 4. A biology lab has a centrifuge that rotates a sample in a circle of radius 6.2 cm at 60,000 revolutions per minute (rpm). What is the acceleration of the sample?

Solution: In each revolution, the sample moves through a circumference $C = 2\pi r$ in a time $T = 1$ minute/60,000 = 60 seconds/60,000 = 10^{-3} s. The velocity of the sample is $v = 2\pi r/T$, so the centripetal acceleration of the sample is

$$a = v^2/r = \frac{(2\pi r/T)^2}{r} = 4\pi^2 \frac{r}{T^2} = 4\pi^2 \frac{6.2\text{ cm}}{\left(10^{-3}\text{ s}\right)^2} = 2.4 \cdot 10^6\text{ m/s}^2.$$

This is about a quarter of a million times the acceleration of an object falling near Earth's surface.

5.5 Angular speed

There is a different way to describe the circular motion of a particle. From the definition of angle in radians, as was illustrated in Fig. 7, the distance s the particle moves as it travels around a circle is $s = r\,\theta$. The radius r is constant, so the derivative of this equation with respect to time is $\frac{ds}{dt} = r\frac{d\theta}{dt}$. We identify the quantity $\frac{ds}{dt}$ as the speed v of the particle around the circle. By analogy, we will call

$$\omega \equiv \frac{d\theta}{dt} \tag{14}$$

the *angular speed* of the particle; ω is the rate at which the angle θ changes with time. The relationship between the speed v and angular speed ω of a particle moving around a circle is therefore

$$v = r\,\omega. \tag{15}$$

The angle θ is measured in radians, so ω (the Greek letter "omega") is measured in *radians/second*. For example, if the radius of the circle is $r = 2$ meters and the speed of the particle remains constant at $v = 2$ meters/second, then $\omega = 1$ radian/second, according to Eq.15. This means that it would take 2π seconds for the particle to go all the way around the 2π radians in the circle. If two particles rotate about the origin with the same angular speed ω, and one is twice as far from the origin as the other, the particle farther out has to move with twice the speed v.

Suppose it takes a time T for a particle to go once around a circle of radius r at constant speed v; we call T the *period* of the motion. Since the particle moves around the circle a distance $2\pi r$, it follows that $v\,T = (r\,\omega)T = 2\,\pi\,r$. Therefore the angular speed and the period are related by

$$\omega = \frac{2\pi}{T}. \tag{16}$$

The larger the angular speed, the shorter the period, and vice versa. We can also write the centripetal acceleration of the particle in terms of ω or T. From Eq. 12,

$$\boxed{a_{\text{centripetal}} = \frac{v^2}{r} = r\omega^2 = r\left(\frac{2\pi}{T}\right)^2.} \tag{17}$$

We have now extended the experimental law $\mathbf{F} = m\mathbf{a}$ to circular as well as linear motion. For uniform (i.e, constant-speed) circular motion, we have found that the inward acceleration can be written in any one of the three ways given in Eq. 17.

Sample Problem 5. A ball of mass m is attached to one end of a string in the Bubble, and whirled around in a circle of radius R so that it completes one revolution in time T.
(a) Find the speed v and angular speed ω of the ball.
(b) Find the force F exerted on the ball by the string.
(c) If the string suddenly snaps, describe the motion of the ball.

Solution: (a) The ball moves around the circumference $2\pi R$ in time T, so its speed is $v = 2\pi R/T$. The angular speed is $\omega = v/R = 2\pi/T$.
(b) The centripetal acceleration of the ball is $a = v^2/R$, so the force on it due to the string is $F = ma = \dfrac{mv^2}{R} = \dfrac{4\pi^2 mR}{T^2}$. For given R, F increases quickly if the period T is decreased. For given T, F increases linearly with R.

(c) If the string breaks, there will be no more force on the ball, so it will move away in a straight line at constant speed, as shown. The speed will be the same as the speed it had before the string broke.

So what have we discovered? We have discovered that $\mathbf{F} = m\mathbf{a}$ works, at least in a few special cases. We have not shown that it must always work; we just have to try it and see. The rule $\mathbf{F} = m\mathbf{a}$ is a *guiding principle*, in which we gain more and more confidence as we carry out more and more experiments that agree with it. It is not possible to prove that it *always* works. We are doing science here, not mathematics. In mathematics you can make whatever assumptions you like and then prove the consequences of that assumption. But the *real world* doesn't care about our assumptions.

In fact, it was found in the twentieth century that $\mathbf{F} = m\mathbf{a}$ does *not* always work. That is very exciting, because limitations on the law tell us something about the way the world is. Roughly speaking, it turns out that the law works very well if the mass m is of macroscopic size and if it moves slowly compared with the speed of light. The law can easily be modified to apply to objects of macroscopic size that move at speeds comparable to the speed of light, but it often falls apart completely (becomes meaningless, in fact) for small particles like individual atoms. Then we have to apply the rules of *quantum mechanics* instead.

State of the Search

1. A Hooke's-law spring exerts a force proportional to its stretch, i.e. $F = k\Delta x$, pulling or pushing so as to try to restore the spring to its equilibrium length.

2. Such a Hooke's-law spring can be used to measure force. When such an ideal spring is attached to an object of mass m, the force applied as the spring is stretched results in an acceleration $\mathbf{a}$ of the object according to the experimental law

$$\mathbf{F} = m\mathbf{a}\,.$$

The force $\mathbf{F}$ is also responsible for changing the momentum of the object, since

$$\mathbf{F} = m\mathbf{a} = \frac{d(m\mathbf{v})}{dt} = \frac{d\mathbf{p}}{dt}.$$

3. If we know the force F on an object as a function of time, then we know the acceleration

$$\mathbf{a}(t) = \frac{\mathbf{F}(t)}{m}$$

as well. We can then find the velocity $\mathbf{v}(t)$ of the object and its position $\mathbf{r}(t)$ by integration. In one dimension, for example, $a = \dfrac{dv}{dt}$ and $v = \dfrac{dx}{dt}$, so we can first find v by integration over a: That is,

$$\int_{v_0}^{v} dv = \int_0^t a\,dt, \quad \text{so} \quad v = v_0 + \int_0^t a\,dt\,.$$

Note that if (and only if!) the acceleration $a = a_0$ is *constant*, a can be brought outside the integral, so then $v = v_0 + a_0 t$. That is, for constant acceleration the velocity changes linearly with time as illustrated.

Having found $v(t)$ we can integrate over v to find $x(t)$:

$$\int_{x_0}^{x} dx = \int_0^t v\,dt \text{ , so } \quad x = x_0 + \int_0^t v\,dt \text{ .}$$

If (and *only* if!) the acceleration $a = a_0$ is *constant*, then $v = v_0 + a_0 t$, so the second integration give $x = x_0 + v_0 t + \frac{1}{2}at^2$.

For constant acceleration, the position changes quadratically with time as illustrated, if the initial velocity is zero.

If the acceleration is constant, two additional formulas are useful as well,

$$x - x_0 = \frac{(v + v_0)}{2}t$$

$$v^2 = v_0^2 + 2a_0(x - x_0),$$

the first involving the time t but not the acceleration, and the second involving the acceleration a_0 but not the time.

4. An object moving in a circle experiences an inward (centripetal) acceleration

$$\mathbf{a}_{\text{centripetal}} = \frac{v^2}{r} = r\omega^2$$

where v is the speed of the object, ω is its angular speed, and r is the radius of the circle. The period of the circular orbit is the time to travel once around, $T = \frac{2\pi}{\omega}$.

PICTURE QUIZ 5

Take the quiz to test yourself; answers are at the end of the text.

1. A spring has force-constant k. If the spring is cut in half, what is the force constant of each half?

(A) $k/2$ (B) k (C) $2k$

2. A spring of force-constant k pulls a block of mass m. If the block starts at rest, the spring stretch is kept constant at x_o, and no other forces act on the block, how long does it take the block to move a distance d ?

(A) $dkm/2x_o$

(B) $\sqrt{x_o/2km}$

(C) $\sqrt{2dm/kx_o}$

(D) None of these

3. The graph shows the position of a bicyclist as a function of time. Various points are labeled A, B, C, etc.

At which points is the net force on the bicyclist

(i) to the right?

(ii) to the left?

(iii) equal to zero?

4. A ping-pong ball and a bowling ball are each rolling toward you. Both have the same momentum, and you exert the same force to stop each. How do the *times* needed to stop them compare?

A. It takes less time to stop the ping-pong ball
B. Both take the same time
C. It takes less time to stop the bowling ball

How do the *distances* needed to stop them compare?

A. The ping-pong ball stops in a shorter distance
B. Both take the same distance
C. The bowling ball stops in a shorter distance

5. A puck of mass m, moving at velocity v, hits an identical puck which is fastened to a pole using a light string of length r. The velocity of the initially-moving puck is perpendicular to the string, as shown. The puck attached to the string then starts revolving around the pole with angular speed ω_0. Suppose we now lengthen the string by a factor of two and repeat the experiment. Compared with ω_0, the new angular speed ω is

A. Twice as great
B. The same
C. Half as great
D. None of the above

6. In problem 5, suppose that after being struck, the puck attached to the string starts revolving around the pole with centripetal acceleration a_0. Suppose we now lengthen the string by a factor of two and repeat the experiment. Compared with a_0, the new centripetal acceleration a is

A. Twice as great
B. The same
C. Half as great
D. None of the above

PROBLEM QUIZ 5

Take the quiz to test yourself; answers are at the end of the text.

1. Starting at time $t = 0$, a cork has the velocity $v = C\sin bt$ as a function of time, where C and b are constants. (a) Find the cork's acceleration $a(t)$. (b) Find the position $x(t)$ of the cork, assuming it started at $x = 0$. (c) At which of the following positions might the cork later be found? (i) $x = C/b$ (ii) $x = 2Cb$ (iii) $x = -C/b$.

2. The light turns green, and a Mercedes starts to move forward with uniform acceleration $a = 2.0$ m/s^2. At that instant a Porsche travelling at constant velocity $v = 20$ m/s passes the Mercedes in the next lane. If they maintain these motions,
(a) how far from the intersection will the Mercedes overtake the Porsche?
(b) How many seconds later will this be?
(c) How fast will the Mercedes be moving at this time?
(d) Sketch both trajectories on a graph of position *vs* time.

3. A proton of mass $m = 1.67 \cdot 10^{-27}$ kg circles in a proton synchrotron of circumference $C = 12$ km at a constant 99.99% the speed of light. (a) How many revolutions does the proton make per second? (b) What is its angular speed? (c) What is its acceleration?

4. Particles A and B each have mass m. A moves counterclockwise in a circle of radius R at constant angular velocity ω, starting at point 1, as shown. B also starts at point 1, but moves purely horizontally, in such a way that at any time its x-coordinate is the same as the x-coordinate of A. As functions of time, find
(a) the force F_A on A;
b) the position $x(t)$ of B.
(c) the velocity $v(t)$ of B;
(d) the force $F(t)$ on B.
(e) Find also the force $F(x)$ on particle B, as a function of its position x rather than in terms of time.
(f) Note that $F(x)$ has the form of a spring force. What is the effective force constant k of this spring force?

Special Topic: Particle Accelerators

Much of what we have discovered about atomic nuclei, their constituent protons and neutrons, and other subatomic particles, has come from experimental observations of high-energy collisions. Collisions can probe nuclear interiors, break nuclei apart, or even create new particles. Often very high velocities are required, achieved by accelerating electrically charged particles like electrons or protons using electric and magnetic fields. If an electric field $\mathbf{E}$ is created in some region of space, for example, it exerts a force $\mathbf{F}_E = q\mathbf{E}$ on a charge q. A proton has a standard positive charge $q = +e$, where $e = 1.602 \cdot 10^{-19}$ Coulombs, and an electron has an equal but opposite negative charge $q = -e$. Therefore the force on a proton is in the same direction as the electric field $\mathbf{E}$, while the force on an electron is in the opposite direction.

A primitive linear accelerator can be built using two electrically charged plates facing one another, as illustrated. The left and right plates are given negative and positive charges, respectively. There is then a force on an electron between the plates which is directed toward the *right*, since electrons are negatively charged, and since like charges repel and unlike charges attract. In the language of electric fields, a field $\mathbf{E}$ is set up between the plates, caused by the plate charges; $\mathbf{E}$ points from the positively charged plate toward the negatively-charged plate, i.e. toward the *left* in the picture. The force on the negatively-charged electron is then $\mathbf{F}_E = q\mathbf{E}$, toward the *right*. If the distance between the plates is small compared with the radius of the plates, this electric force is constant. Electrons boil off a hot filament near the left plate, accelerate toward the right, and then pass through a hole in the right plate. They can then be used to strike a target.

The scattering of electrons by a target can be used to learn properties of the target. For example, the electrons may scatter off atomic nuclei; the number of scattered electrons can be counted as a function of the scattering angle Θ, as shown. This information can be used to determine the nature of the force between electrons and nuclei.

Sample Problem S1. Suppose the force on an electron between the plates of a primitive linear accelerator is $F_0 = 1.6 \cdot 10^{-15}$ N, which corresponds to the force exerted on an electron that is between two plates a distance $d = 10$ cm apart, with a voltage difference of 1000 Volts. The electron mass is $m = 9.1 \cdot 10^{-31}$ kg.
(a) What is the speed v of an electron as it passes through the hole, assuming it has negligible speed when it starts out at the filament?
(b) How long a time t does it take the electron to reach the hole?

Solution: The force on the electron is constant between the plates, so its acceleration is constant. Therefore we can use Eqs. 9 from *Day Five* to find the motion.
(a) Eq. 9d, $v^2 = v_0^2 + 2a_0(x - x_0)$, is the easiest formula to use to find the electron's speed, because we know its initial speed ($v_0 = 0$), its acceleration

$$a_0 = \frac{F}{m} = \frac{1.6 \cdot 10^{-15}\ \text{N}}{9.1 \cdot 10^{-31}\ \text{kg}} = 1.76 \cdot 10^{15}\ \text{m/s},$$

and the distance it moves, $x - x_0 = 10$ cm. So

$$v^2 = 2a_0(x - x_0) = 2\,(1.76 \cdot 10^{15}\ \text{m/s})\,(0.1\text{m}) = 3.52 \cdot 10^{14}\ \text{m}^2/\text{s}^2,$$

giving $v = 1.88 \cdot 10^7$ m/s, or $1.9 \cdot 10^7$ m/s to two significant figures.

(b) We can find the time from the equation $v = v_0 + a_0 t$, because we know v_0, a_0, and now v from part (a). So $t = \dfrac{v - v_0}{a_0} = \dfrac{1.88 \cdot 10^7\ \text{m/s}}{1.76 \cdot 10^{15}\ \text{m/s}^2} = 1.1 \cdot 10^{-8}$ s.

Much higher electron velocities can be achieved if electrons can be given repeated accelerations without requiring excessive charges on two plates. The basic technique is to build hollow metal tubes, called "drift tubes", as shown, upon which negative or positive

charge can be placed. An electron is injected from the left along the axis of Tube 0, so that it drifts along the axis at speed v_0 toward the right. While it is drifting through Tube 0, a negative charge is placed on Tube 0 and a positive charge is placed on Tube 1. This does not cause a force on the electron until it reaches the right end of Tube 0. Then in the gap between Tubes 0 and 1 the electron experiences a force toward the right, accelerating it into Tube 1. It then drifts at constant speed v_1 through Tube 1, feeling no force while the charge on Tube 1 is changed from positive to negative and Tube 2 is given a positive charge. When the electron reaches the gap between Tubes 1 and 2, it again experiences a force, accelerating it into Tube 2. This process of alternating drifts (while the electron is

within a tube) and accelerations (while the electron is between tubes) can be used to give the electron very high velocities. (See problem 5P30.) This is the basic process used to accelerate electrons up to very high velocities in linear accelerators, such as in the two-mile long electron linear accelerator at Stanford University.

An alternative type of accelerator is one in which the electron, proton, or other charged particle is whirled faster and faster in a circle. Early circulating machines were called *cyclotrons*, in which protons injected near the middle of a circular machine gained speed while they spiraled outwards toward the cyclotron's rim.

A charged particle can be caused to move in a circle by a magnetic field **B** . The field exerts a force of magnitude $F_B = q\,v\,B$ on a particle of charge q , where v is the velocity of the particle, assumed to be perpendicular to **B**, and B is the magnitude of **B**. The force, the velocity, and the magnetic field are in fact all perpendicular to one another. If v is in the x-direction, for example, and **B** is in the y-direction, then F_B is in the z-direction, as shown, if q is positive. (If q is negative, F_B is in the *negative* z-direction.)

Sample Problem S2. A proton of electric charge e , moving with velocity **v** , enters a region of space in which the magnetic field **B** is constant, and perpendicular to **v** .
(a) Show that the proton moves in a circle.
(b) Find the radius r of the circle, in terms of e, v, B, and the proton's mass m.
(c) Show that the period T of the proton's motion is independent of r .

Solution: (a) The force $\mathbf{F}_B$, and therefore the proton's acceleration $\mathbf{a} = \mathbf{F}_B/m$, is perpendicular to the proton's velocity. Therefore in a short time interval Δt, the change in the velocity is $\Delta\mathbf{v} = a\Delta t$, as shown: The new velocity $\mathbf{v}_1$ is the original velocity $\mathbf{v}_0$ plus the change in velocity $\Delta\mathbf{v}$. For infinitesimal $\Delta\mathbf{v}$, $\mathbf{v}_1$ has the same magnitude as $\mathbf{v}_0$, but is in a slightly different direction. During the next time interval Δt, the acceleration is perpendicular to $\mathbf{v}_1$, so the velocity vector keeps changing its direction, but not its magnitude, as shown. Now if the particle starts at the point 0, it will move to $\Delta r_0 = \mathbf{v}_0\Delta t$ in time Δt, and then will move the additional displacement $\Delta r_1 = \mathbf{v}_1\Delta t$, and so on, as shown. In each time interval Δt it moves the same *distance,* because each $\Delta\mathbf{v}$ has the same magnitude, but the *direction* of the Δr's changes slightly each time, and always by the same angle. After a sufficient number of steps, the proton will have gone around a complete circle.

(b) The particle moves in a circle, so its inward acceleration is $a = v^2/r$. Therefore the equation $F = ma$ is $evB = mv^2/r$. Solving for r, we find

$$r = \frac{mv}{eB} = \frac{p}{eB}$$

where p is the proton's momentum. The larger the momentum, the larger the circle radius.
(c) The period T, the time to circle once around, is the circumference of the circle $2\pi r$ divided by the particle's speed v. So

$$T = \frac{2\pi r}{v} = \frac{2\pi(mv/eB)}{v} = \frac{2\pi m}{eB},$$

which is independent of both r and v. Suppose one proton has velocity v and the other has velocity $2v$. The orbit of the second particle has twice the radius of the first particle's orbit, but the orbital periods are the same. The orbital frequency ν (the greek letter "nu"), called the *cyclotron frequency*, is the number of revolutions/second, i.e., $\nu = 1/T = eB/2\pi m$. The cyclotron frequency depends only the magnetic field B and the charge-to-mass ratio e/m of the proton; it does not depend upon the proton's velocity.

Note that since the angular velocity of the particle around the circle is $\omega = 2\pi/T$, the orbital frequency ν and angular velocity ω are related by $\omega = 2\pi\nu$. For every rotation, the particle turns through 2π radians.

In a cyclotron, charged particles move within two "D" -shaped hollow metal cavities, as shown. A constant magnetic field **B** is established perpendicular to the "dees", so a particle moves in a semicircular path when it is within one of them, as described in *Sample Problem* S2. When the particle reaches the gap between the dees, it is subjected to an electric force in its direction of motion, which speeds it up. It then circles within a dee, and speeds up again when it next traverses the gap. When it finally reaches the outer rim of the cyclotron, the particle can be made to crash into a target whose properties are under investigation.

Cyclotrons require huge quantities of steel to form the dees, since they must enclose the entire spiralling path of the protons. A more efficient circular accelerator called a *synchrotron* was subsequently invented, requiring much less construction cost for the same final energy and with other advantages as well. In a synchrotron, protons move always at essentially the same radius. As they speed up, their centipetal acceleration v^2/r increases with time, so the force $F = evB$ must increase with time as v^2. This means that the magnetic field B must be made to increase as the particles speed up. Huge synchrotrons, such as those at Fermilab near Chicago and at CERN near Geneva, Switzerland, have been used to discover quarks and other fundamental particles.

Questions and Problems from Day Five

5Q1. A jet plane is propelled forward by a force F due to the burning of fuel. What is the source of the force on the burned fuel, that pushes it out the rear as exhaust?

5Q2. Consider the statement "If **F** = *m***a** is correct in one inertial frame, it is correct in all inertial frames." Is the statement true or false? What assumptions do you have to make or what experiments would you have to carry out to verify this?

5Q3. One end of a spring is attached to a ball, and the other end is pulled by an explorer so the spring stretch stays constant. If the spring exerts a force F_1 on the ball, what force is exerted by the ball on the spring? If the explorer pulls on the spring with force F_2, what force is exerted on the explorer by the spring? Which is larger, F_1 or F_2? When would these forces be nearly equal?

5Q4. Is it possible for a car to accelerate while moving at constant speed?

5Q5. In the English system of units, the fundamental dimensions of mechanics are *force* (in pounds), *length*(in feet), and *time* (in seconds). In this system, what are the dimensions of (a) velocity (b) acceleration (c) mass (d) momentum (e) impulse?

5Q6. A particular problem features the quantities F, m, ℓ, T, v, and a, representing a force, a mass, a length, a time, a velocity, and an acceleration. Which of the following equations cannot be valid for this problem in all systems of units?
(a) $F = \ell m/T^2$ (b) $T = \mathrm{v}F/m$ (c) $a + (m/F)\mathrm{v}^2/T^2 = 0$ (d) $\mathrm{v} = \sqrt{FT/m^2}$

5Q7. According to quantum mechanics, the vacuum is not empty, but is filled with a sea of "virtual" particles. One consequence of this is the *Casimir force*, a tiny attractive force between two flat parallel metal plates. The force per unit area (pressure) between the plates depends upon Planck's constant $\hbar$ (with units ML^2/T), the speed of light c, and the distance d between the plates. If the plates are initially a certain distance apart, how much does the force/area increase if the distance is cut in half?

Section 5.1 Springtime in the Bubble

5P1. Two springs 1 and 2, with force constants k_1 and $k_2 = 2k_1$, are attached to opposite sides of a block. The other end of spring 2 is attached to a wall, as shown. Spring 1 is pulled to the left so that in equilibrium is has been stretched a distance d. (a) In terms of d, what is the stretch in spring 2? (b) How far must spring 1 be pulled to the left so that its stretch is d?

5P2. Two springs 1 and 2, with force constants k_1 and $k_2 = k_1/2$, are attached to opposite sides of a block. The other end of spring 2 is attached to a wall, as shown. How far must we pull the left end of spring 1 so the block willl have moved a distance d when the system comes to equilibrium?

5P3. Six identical Hooke's-law springs are attached to two blocks.. Spring 1 is attached to the left-hand block only, springs 2 and 3 are attached between both blocks, and

springs 4, 5, and 6 are attached to the right-hand block and to a wall. Spring 1 is now pulled to the left, so that in equilibrium it has been stretched by Δx. In terms of Δx, (a) what then is the stretch in springs 2 and 3? (b) in springs 4, 5, and 6? Explain. (c) How far must the left end of spring 1 be pulled to the left so that its stretch is Δx ?

5P4. Two Hooke's-law springs, each having force-constant k, are connected in parallel, as shown. What is the overall force constant of the combination? That is, if the two springs were replaced by a single spring whose restoring force equaled that of the original two acting together, what would be the force-constant of the single spring?

5P5. Two Hooke's-law springs, each with force-constant k, are connected in series, as

shown. What is the overall force constant of the combination? That is, if we were to replace the combination by a single spring, what would be its force constant?

5P6. Each of three identical nonlinear springs exerts a restoring force $F = k(\Delta x)^n$, where Δx is the spring stretch, and k and the exponent n are unknown constants. One spring is attached to the right side of a block. The other two are connected in parallel between a wall and the left side of the block, as shown. Now the single spring is

pulled to the right so that it stretches an amount Δx; this causes each of the other springs to stretch an amount $\Delta x/3$ in equilibrium. (a) What is the exponent n? (b) If the single spring is now pulled farther, so its stretch is $2\Delta x$, what now is the stretch of each of the other two springs?

Section 5.2 Is it true that $\mathbf{F}_{spring} = d\mathbf{p}/dt$?

5P7. In the experiment in which a 1.0-kg ball was pulled by a spring with constant stretch $\Delta x = 5.0$ cm, times were measured when the ball passed markers placed every 25 cm. The data is shown in the table.
(a) Use the crude approximation $v = \Delta x/\Delta t = [x(t_2) - x(t_1)] / (t_2 - t_1)$ to estimate the velocity of the ball at times intermediate to those measured. Present the results in a table. (b) Plot the results as as graph of v *vs* t. (c) Draw the best straight line through the data and find the acceleration a. (d) (Optional) Plot the x *vs* t data directly on log-log graph paper, and estimate a using the best straight line through the data.
(e) Estimate the force-constant k of the spring in units Newtons/meter.

x(cm)	t(sec)
0	0
25	14.1
50	19.9
75	24.5
100	28.2
125	31.5
150	34.5
175	37.4
200	39.9

5P8. The figure shows the position $x(t)$ of a truck. Make a rough sketch of the truck's velocity as a function of time.

5P9. The figure shows the acceleration $a(t)$ of a particle. Sketch the particle's velocity as a function of time, assuming it starts at rest at time $t = 0$.

5P10. Rockets exert a force $F = \alpha t^2$ on a spaceship of mass M, where α is a constant and t is the time. If the ship starts at position x_0 with velocity v_0 at time $t = 0$, find (as functions of time) the ship's (a) velocity v (b) position x.

5P11. The force on a ball of mass m increases exponentially with time, $F = F_0 e^{\alpha t}$, where F_0 and α are constants. Find (a) the acceleration a (b) the velocity v (c) the position x of the ball, all as functions of time, assuming the ball starts at position x_0 with velocity v_0 at time $t = 0$.

5P12. An oscillating electric field exerts a force $F = F_0 \sin\omega t$ on an atomic nucleus of mass m, where F_0 and ω are constants. Find (a) the acceleration a (b) the velocity v (c) the position x of the nucleus, all as functions of time, assuming the nucleus starts at rest at $x = 0$.

5P13. We want the position x of a ball bearing of mass $m = 0.20$ g to vary with time according to

$$x = 3.0 \text{ m} - (2.2 \text{ m/s}^2)\, t^2 + (7.6 \text{ m/s}^4) t^4.$$

As a function of time, what force must we exert on the ball bearing?

5P14. As a function of time, the velocity v of an automobile of mass m moving along a straight road is

$$v = \alpha - \beta t^3 + 3\gamma \sin \omega t$$

where α, β, γ, and ω are constants. In terms of constants and time, find (a) the position x of the car, if it is located at $x = 0$ at time $t = 0$ (b) the force F on the car.

5P15. Freeman Dyson has estimated that the impulse given an Orion starship by a single nuclear explosion would be delivered over a time interval $\Delta t = 10$ milliseconds.
(a) For the explosion parameters given in the *Special Topic* of *Day Four,* with ejected particle velocity $v = 3000$ km/s, what would be the average force F on the ship during the 10 ms?
(b) What would be the average acceleration a of the ship during this time interval? Assume the ship is fully loaded.
(c) If instead the same impulse could be delivered in only $\Delta t = 1$ ms, how would that change the average force and acceleration of the ship, and how would it change the velocity gained by the ship as a consequence of the explosion?

5P16. Suppose the momentum of a particle of mass m is given by the formula $\mathbf{p} = m\,\mathbf{v}/(1 - v^2/c^2)^{1/2}$ rather than $\mathbf{p} = m\mathbf{v}$, where c is a constant with dimensions of velocity. That is, suppose that measurements show that $\mathbf{p}_{\text{total}}$ is conserved in collisions if the new formula is used, but not if $\mathbf{p} = m\mathbf{v}$ is used.
(a) Show that at least one of the equations $\mathbf{F} = m\mathbf{a}$ and $\mathbf{F} = d\mathbf{p}/dt$ must be false.
(b) If one of these equations is true, which one must it be, if the momentum of a particle is always given by a real number? (c) Using the law selected in (b), prove that if the particle starts with some speed $v < c$, for finite forces and times it is impossible for the particle to reach the speed $v = c$.

Section 5.3 Uniformly accelerated motion

5P17. A car of mass m begins traveling on a level road at time $t = 0$ with the speed v_0 in the positive x-direction. No horizontal forces are exerted until time t_1, when a wind-force F_0 begins to act in the *negative* x-direction.
(a) How far has the car moved from its starting point by the time it comes instantaneously to rest? (b) At what time is the car's velocity equal to $-v_0$?

5P18. A car traveling at speed $v = 50$ km/hr (about 31 mi/hr) runs into a telephone pole.
(a) What (assumed constant) force F acts on the driver's upper body (of mass $m =$ 35 kg) when it is stopped by an air bag as it moves forward a distance $d = 150$ cm relative to the road? (b) How long does it take the driver to come instantaneously to rest?

5P19. An engineer designs an aircraft which, in level flight, can reach the speed of sound at sea level ($v_S = 330$ m/s) in a distance $x = 1.0$ km at constant acceleration a, starting with speed $v_O = 50$ m/s. (a) What is a? (b) How long a time does it take to reach v_S from v_O?

5P20. Two airplanes land simultaneously at opposite ends of a narrow runway, a distance $D = 1250$ m apart. The two planes touch down with speeds $v_1 = 65.0$ m/s and $v_2 = 75.0$ m/s, and each brakes with the same constant deceleration $a = -4.00$ m/s^2. (a) Do they stop before they crash into one another? (b) If so, how far apart are they when they both have stopped? If not, what is their relative speed when they crash?

5P21. A VW Microbus and a Lamborghini racing car are stopped side-by-side. When the light turns green, the VW moves forward at constant acceleration $a_{VW} = 0.50$ m/s^2. The Lamborghini, with transmission problems, finally gets going 30 seconds later, at constant acceleration $a_L = 4.0$ m/s^2. When the Lamborghini overtakes the VW, (a) how far from the stoplight are they? (b) how long a time has it been since the light turned green? (c) Sketch the position $x(t)$ of each car on the same graph.

5P22. A car and truck are both traveling at a steady 22 m/s (50 mi/hr) as shown, the car 15 m behind the truck. The driver of the car decides to pass the truck, so accelerates with constant acceleration $a = 0.80$ m/s^2, passing the truck and pulling back into the correct lane when it is 25 m ahead of the truck, when it stops accelerating. (a) How far does the car travel while accelerating? (b) How long a time does the car accelerate? (c) What is its final speed?

5P23. A motorcycle policeman, parked beside the road, sees a car pass by at 36 m/s (80 mi/hr). Ten seconds later the policeman starts to give chase, accelerating at $a = 1.50$ m/s^2 until he reaches 45 m/s (100 mi/hr), when he stops accelerating and travels at constant speed. (a) How long will it take him to overtake the car if the car keeps moving at constant speed? (b) How far will he have moved during this time?

5P24. Beginning with Eqs. 9a and b, derive (a) Eq. 9c (b) Eq. 9d

5P25. Starship *Enterprise* observes a Klingon vessel a distance $d = 12$ light-hours away, at the same time the Klingon vessel observes the *Enterprise*. They are at rest relative to one another. The Klingons head for their base, a distance $D = 24$ light-hours away from them, with constant acceleration $a_K = 2.5$ km/s^2. The *Enterprise* gives chase, with constant acceleration $a_E = 5.0$ km/s^2. Does the *Enterprise* catch the Klingons before they reach their base? If so, how far from the base is this? If not, how far is the *Enterprise* from the base when the Klingons reach it? (1 light-hour is the distance light travels in one hour, at speed $c = 3.0 \cdot 10^8$ m/s)

5P26. An electric field $\mathbf{E}$ exerts force $\mathbf{F} = q\mathbf{E}$ on a charge q. An electron of mass m, charge $q = -e$, and initial velocity $\mathbf{v} = v_O\,\hat{\mathbf{x}}$ enters a uniform electric field $\mathbf{E} = -E_O\,\hat{\mathbf{y}}$ established within a parallel plate capacitor; v_O and E_O are constants. Find (a) the force components F_x and F_y on the electron; (b) its velocity components $v_x(t)$ and $v_y(t)$; (c) its position components $x(t)$ and $y(t)$, assuming it was located at the origin at time $t = 0$; (d) the shape of its path, $y(x)$.

5P27. A spring with force constant $k = 0.2$ N/m pulls a ball of mass $m = 0.5$ kg. The spring stretch is maintained at $\Delta x = 10$ cm. If the ball starts at $x_0 = 0$ with velocity $v_0 = 0$, (a) how long a time t does it take the ball to reach the position $x = 2$ m ? (b) what velocity v does it have when it arrives ? (c) what distance x has it traveled when its velocity is $v = 8$ m/s?

5P28. An elevator E is suspended from the axle of a pulley P_1. The pulley rides on a cable, one end of which is attached to a fixed point on the roof while the other end is strung over a second pulley P_2 and then to a counterweight C, as shown. The axle of the second pulley is attached to the roof. (a) If the elevator is moving upward with constant velocity v_E, what is the velocity of the counterweight C? (b) If the elevator is moving upward with constant acceleration a_E, what is the acceleration of the counterweight?

Section 5.4 Circular motion

5P29. Find (a) the rotation frequency ν (b) the angular speed ω, of the second hand of an analog electric clock. Find also (c) the speed v (d) the centripetal acceleration a of the pointer on the second hand, if it is a distance $r = 3.5$ cm from the rotation axis.

5P30. A unicycle wheel of radius $r = 21$ cm is spinning with period $T = 0.16$ s. Find the wheel's (a) frequency of rotation ν (b) angular speed ω. Find also (c) the speed v (d) the centripetal acceleration a of a point on the wheel's rim.

5P31. A carnival ride consists of a car on the end of a boom of length L, tilted at angle θ to the horizontal, circling around a vertical axis with angular speed ω, as shown. Find (a) the velocity v (b) the acceleration a of a passenger inside the car, both in terms of L, θ, and ω.

5P32. A massless spring of force-constant k has unstretched length d. A ball of mass m is attached to one end of the spring, and the assembly is made to rotate in a circle about the other end of the spring, which is attached to a fixed point. If the spring stretch is Δx, what is the period of rotation T, in terms of given constants?

5P33. A kitchen "lettuce spinner" acts like a centrifuge to "pull" water off washed lettuce meant for a salad. If a lettuce leaf at radius $R = 4.0$ inches from the rotation axis is spun at frequency $\nu = 16$ revolutions/second, find (a) the speed v of the leaf (b) the period of rotation T (c) the angular speed ω (d) the acceleration a of the leaf, in feet/s^2. (e) One "g" is the acceleration of gravity at the Earth's surface, $1\ g = 32$ ft/s^2. How many "g's" does the leaf experience?

5P34. Four children, each of mass m, stand on a merry-go-round, which is turning with constant angular speed ω. Child #1 stands at the center, #2 stands a distance $R/3$ from the center, #3 stands a distance $2R/3$ from the center, and #4 stands at the rim, a distance R from the center. Find the horizontal force acting on child (a) #1 (b) #2 (c) #3 (d) #4.

5P35. (a) Earth, with radius $R = 6400$ km, turns about its axis in time $T = 1$ day. Find the speed v of a point on the equator (in m/s), and the centripetal acceleration a_{cent} of this point (in m/s^2). (b) Pulsars are rapidly-rotating neutron stars. Pulsar 1913+16 rotates with a period $T = 0.059$ s, for example. Find the speed v of a point on this pulsar's equator, in m/s and as a fraction of the speed of light, $c = 3.0 \cdot 10^8$ m/s. Also find the centripetal acceleration a_{cent} of this point, in m/s^2. Assume the pulsar's radius is $R = 10$ km.

5P36. A chunk of ice of mass $m = 3.0$ kg in one of Saturn's rings orbits the giant planet at a radius $r = 1.0 \times 10^5$ km with a period $T = 0.39$ days, caused by Saturn's attractive gravitational force. (a) What force F does the chunk of ice exert on Saturn? (b) What is the ratio of the acceleration of Saturn due to this force, to the acceleration of the ice chunk due to Saturn? (Saturn's mass is $M = 5.7 \cdot 10^{26}$ kg.)

5P37. Planets have orbits around the Sun that are roughly circular. Using the table below of the orbital radii and periods of the inner planets, how does the centripetal (i. e., inward) acceleration of the planets depend upon their orbital radii? That is, if $a \propto r^n$, what is the exponent n? (note that 1 A. U. = 1 astronomical unit, the mean Sun-Earth distance)

planet	orbital (mean) radius in A.U.	period (years)
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.523	1.881

5P38. It typically takes about 10 months for spacecraft to travel from Earth to Mars. This may be too long for humans to make the trip in zero "gee", because of the loss of bone mass and other physical problems known to affect astronauts who remain in Earth orbit for long periods. One proposed idea is to tie a long cable between the spacecraft and an equal-mass counterweight, and make them both rotate about their mutual center of mass halfway along the cable, as shown. There would then be an effective gravity within the spaceship equal to its centripetal acceleration.

(a) Find a formula for g_{eff} on board the spacecraft, as a function of the length of the cable L and the speed v_{rel} of the craft relative to the CM of the system.

(b) What is v_{rel} if $L = 1$ km and $g_{eff} = 9.8$ m/s^2, equal to gravity on Earth's surface? (If L is much less than 1 km, astronauts would notice unsettling differences between the on-board artificial gravity and gravity on Earth.)

5P39. A linear accelerator (linac) is built with a large number of hollow cylindrical "drift tubes", as described in the *Special Topic*. Suppose each gap has width d, the force F is constant within each gap, and F is the same in each gap. (a) Find the electron's speed v_N in the Nth tube in terms of its speed v_{N-1} in the $(N-1)^{th}$ tube, and the parameters F, d, and m, where m is the electron mass. (b) Find v_N in terms of v_0, the electron's speed in the 0th tube. (c) It is easiest to exchange the positive and negative charges on drift tubes at constant frequency, which means that the time spent within each tube must be the same. Find then the length of the Nth tube as a multiple of the length of the 0th tube, in terms of N, F, d, m, and v_0. (d) Suppose $Fd >> mv_0^2$. What is the ratio ℓ_{16}/ℓ_1 of the lengths of tubes 16 and 1? What is the ratio v_{16}/v_1 of electron speeds in tubes 16 and 1? (*Note*: Varying the lengths of drift-tubes so that electrons spend the same time inside each tube is an essential design feature in linacs. The assumption of constant electrical force in the gaps and zero force within a tube is a simplified model.)

5P40. (a) Suppose the width of the gap in a cyclotron is ℓ, and the electrical force F_0 on a charged particle is constant while the particle is within the gap (see the description of cyclotrons in the *Special Topic*). Assuming it starts from rest, find the particle's speed v_1 after its first acceleration, in terms of F_0, ℓ, and its mass m. (b) Find the particle's speed v_N after it has traversed the gap N times, in terms of N, F_0, ℓ, and m. (c) Find the radius r_N of the circular path after N gap traversals, in terms of N, F_0, ℓ, m, B(the magnetic field) and q (the particle's charge).

5P41. A nice feature of a cyclotron is that even though the charged particles move within the "D's" in circles of increasing radius as they speed up, the time to return to the gap is the same regardless of the radius, so the alternating current applied to the D's can have constant frequency and so the particles are accelerated properly within the gaps. This is true for *nonrelativistic* particles, that is. For *relativistic* particles, whose velocities are an appreciable fraction of the speed of light, this nice property is no longer true. The centripetal force on relativistic particles due to the applied magnetic field B is still $F = qvB$, where q is the charge and v is the particle's speed. It is also still true that $F = dp/dt$, where p is the momentum of the circulating charge. However, the momentum of as relativistic particle is $p = mv/\sqrt{1 - v^2/c^2}$ rather than simply mv. Figure out two or three possible ways one might compensate for this effect, so that relativistic particles could continue to be accelerated in a cyclotron, i.e. so that they would continue to be given a push in the correct direction every time they reach the gap. (One of these ways has been used to build so-called "synchrocyclotrons" which have successfully accelerated relativistic particles.)

5P42. In a particular synchrotron, protons of mass m and electric charge e move in a circle of fixed radius R. For a while the speed of the proton increases with time t according to $v = v_0 + \alpha t$, where α is a constant (the component of acceleration tangent to the circle.) As functions of time, find (a) the tangential component F_{tan} and (b) the radial (i.e., the centripetal) component F_{rad} of the force on the proton. (c) Find the magnitude of the magnetic field B within the synchrotron.

Day Six

Marshalling our Forces: Isolation Diagrams

"Science is the attempt to make the chaotic diversity of our sense-experience correspond to a logically uniform system of thought." – Albert Einstein

6.1 Multiple forces

During *Day Five* we observed what happens when we pull on a ball with a single spring. We *defined* force by the stretch of a standard spring, and then we found that the ball's acceleration was proportional to the spring force. Now we need to take up two questions: (1) What happens if the force is *not* due to a spring? (2) What happens if *more* than one force acts? The answers to these two questions will go a long way toward generalizing the special cases we described yesterday.

Throughout *Day Five* we assumed that if a ball remains at rest, the total force acting on it is the vector sum of the individual forces, and that this vector sum is *zero*. At least that's what we assumed for forces in one dimension, when we attached a single spring to one side of a ball and two springs to the other side, in order to show that our spring forces were proportional to the stretch.

We can check this statics condition $\sum \mathbf{F} = 0$ for forces acting in two or three dimensions. Take the hook-up shown in Fig. 1a, for example, with three similar springs attached at 120° angles around a ball. From symmetry we expect that each spring should be stretched the same amount, so that the magnitude of each of the three forces on the ball is the same, as shown in Fig. 1b.

Fig. 1. (a) A ball in equilibrium with three springs attached
(b) Three forces on the ball (c) The forces sum to zero

The three force vectors form the sides of an equilateral triangle, as in Fig. 1c. The sum of the three vectors is therefore zero, as we expected. We can try other set-ups, with various numbers of springs, including some where not all of the springs are in the same plane. In every case the vector sum of the forces on an object, which we call the total force or *net* force, is *zero* if the object is in equilibrium.

Sample Problem 1. Three springs, all with the same force-constant k, are attached to a ball in the Bubble. The ball stays in equilibrium. One of the springs pulls "upward", as shown; the other two pull at angles of $\theta = 45^{\circ}$ to the vertical, also as shown. If the stretch in the vertical spring is $s = 4.0$ cm, what is the stretch d in each of the other two springs?

Solution: The net force on the ball is zero. The upward force due to the vertical spring is $F_{\text{up}} = ks$, where $s = 4.0$ cm. The downward force of each of the other springs is $F_{\text{down}} = kd\cos\theta = kd/\sqrt{2}$, so to balance the vertical forces we must have

$$2\,F_{\text{down}} = 2kd/\sqrt{2} = F_{\text{up}} = ks = k(4.0\text{ cm}).$$

Therefore the stretch d in each of the lower springs is $d = 4.0\text{ cm}/\sqrt{2} = 2.8$ cm.

A useful feature of the vector addition of forces is that it gives us a way to measure the force exerted on a ball by agents other than springs, if we assume that static equilibrium means always that $\sum \mathbf{F} = 0$. So far we don't know anything about the force caused by an attached *string*, for example, or the *force of contact* with another ball or with a wall.

A. String tension

We attach a string and a spring to opposite sides of a ball, as shown in Fig. 2(a). The force exerted by the string must be equal but opposite to the force exerted by the spring, if the ball stays in equilibrium. That is, the magnitude T of the force exerted on the ball by the string, which we will call the string *tension*, must be $T = k\Delta x$, where Δx is the stretch of the spring on the other side. According to the force-counterforce rule, the ball pulls on the string with an equal but opposite force T, so since the string itself in in equilibrium it follows that the wall on the right must pull on the string with a force of equal strength T, as shown in Fig. 2(b).

Fig. 2. (a) A ball in equilibrium with a spring force and a string tension. (b) The string in equilibrium has equal but opposite force T upon it. (c) The tension T exists throughout the string.

In fact, this same tension force T must exist *throughout* the string. Consider the typical piece of string shown in Fig. 2(c). This piece is in equilibrium, so the forces on it to the left and to the right must have equal magnitudes. The force to the left, due to the rest of the string at the left, must also be T, because the rest of the string at the left is in equilibrium, and we already know the ball pulls on the left-hand end with force T. Likewise the force to the right must be T as well. Therefore this tension T exists throughout the string, and is the magnitude of the force with which the string pulls on things it is attached to. The tension can have any positive value up to some maximum tension, above which the string breaks.

Sample Problem 2. An old, broken-down spacehulk of mass m is towed by a spacetug using a light cable attached to the hulk, as shown. The cable will snap if the tension T within it exceeds T_0. What is a_{max}, the maximum achievable acceleration of the hulk?

Solution: It is the cable tension T which accelerates the hulk*, with $F_{net} = T = ma$. If T_0 is the maximum allowed tension, it follows that the maximum acceleration of the hulk is

$$a_{max} = T_0/m.$$

*Note that we have assumed that the mass of the cable is negligible compared with the mass of the hulk! Otherwise the tug will have to accelerate the massive cable as well as the hulk, so the cable tension at the right-hand end of the cable will have to be *larger* than at the left-hand end.

B. Contact forces: the normal force

If a ball is pushed against a wall, the wall has to push back against the ball, with a force of contact. We can jam a spring and ball between two balls, as shown in Fig. 3(a). The contact force exerted on the ball by the wall must then be equal but opposite to the force exerted by the spring, if the ball stays in equilibrium. The compression of the spring can therefore be used to measure the magnitude of the contact force. If we try to do the experiment so that the spring is *not* perpendicular to the walls, as shown in Fig. 3(b), the ball and spring start moving sideways. The reason must be that the contact force is perpendicular to the wall, so the ball can be in equilibrium in Fig. 3(a) but not in 3(b). So we can also call this force a normal force, meaning that it is perpendicular to the wall. (In our experiments there is no friction between the ball and wall. If there were, the force of contact would have a component parallel to the wall as well.) As with a string, whose

Fig. 3. A spring and ball between two walls. (a) If the spring is perpendicular to the walls, the forces are balanced. (b) If the spring is not perpendicular to the walls, the forces are not balanced, because the force due to the wall is a normal force. (c) Measuring the normal force using springs behind the wall.

tension can be arbitrarily large as long as it doesn't break, the normal force on a ball can have arbitrary magnitude unless something gives way or breaks.

Another way to measure the normal force of a wall on a ball in equilibrium would be to put some springs behind the wall, as shown in Fig. 3(c). The compression of the springs could measure the normal force.

Sample Problem 3. A theme park features a large horizontal wheel of radius R which can spin around its vertical axis of symmetry. Surrounding the wheel is a vertical wall which spins with the wheel. A customer of mass M steps onto the wheel, which is then slowly spun up to a final steady angular velocity ω. The customer finds himself pressed against the wall, and can even lift his feet while staying fixed relative to the wall. What is the normal force N exerted on the customer by the wall?

Solution: While moving in a circle at constant angular velocity the customer has inward acceleration $a_{\text{centripetal}} = R\omega^2$. The normal force is the only horizontal force on the customer, so from $F = ma$ in the horizontal direction we have

$$N = Ma = MR\omega^2.$$

6.2 Is $\mathbf{F}_{net} = m\mathbf{a}$?

What happens if $\mathbf{F}_{net} \neq 0$, so the ball is *not* in equilibrium? If only a single spring force acts, we know that a ball of mass m accelerates according to $\mathbf{F} = m\mathbf{a}$. So the natural guess when many forces act is that the ball accelerates according to $\mathbf{F}_{net} = m\mathbf{a}$, where $\mathbf{F}_{net}$ is the vector sum of all the individual forces acting on it, whether they are caused by springs, strings, normal force, or whatever. We try an experiment to see if this works.

We pull two balls of equal mass $m_1 = m_2 \equiv m$ with springs, by means of an external force $\mathbf{F}_0$. We keep all the accelerations the same as one another and also constant, as in Fig. 4. The right-hand spring has constant stretch Δx_1, and the left-hand spring has constant stretch Δx_2. Ball m_2 has only one spring attached to it. That spring is stretched by Δx_2, so pulls on m_2 to the right with force $k\Delta x_2$. We should therefore have

$$k\Delta x_2 = m_2 a_2 = m a_2. \tag{1}$$

Fig. 4. Two balls and two springs with constant acceleration

Two forces act upon ball m_1, $k\Delta x_1$ to the right, due to the right-hand spring, and $k\Delta x_2$ to the left, due to the left-hand spring. *Note that the external force F_0 acts on the spring, not on m_1, so is not included as one of the forces acting on m_1.* The net force on m_1 is therefore $k\Delta x_1 - k\Delta x_2$, so

$$k\Delta x_1 - k\Delta x_2 = m_1 a_1 = m a_1. \tag{2}$$

We pull so that both balls accelerate equally to the right by keeping the stretches Δx_1 and Δx_2 constant. (That is, if the two balls are always the same distance apart they can't have any relative velocity ro relative acceleration, so $a_1 = a_2$.) The left-hand sides of Eqs. 1 and 2 must therefore be equal to one another,

$$k\Delta x_2/m = (k\Delta x_1 - k\Delta x_2)/m\,, \tag{3}$$

which gives $\Delta x_1 = 2\Delta x_2$. The right-hand spring should be stretched *twice* as much as the left-hand spring. Observations verify that this is true when the whole system moves as a unit, both balls with the same acceleration. We conclude that $\mathbf{F}_{net}$ *is* equal to $m\mathbf{a}$ for each of the balls. The law $\mathbf{F}_{net} = m\mathbf{a}$ has survived this test.

6.3 Isolation diagrams*

There is an organized way to keep all the masses and forces straight, to make sure we have the right forces acting on the right masses. The steps are as follows, for the problem of two balls and two springs:

1. Draw a picture of the physical situation, labeling everything.
For our problem, the picture is:

2. Draw a separate *isolation diagram* for each body of interest. That is, draw a picture of one of the bodies, and the forces acting on that body only, using an arrow to designate the direction of each force. Repeat the procedure for each body of interest.

In an isolation diagram, we draw a single body and only the forces acting on that body. In other words, we isolate that body from other parts of the system, and show only forces acting directly on that body, which are due to things *touching* it, like a spring, a string, a surface, or air. It is also convenient to label and assumed positive-acceleration direction by a double-arrow, and put it near but not touching the object. We use a *double*-arrow to distinguish acceleration from force.

In this problem, we have two isolation diagrams:

Note that m_2 has only one spring touching it, so only that one spring can exert a force on it.

3. Choose coordinate axes, and write out the component equations of $\mathbf{F} = m\mathbf{a}$ for each isolation diagram, where $\mathbf{F}$ is the vector sum of all the forces acting on the body.

There is only one interesting dimension in this problem, so there is only one equation for each diagram. We take the net force in each diagram to be in the direction of the double-arrow, the assumed positive direction of acceleration. The equations are

*Also often called "free-body diagrams" or "force diagrams". We use the term *isolation diagram* because it emphasizes the fact that we are isolating some portion of a system for analysis.

(1) $$k\Delta x_1 - k\Delta x_2 = m_1 a = ma \qquad (4)$$

(2) $$k\Delta x_2 = m_2 a = ma \qquad (5)$$

4. Try to solve the equations!

In this case, it is easy. We have arranged that the balls accelerate equally, with known acceleration a, so the unknowns are Δx_2 and Δx_1, which are easy to find:

$$\Delta x_2 = ma/k \quad \text{and} \quad \Delta x_1 = 2ma/k\,. \qquad (6)$$

5. Check that the solutions are reasonable.

We verify that the answers ma/k (and $2ma/k$) have the dimensions of length. Our physical intuition tells us that if the acceleration is zero, both stretches should be zero, and the answers agree that they are; we also believe that stiffer springs should give smaller stretches, and they do, according to the solution.

Sample Problem 4. A spring of force-constant k and a string are attached to opposite sides of a ball of mass m, as shown. The other end of the spring is kept fixed. At time $t = 0$ the spring is unstretched and the ball is at rest. The far end of the string is pulled to the right with constant acceleration a. Using the isolation diagram approach, find the string tension T as a function of time.

Solution: 1. The physical picture is already drawn above; the remaining steps are:

2. The isolation diagram of the ball is

where x is the spring stretch at time t.

3. Newton's second law $F = ma$ then gives $T - kx = ma$, so $T = ma + kx$.

4. The acceleration a is constant, so $x = \frac{1}{2}at^2$. The tension therefore grows with time according to the equation

$$T = \left(m + \tfrac{1}{2}kt^2\right)a\,.$$

(This result is valid until the spring exceeds its elastic limit, and no longer obeys Hooke's law, or else either the spring or string breaks.)

5. The force-constant k has dimensions of force/length = M/T^2, so $kt^2/2$ has dimensions of mass, and the entire solution is dimensionally consistent. We would expect the tension to grow with time, and to be larger for larger accelerations, consistent with the solution.

Sample Problem 5. A locomotive pulls two train cars along a horizontal track, exerting a contant force F_o on the leading car. The leading car has mass m_1 and experiences a retarding frictional force F_{f1}; the trailing car has mass m_2 and experiences a retarding frictional force F_{f2}. Find (a) the acceleration a of the train, and (b) the force of coupling F_c between the two cars.

Solution: (1) *The first step is to draw a picture of the physical situation:*

(2) *The second step is to draw an isolation diagram for each body of interest.*

We draw isolation diagrams of (*i*) the two train cars as a unit; (*ii*) the leading car by itself; (*iii*) the trailing car by itself, all as shown. We are able to lump both cars together as a unit in (*i*), because both have the same acceleration.

(3) and (4) *Now we can write out the $F = ma$ equation corresponding to each diagram and solve them for the quantities required.*

For the first diagram (*i*), the net force is $F_o - F_{f1} - F_{f2}$. Note that we need only the forces acting from *outside* the system. The force of coupling F_c is an *internal* force, within the system, so is not needed. (The internal forces cancel out, because there is a force F_c to the right on the trailing car, and an equal but opposite force $-F_c$ acting on the leading car.) The $F = ma$ equation is therefore $F_o - F_{f1} - F_{f2} = (m_1 + m_2)\,a$, so

$$a = \frac{F_o - F_{f1} - F_{f2}}{(m_1 + m_2)}.$$

This is the answer to part (a) of the question.

For the *third* diagram (*iii*), the diagram for the *trailing* car, the net force is $F_c - F_{f2}$, so $F_c - F_{f2} = m_2 a$. We can then solve for the coupling force F_c,

$$F_c = F_{f2} + m_2 a = F_{f2} + m_2 \frac{F_o - F_{f1} - F_{f2}}{(m_1 + m_2)},$$

where we have used the expression for the acceleration previously found. This is the answer to part (b); we have found the coupling force F_c in terms of given quantities.

We have already found answers to the questions posed. Nevertheless, there is an isolation diagram we have not used so far, the second diagram (*ii*), for the *leading* car. Let us see what that diagram can tell us. The net force on the leading car is $F_o - F_c - F_{f1}$, so $F_o - F_c - F_{f1} = m_1 a$. We can solve this equation for the coupling force F_c; this gives

$$F_c = F_o + F_{f1} - m_1 a = F_o + F_{f1} - m_1 \frac{F_o - F_{f1} - F_{f2}}{(m_1 + m_2)}.$$

This result for F_c *looks* quite different from the result found from diagram (iii) above, but a little algebra (combining the two F_o terms and the two F_{f1} terms) shows that the two results are in fact the same. So the $F = ma$ equation based on diagram (*ii*) is a check on the results of the other two $F = ma$ equations.

(5) *Check that the answers are reasonable.*

We first check that the equations for a and F_c are dimensionally consistent, which they are. To check the results in a special case, suppose both frictional forces are zero and that each car has the same mass, $m_1 = m_2 = m$. Then the acceleration is

$$a = F_o / 2m,$$

which is correct, since the force F_o of the locomotive is pulling a total mass of $2m$. The force of contact is

$$F_c = F_o - m \frac{F_o}{(m + m)} = F_o / 2$$

which is also correct, since the force F_c to pull the trailing car should be half that to pull both cars, if their masses are equal and there is no friction.

6.4 Systems of particles

Can we use $\mathbf{F} = m\mathbf{a}$ for a large number of particles? In fact, suppose there is a system of N particles with masses m_i $(i = 1, \ldots N)$. The particles could be small parts of a rigid object like a wheel, or air molecules in a room, or stars in a galaxy. How would $\mathbf{F} = m\mathbf{a}$ work for such systems?

We can write the total force on the ith particle in the form

$$\mathbf{F}_i = (\mathbf{F}_i)_{ext} + \sum_j (\mathbf{F}_i)_j \tag{7}$$

where $(\mathbf{F}_i)_{ext}$ is the total force on the ith particle due to influences external to the system, and $(\mathbf{F}_i)_j$ is the force on the ith particle due to the jth particle within the system. The sum $\sum_j (\mathbf{F}_i)_j$ over j should *not* include the term with $j = i$, because the ith particle does not exert a force on itself.

The net force on the entire system of particles is the vector sum of the forces on the individual N particles,

$$\sum_i \mathbf{F}_i = \sum_i (\mathbf{F}_i)_{ext} + \sum_i \sum_j (\mathbf{F}_i)_j \,. \tag{8}$$

The terms in the double sum on the far right cancel in pairs by Newton's third law, the force-counterforce rule. For example, $(\mathbf{F}_1)_2 = -(\mathbf{F}_2)_1$: The force on particle 1 due to particle 2 is equal but opposite to the force on particle 2 due to particle 1. Therefore there is no net internal force. Only the external forces count for the system as a whole, so

$$\sum_i \mathbf{F}_i = \sum_i (\mathbf{F}_i)_{ext} \equiv \mathbf{F}_{ext} \,; \tag{9}$$

the net force on the entire system is just the sum of the external forces alone.

By Newton's second law, the net force on the ith particle obeys

$$\mathbf{F}_i = \frac{d\mathbf{p}_i}{dt}; \tag{10}$$

the force is equal to the time rate of change of the particle's momentum. Summing this equation over all of the particles, we get

$$\sum_i \mathbf{F}_i = \sum_i \frac{d\mathbf{p}_i}{dt} = \frac{d\mathbf{P}}{dt} \tag{11}$$

where $\sum_i \mathbf{p}_i = \mathbf{P}$, the total momentum of the system of particles. Combining Eq. 11 with Eq. 9, we find that

$$\boxed{\mathbf{F}_{ext} = \frac{d\mathbf{P}}{dt}.} \tag{12}$$

The entire system of particles obeys Newton's second law $\mathbf{F} = \frac{d\mathbf{p}}{dt}$, *as long as the force is taken to be the sum of the external forces, and the momentum is taken to be the total momentum of the system, which is the sum of the momenta of the individual particles in the system.*

One important consequence of this result is that it is not necessary to define exactly what is meant by a "particle". If $\mathbf{F} = \frac{d\mathbf{p}}{dt}$ applies for each particle in a system, then it applies also for the system as a whole, where only external forces are counted. So the system (a metal ball, for example) can also be thought of as a single particle, at least when finding its momentum change due to external forces. Likewise, the net external force on all the air molecules in a room is equal to the time rate of change of the total momentum of all of the molecules.

Sample Problem 6. Show that

$$\mathbf{F}_{\text{ext}} = M\mathbf{a}_{\text{CM}}$$

for a system of particles, where $\mathbf{F}_{\text{ext}}$ is the total external force on the system, $M = \sum_i m_i$ is the total mass of the system, and $\mathbf{a}_{\text{CM}}$ is the acceleration of the system's center of mass.

Solution: The position of the CM of a group of particles is defined to be $\mathbf{R}_{\text{CM}} = \dfrac{\sum_i m_i \mathbf{r}_i}{M}$, where $\mathbf{r}_i$ is the position of some mass m_i (see *Day Four.*) The derivative is

$$\mathbf{V}_{\text{CM}} \equiv \frac{d\mathbf{R}_{\text{CM}}}{dt} = \frac{\sum_i m_i \dfrac{d\mathbf{r}_i}{dt}}{M} = \frac{\sum_i m_i \mathbf{v}_i}{M} = \frac{\mathbf{P}}{M}.$$

The acceleration of the CM is a second derivative,

$$\mathbf{a}_{\text{CM}} \equiv \frac{d\mathbf{V}_{\text{CM}}}{dt} = \frac{1}{M}\left(\frac{d\mathbf{P}}{dt}\right) = \frac{\mathbf{F}_{\text{ext}}}{M}$$

using Eq. 12. That is,

$$\mathbf{F}_{\text{ext}} = M\mathbf{a}_{\text{CM}}$$

for the system of particles. Therefore if $\mathbf{F} = m\mathbf{a}$ works for each particle of a system, it also works for the system as a whole, if $\mathbf{F}$ is the net external force and the acceleration is that of the center of mass. This again means we can consider a system of particles to be a single particle, if we pretend the system is all located at the CM. This is easy to imagine for a ball bearing, but it is also true for a spinning tennis racquet, for a particular gallon of water flowing down a river, or for an exploding star.

Sample Problem 7. A set of N iceskaters ($N > 1$), each of mass m, stand in a circle of radius R on a frictionless ice rink. Each skater holds one end of a rope of negligible mass in each hand, so with N equal-length ropes the skaters are connected together as illustrated at the right for $N = 6$. The entire assembly is then caused to rotate about the center of the circle with angular speed ω. Find the tension in each rope.

Solution: (1) Including the rotational motion, the physical picture is as shown at the right. Note that the angle between two adjacent skaters is $\theta = 2\pi/N$. A diagram of any two adjacent skaters is drawn below. Note from the geometry of the triangles that the angle between a rope and a line that is perpendicular to the radius vector of a skater is half of θ, i. e., $\theta/2 = \pi/N$.

(2) An isolation diagram of a single skater is therefore as shown below.

(3) From the isolation diagram it is clear that the net *horizontal* force is zero. The net vertical force, i. e., the net force toward the center of the circle, is $2T\sin(\theta/2)$. The $F = ma$ equation is therefore $2T\sin(\theta/2) = 2T\sin(\pi/N) = ma$, where the inward (centripetal) acceleration is $a = R\omega^2$, since the skater is moving in a circle.

(4) We can now solve for the tension in any cord; the result is

$$T = \frac{mR\omega^2}{2\sin(\pi/N)}.$$

(5) The right-hand side has the correct dimensions of force. We can also check it in the special case $N = 2$: Then the formula gives $T = mR\omega^2/2$, which is correct, recalling that there are *two* ropes connecting the two skaters, so the total inward force acting on a skater is $2T = mR\omega^2$.

State of the Search

An object can only be in equilibrium if the net force on it is zero, where $\mathbf{F}_{net}$ is the vector sum of all the forces acting on the object.

If the net force on a particle of mass m is not zero, the particle accelerates according to Newton's second law $\mathbf{F}_{net} = m\mathbf{a}$. We apply the law as follows to a single particle or to a system of particles:

1. We draw a picture of the physical situation, labeling everything.

2. We draw a separate isolation diagram for each body of interest. That is, we draw a picture of one of the bodies, and the forces acting on that body only, using an arrow to designate the direction of each force. We include a double-arrow representing the positive acceleration direction for that body. We repeat for each body of interest.

3. We choose coordinate axes, and write out the component equations of $\mathbf{F} = m\mathbf{a}$ for each isolation diagram, where $\mathbf{F}$ is the vector sum of all the forces acting on the body.

4. We try to solve the equations.

5. Finally, we check that the solutions are reasonable.

When we pull on a taut string, a tension T develops within the string. The string exerts this tension force back on us. The tension can have any magnitude up to some maximum value, when the string breaks.

If an object pushes against a surface, the surface pushes back on the object, with a contact force or "normal force". If there is no friction between the object and the surface, the only force between them is this normal force $\mathbf{N}$, directed perpendicular to the surface.

For a system of particles, internal forces between particles of the system cancel by Newton's third law, the law of forces and counterforces. If $\mathbf{F}_{ext}$ is the vector sum of the external forces on the system, it follows that $\mathbf{F}_{ext} = \dfrac{d\mathbf{P}}{dt} = M\mathbf{a}_{CM}$, where $\mathbf{P}$ is the total momentum of the system, M is the total mass, and $\mathbf{a}_{CM}$ is the acceleration of the center of mass.

PICTURE QUIZ 6

Take the quiz; answers are given at the end of the text.

1. A spacetug rocket of mass M is hauling a spacewreck of mass m on the end of two cables, where each cable is aligned at angle $\theta = 30^o$ to the forward direction, as shown. If the spacetug has the forward acceleration a_o, the tension in each cable is

(A) $\frac{1}{\sqrt{3}}ma_o$ (B) $\frac{2}{\sqrt{3}}ma_o$ (C) $\frac{1}{2}ma_o$ (D) $\frac{1}{2}Ma_o$ (E) $\frac{1}{\sqrt{3}}(m+M)a_o$

2. Blocks of mass M and mass m are placed in contact with one another in the Bubble. We push on M with a force F_o, as shown at the right. Which of the following is a set of isolation diagrams sufficient to find the acceleration of the blocks and the force of contact N between them?

3. In the Bubble, a ball of mass m attached to a string is whirled around at constant speed v in a circular path of radius r, as shown at the right. Identify which of the following isolation diagrams is correct, and state what is wrong with each of the others.

4. Two stars, of mass m and mass $M = 2m$, form a double-star system. The stars are too far away from other stars to feel any appreciable force from them. Which one of the following sequences of pictures, drawn at equal time intervals, could represent the actual motion of the system? (The point O is a fixed point in some inertial frame.)

PROBLEM QUIZ 6

Answers are given at the end of the text.

1. A spacetug pushes a cargo container of mass $m = 25$ tonnes. If the acceleration of the container is $a = 0.60$ m/s^2, what is the force of contact (the normal force) of the spacetug on the container?

2. A spacetug pushes *two* cargo containers arranged one in front of the other, as shown below. The front container has mass m_1 and the rear container has mass m_2. The normal force of the tug on the rear container is N_0.

(a) Draw an isolation diagram of the two containers taken as a unit, showing all external forces acting on this system.
(b) Referring to the diagram, what is the net force on the system of two containers?
(c) Using Newton's second law, what is the acceleration of the system?
(d) Draw an isolation diagram of the front container by itself, showing all forces acting on that container.
(e) Using the diagram, Newton's second law, and the results of part (c), what is the normal force on the front container due to the rear container?
(f) Draw an isolation diagram of the rear container by itself, showing all forces acting on that container.
(g) Use the diagram, any or all of m_1, m_2, and N_0, and the result of part (c), to find the normal force on the rear container due to the front container.
(h) Compare the answers to parts (e) and (g). If the answers are the same, why should they be the same? If they are different, why should they be different?

3. Two small balls, each of mass m, are attached to two strings, each of length ℓ, in the manner shown. Keeping the strings straight and the balls in a line, the system is caused to rotate about the free end of the left-hand string with constant angular speed ω. Using the isolation diagram method, find
(a) the tension T_1 of the outer string (string 1)
(b) the tension T_2 of the inner string (string 2)

Special Topic: Rockets

What makes a rocket take off?

It is the force of burning fuel in the combustion chamber that pushes on the rocket. Said another way, the rocket accelerates because of a reaction force: As the exhaust is driven out the rear of the rocket, the exhaust pushes with an equal but opposite force back on the rocket.

The details of the motion are a bit tricky to work out, because the rocket's mass m keeps decreasing as fuel is burned and expelled. How do we use $F = ma$ when m is not constant ?

We start by simplifying the problem as much as we can. We suppose that the rocket is in essentially gravity-free empty space, so there is no force on it apart from the pressure of burning fuel in its combustion chamber. We also consider a short time interval during which the rocket expels a small amount of exhaust Δm, all of which emerges with essentially the same speed u relative to the ship. Beforehand the rocket has mass $M + \Delta m$ and is at rest; after expelling the exhaust, the rocket gains the small velocity Δv. Fig. S1 shows before-and-after pictures.

Fig. S1. A rocket just before and just after it has ejected a bit of exhaust Δm at speed u.

The momentum of the rocket is zero beforehand; after ejecting the exhaust the total momentum of the system is $M\Delta v - \Delta m\, u$, which must still be zero by momentum conservation. Therefore

$$M\Delta v = \Delta m\, u, \tag{S1}$$

so the rocket gains the speed $\Delta v = \dfrac{\Delta m\, u}{M}$ during the short time interval. The rocket gains momentum because the exhaust gains momentum in the opposite direction.

The ship keeps on ejecting bits of exhaust Δm, always at speed u relative to the ship, the total velocity of the ship becomes

$$v = \Delta v_1 + \Delta v_2 + \Delta v_3 + \ldots = \left[\frac{\Delta m}{M_1} + \frac{\Delta m}{M_2} + \frac{\Delta m}{M_3} + \ldots\right] u\,, \tag{S2}$$

accounting for the fact that the mass M keeps changing. Here $M_2 = M_1 - \Delta m$, $M_3 = M_2 - \Delta m = M_1 - 2\Delta m$, etc., where M_1 is the rocket's mass after the first ejection. The rocket gets a bigger velocity boost each time, because its mass keeps getting smaller.

Overall mass of the rocket plus exhaust is conserved, so the mass change of the rocket in each boost is $\Delta M = -\Delta m$; ΔM is negative because the rocket loses mass. The sum on the right of Eq. S2 is therefore

$$\sum_{i=1}^{N} \frac{\Delta m}{M_i} = -\sum_{i=1}^{N} \frac{\Delta M}{M_i} \tag{S3}$$

for N boosts. The ejection of exhaust is essentially continuous, however, so we should sum over an infinite number of infinitesimal boosts, replacing the sum $\sum_{i=1}^{N} \frac{\Delta M}{M_i}$ by the integral $\int_{M_0}^{M} \frac{dM}{M}$. We integrate from M_0 , the initial mass of the rocket, to M, the rocket's mass when it has achieved speed v. The integral is

$$\int_{M_o}^{M} \frac{dM}{M} = \ell n M\Big|_{M_o}^{M} = \ell n M - \ell n M_o = \ell n \frac{M}{M_o} = -\ell n \frac{M_o}{M}. \qquad \text{(S4)}$$

Therefore at any time the velocity v of the rocket (if it starts at rest) is

$$v = u \sum_{i=1}^{N} \frac{\Delta m}{M_i} = -u \int_{M_o}^{M} \frac{dM}{M} = u\, \ell n \frac{M_o}{M}, \qquad \text{(S5)}$$

where M is the total rocket mass at that time, M_o is the initial mass, and u is the exhaust speed relative to the rocket. If the rocket begins with velocity v_o, then its velocity at a later time is*

$$v = v_o + u\, \ell n \frac{M_o}{M} \qquad \text{(S6)}$$

The velocity of the rocket increases as its mass M decreases. A plot of v/u as a function of M_o / M (assuming $v_o = 0$) is the logarithmic function shown in Fig. S2.

Fig. S2. The velocity of a rocket as a multiple of the exhaust speed u, as a function of the mass ratio M_o/M.

*Equation S6 is sometimes called the *rocket equation.*

Sample Problem S1. A particular rocket expels its exhaust at the *constant rate* $\left|\frac{dM}{dt}\right| = \lambda$, with constant speed u relative to the rocket. As functions of time, find the rocket's mass M and its acceleration a.

Solution: The mass decreases with time according to $\frac{dM}{dt} = -\left|\frac{dM}{dt}\right| = -\lambda$, so $dM = -\lambda\, dt$, whose integral is

$$M = \int dM = -\int \lambda\, dt = M_0 - \lambda t\,,$$

where the constant of integration M_0 is the initial mass of the ship. The ship's acceleration is

$$a = \frac{dv}{dt} = u\frac{d}{dt}\left[\ell n\left(\frac{M_0}{M}\right)\right] = u\frac{d}{dt}\left[\ell n M_0 - \ell n M\right] = -\frac{u}{M}\frac{dM}{dt}$$

using the chain rule. Therefore

$$a = \frac{u\lambda}{M} = \frac{u\lambda}{M_0 - \lambda t}\,.$$

The denominator keeps getting smaller as time goes by, so the acceleration of this rocket keeps increasing until all the fuel is used up, as shown. The acceleration drops to zero once all the fuel is burned. The illustration assumes that fuel makes up 90% of the initial mass.

Sample Problem S2. Find the rate at which fuel must be burned if a rocket is to have the *constant acceleration* a_0.

Solution: If an object starts at rest and then accelerates uniformly, its velocity is $v = a_0 t$. The rocket equation then becomes

$$v = u\,\ell\mathrm{n}\frac{M_0}{M} = a_0 t\,.$$

Dividing by u and exponentiating each side, we find that

$$M = M_0 e^{-a_0 t/u}\,;$$

the mass of the ship decreases exponentially with time. Taking the time derivative,

$$\frac{dM}{dt} = -\frac{a_0}{u} M_0 e^{-a_0 t/u} = \left.\frac{dM}{dt}\right|_0 e^{-a_0 t/u}\,,$$

where $\left.\frac{dM}{dt}\right|_0$ is the initial rate of fuel burning, the rate at time $t = 0$. *Therefore the rocket accelerates uniformly if fuel is burned at an exponentially* <u>*decreasing*</u> *rate.*

We can find the acceleration of a rocket using the rocket equation

$$v = v_0 + u\,\ell\mathrm{n}\frac{M_0}{M} \equiv v_0 + u(\ell\mathrm{n}M_0 - \ell\mathrm{n}M)\,. \tag{S7}$$

Taking the time derivative, the rocket's acceleration a is

$$a = \frac{dv}{dt} = -\frac{u}{M}\frac{dM}{dt}\,. \tag{S8}$$

Multiplying by M and remembering that $\frac{dM}{dt}$ is negative (because the rocket is losing mass), we find that

$$Ma \equiv M\frac{dv}{dt} = u\left|\frac{dM}{dt}\right| \tag{S9}$$

where $\left|\frac{dM}{dt}\right|$ is the magnitude of $\frac{dM}{dt}$. Equation S9 looks like $F = ma$, if we identify the right-hand side as the force on the rocket. This force is called the *thrust*

$$\boxed{\text{Thrust} = u\left|\frac{dM}{dt}\right|,} \tag{S10}$$

which is due to the pressure of burning fuel pushing on the combustion chamber.

There is another interesting difference between Eq. 9 and the traditional $F = ma$ equation, in that the mass M of the rocket varies with time. The rocket problem is therefore an example of what is called a *variable mass* problem. Other examples of variable mass problems are given in Appendix I at the end of this text.

What happens if there are also external forces F_{ext} (like gravity or air resistance) on the rocket in addition to the thrust? We would expect that then

$$\boxed{Ma = u\left|\frac{dM}{dt}\right| + F_{\text{ext}},} \tag{S11}$$

adding these external forces F_{ext} to the thrust to get the total force on the rocket. This equation is correct for one-dimensional motion, as can be shown using techniques developed in Appendix I.

Questions and Problems from Day Six

6Q1. What experiments involving throwing a ball at a wall could be used to check that the force of the wall on the ball is a *normal* force?

6Q2. During *Day Six* we worked out the motion of balls with masses m_1 and m_2 connected to two springs, as shown. We drew the isolation diagram for m_2 as

shown at the right, with only a *single* force acting. However, shouldn't the stretch of the right-hand spring also affect m_2? For example, if Δx_1 is large, wouldn't m_2 accelerate more than it would if Δx_1 were small? So why don't we need to apply a *second* force to m_2, due to the right-hand spring?

6Q3. A bomb drifts along at constant speed through empty space, as viewed from the inertial frame of the Bubble. If it explodes into a million pieces, what is the motion of its CM?

6Q4. Two tug-of-war teams pull on a rope in opposite directions. According to the force-counterforce law, the force of the rope on one team is equal but opposite to the force of the rope on the other team, if the rope has negligible mass compared with the teams. So how can either team beat the other?

6Q5. David uses a cord of length ℓ for his slingshot. He puts a stone in a pocket on one end of the cord, and whirls it around his head. However, the cord breaks before he gets the stone up to the speed he would like it to have. Would he have achieved higher speeds if he had made the cord *shorter* or *longer*? If he wants to double the breaking speed, by how much should he shorten or lengthen the cord?

6Q6. A ball of mass m is attached to a string of fixed length r and a spring as shown. The other ends of the string and spring are held fixed. The ball is pulled to one side and released; it then swings back and forth in circular arcs like a pendulum, keeping the string taut. If F_S and T are the spring force and the string tension, draw the isolation diagram of the ball, assuming the spring is stretched.

6Q7. Motors for model rockets are typically solid-fuel (blackpowder) motors which can propel a rocket weighing a few ounces up to altitudes of 1000 ft or so. A typical curve of thrust *vs* time is shown. (a) Describe qualitatively how the rocket will behave in response to the thrust curve shown. (b) Estimate the total impulse generated by the motor, in Newtons· seconds.

6Q8. What is the advantage of a multistage rocket over a single-stage rocket?

6Q9. Sometimes parallel staging is used to propel rockets, in which booster rocket engines are attached to the main rocket engine, and then ejected after their fuel is expended. (Space-shuttle launches are an example.) The main rocket burns at the same time as the boosters do, as opposed to serial staging, in which the stages burn sequentially. What is the advantage of parallel staging over a single-stage rocket?

6Q10. Robert H. Goddard, former Professor of Physics at Clark College in Worcester, Massachusetts, did most of the U. S. work on rocket propulsion prior to World War II. What response might one give to the editor of the prestigious New York newspaper which published the following 1921 editorial?

> *"That Professor Goddard with his 'chair' at Clark College and the countenance of the Smithsonian Institution does not know the relation between action and reaction, and the need to have something better than a vacuum against which to react - to say that would be absurd. Of course he only seems to lack the knowledge ladled out daily in high schools."*

Section 6.1 Multiple forces

6P1. One end of spring 1, of force constant k and unstretched length ℓ, is attached to a wall at $x = 0$. One end of spring 2, of force constant $2k$ and unstretched length ℓ, is attached to a wall at $x = D$ $(D > 2\ell)$. If the free ends of the two springs are brought together and attached with a string, what is the coordinate x of the junction, if the springs are in equilibrium, in terms of any or all of k, ℓ, and D?

6P2. Three springs, all with the same force-constant k, are attached to a ball, which is in equilibrium. One of the springs pulls "upward", as shown; the other two pull at angles of $\theta = 30^\circ$ to the vertical, also as shown. If the stretch in the vertical spring is Δx, what is the stretch in each of the other two springs?

6P3. Three springs, each with the same unstretched length ℓ, are attached together at one end; the other ends are attached to walls. Spring 1 has force constant k, and springs 2 and 3 each have force constant $2k$. In equilibrium the springs are oriented 120° apart, as shown. If the stretch in spring 1 is Δx, what are the stretches in springs 2 and 3?

A. String tension

6P4. A string is attached to two springs as shown. The springs are stretched, and their other ends are attached to fixed points. The system is in equilibrium.

(a) If each spring has force-constant k and is stretched by Δx, what is the tension T in the string? (b) If instead the left and right-handed springs have force-constants k_1 and k_2, respectively, and the left-hand spring is stretched by Δx_1, what is the stretch Δx_2 of the right-hand spring? What is the tension in the string in this case?

6P5. Two springs pull on a ball, as shown. Spring 1, with $k_1 = 1.0$ N/m, is stretched $\Delta x_1 = 2.5$ cm, and spring 2, with $k_2 = 0.50$ N/m, is stretched $\Delta x_2 = 4.0$ cm. Spring 1 is oriented along the positive x-axis, and spring 2 is oriented along the positive y-axis. A string is also attached to the ball to keep it in equilibrium. What angle θ must the string have relative to the x-axis, and what is its tension T?

6P6. Two strings and one spring are attached to a mass m as shown. (a) If the spring has force-constant k and Δx is its stretch, what is the tension in each string if m remains at rest? (b) What is the tension in each string if the entire assembly is moving upward in the figure at constant velocity? (The spring stretch stays the same.) (c) What is the tension in each string if the entire assembly moves upward, with uniform acceleration a? (The spring stretch stays the same.)

6P7. A small ball of mass m is attached to a string of length ℓ, and the other end of the string is held fixed. (a) If the ball is given a tangential velocity v, what is ω, the angular speed of rotation? (b) If the string will break when its tension T exceeds T_0, what is the maximum tangential velocity v_0 which can be given to the ball without breaking the string?

6P8. One proposal for achieving artificial gravity for astronauts on long-distance spaceflights is to attach the spacecraft to an equal-mass counterweight, using a long, light tether, and then spin up the assembly about its center of the tether, as shown. An artificial gravity in the spacecraft is thereby established outward (away from the tether center), equal in magnitude to the spacecraft's centripetal acceleration. (a) What tether length ℓ is required if the assembly rotates once/minute and there is an artificial gravity of 10 m/s^2 within the spacecraft? (b) What is the tension T in the tether if the spacecraft's mass is 100 metric tonnes (1 tonne = 1000 kg)?

6P9. Let the spacecraft (and the counterweight) in 6P8 each have mass M, and let the rotation period be P and the spacecraft's artificial gravity be g. Suppose now that the tether has mass m, uniformly distributed over its length. (a) Find the tether length in terms of given parameters. (b) Find the tension within the tether as a function of the distance x from the tether's center. *Hint*: Draw an isolation diagram of a small piece of the tether, one end of which is at x and the other at $x + \Delta x$. The tension on the first end is T, and that on the other end is $T + \Delta T$. Then use $F = ma$.
(c) Where along the tether's length is the tension greatest? (This would be the spot where it is most likely to break.)

6P10. Starting at time $t = 0$, two teams of dogs start pulling in opposite directions on a dogsled of mass 88 kg, by means of light ropes. One team pulls to the left with the force 176 N, and the other pulls to the right with the force 220 N. (a) How fast will the sled be moving at the end of 10 s? (b) At the end of 10 s, the rope on the right breaks. At what time will the sled again come momentarily to rest?

B. Contact forces: the normal force

6P11. A string attached to a block is threaded through a hole in a wall; the other end of the string is attached to a spring of force constant k. The spring is pulled to the right so that the normal force between the block and wall is N. In terms of N, what is (a) the tension T in the string? (b) the stretch Δx in the spring?

6P12. A block is placed in a corner where two walls come together, and a spring of force-constant k is pushed against the block so that the spring is compressed by an amount d. The spring is oriented at angle $\theta = 30^o$ to one of the block faces, as shown. In terms of k and d, find (a) the normal force N_1 exerted on the block by wall 1 (b) the normal force N_2 exerted on the block by wall 2.

Section 6.3 Isolation diagrams

6P13. Two blocks, of mass m and mass M, are placed together, as shown. A spring of force constant k is pressed against m, and the compression Δx in the spring is kept constant. Use the five steps of the isolation diagram approach to answer the following: (a) What is the acceleration of the two blocks? (b) What is the normal force N between the two blocks?

6P14. Two balls with masses m_1, m_2 are pulled by two light springs k_1, k_2 so that both balls have the same constant acceleration a. Find Δx_1 and Δx_2, the stretches in each spring, in terms of the m's, the k's, and a. Show clearly all five steps of the isolation diagram approach.

6P15. Two blocks, of masses m_1 and m_2, are tied to opposite ends of a string, and the string is slipped over a frictionless peg in gravity-free space. Light springs with force-constants k_1 and k_2 are also attached to m_1 and m_2, as shown. Suppose k_1 is stretched by Δx_1 and k_2 by Δx_2. (a) Draw isolation diagrams for m_1, m_2, and for the string. (The sum of all the contact forces by the peg on the string is equivalent to a single "normal force" at the midpoint.) Then in terms of given quantities, find (b) the accelerations of m_1 and m_2, and (c) the tension in the string.

6P16. A chain is tied tightly between two trees and a force F_0 is applied at right angles to the chain at its midpoint, as shown in a view from above. (a) Consider the short length of chain shown upon which F_0 acts. What *other* forces act on this piece of rope? (b) Draw an isolation diagram for this short length of chain. (c) When the chain is in equilibrium, and the chain ends at an angle θ from the straight line between the trees, what is the tension in the chain in terms of F_0 and θ? (d) Based on the result, suggest a way to move a heavy object along the ground, using one tree and a chain.

6P17. Three balls, of masses m_1, m_2, m_3, are pulled by three springs k_1, k_2, k_3 as shown, so that all three masses have the same constant acceleration a. Find Δx_1,

Δx_2, and Δx_3, the stretch in each of the three springs, in terms of the m's, the k's, and a. Assume the springs have negligible mass compared with that of the balls. Show clearly all five steps of the isolation diagram approach.

6P18. Three boxes, with masses m_1, m_2, m_3, are pushed along a frictionless frozen lake by a force F_0, as shown. Find the magnitude and direction of

(a) the acceleration a of the three boxes, using an isolation diagram for the entire system. (b) the force F_{21} box 1 exerts on box 2. Use an isolation diagram. (c) the force F_{23} box 3 exerts on box 2. Use an isolation diagram.

6P19. Two balls in the Bubble, with masses m_1 and m_2, are attached by a string as shown. A second string is used to pull ball 1 with force F_0. The string masses are negligible.

Find the acceleration of the system and the tension in each string, in terms of m_1, m_2, and F_0, showing clearly all five steps of the isolation diagram approach.

6P20. In 6P19, suppose the string connecting ball 1 with ball 2 is instead a *cable* of length

L and mass M. The string attached to ball 1 alone still has negligible mass. Find the tension in the cable at points (a) A (b) B (c) C.

6P21. Three balls, with masses m_1, m_2, and m_3, are connected by light cords 1 and 2, as shown. The system is pulled to the right with a third cord having tension T_0,

connected to ball 1. If the tension in cord 1 is $T_0/2$, and the tension in cord 2 is $T_0/4$, find m_1, m_2, and m_3 in terms of T_0 and a, the acceleration of the system. Use isolation diagrams.

6P22. A chain consisting of four links, each of mass $m = 40$ g, is pulled in the Bubble with a force F_0 acting on the first link, as shown, so that the chain accelerates with $a = 0.5$ m/s^2.

Find the force (a) F_{21} exerted by link 1 on link 2 (b) F_{34} exerted by link 4 on link 3 (c) F_{23} exerted by link 3 on link 2. Also find (d) The force F_0, and (e) the net force on each link. Use isolation diagrams.

6P23. An astronaut with a jet-pack on his back pushes four toolboxes through space as shown. Each box has mass $m = 10$ kg; they all accelerate together toward the right with acceleration $a = 0.50$ m/s^2.
(a) Find the force F_{32} box 2 exerts on box 3.
(b) Find the force F_{12} box 2 exerts on box 1.

6P24. The starship *Enterprise* is towing a couple of broken-down spaceships, with masses m_0 and $2m_0$, by cables as shown. The *Enterprise* keeps the tension in the cable between itself and the first relic to be constant at T_0.

(a) Find the acceleration a of the two relics, in terms of m_0 and T_0. Start by drawing an appropriate isolation diagram. (b) Find the tension T_1 in the cable between the two relics, in terms of m_0 and T_0. Again, draw an appropriate isolation diagram. (c) When each ship has traveled a distance d, what will be the speed v of each, if they started at rest?

6P25. A space tug-ship is guiding three passenger spaceships, each of mass $m = 5.0 \cdot 10^6$ tonnes, through a field of asteroids. The tug is connected to the (unpowered) spaceships by cables of negligible mass, as shown; by pulling on the first cable it gives all the ships an acceleration $a = 3.0$ m/s^2. Find the tensions (a) T_1 in cable 1 (b) T_2 in cable 2 (c) T_3 in cable 3. (d) If cable 1 snaps after a time $t_1 = 2.0$ minutes, how far a distance D will each ship drift during the *next* 2.0 minutes, assuming the ships started at rest when the tug began to pull?

6P26. Three small balls 1, 2, and 3 are attached to cords a, b, and c as shown. They

all move in circular orbits of period P about the left-hand end of cord a. If each cord has length d, and their tensions are related by $T_a = 2T_b = 4T_c$, find the masses m_1, m_2, and m_3, in terms of P, d, and T_a. Use isolation diagrams.

6P27. Two space-pigs, of equal mass m, are attached to opposite ends of a light spring of natural length d and force-constant k. The pigs are circling their mutual center of mass with period P. (a) Draw an isolation diagram of one of the pigs. (b) Find the speed v of the pig, in terms of given parameters. (c) Find v (in terms of d, k, and m *only*, if the spring's length is $2d$.

6P28. The rims of two metal hemispherical shells are placed together to form a spherical shell, as shown. The air within is pumped out, so the hemispheres are pressed together by the air pressure outside, p = force/area = $1.01 \cdot 10^5$ N/m^2 (= 14.7 lb/in^2). (a) If the hemispheres have diameter $d = 1.00$ m, what force F does the right hemisphere exert on the left hemisphere? (*Hint:* Suppose the right hemisphere were hung by a rope, with a flat metal plate covering the left-hand side, as shown, so the air in it could be pumped out. What would be the net force on the flat plate due to the air outside? Therefore what would the net force on the hemispherical surface have to be, due to the air, if the apparatus were in equilibrium?) (b) Teams of horses pull on each hemisphere, as shown. If each horse can pull with force $F_0 = 4000$ N (about 900 lb), how many horses must be on each team to pull the hemispheres apart?

(c) Suppose one rope is attached to a (very) strong wall instead, as shown. How

many horses are needed in the single remaining team to pull the hemispheres apart?

6P29. Two balls, with equal masses m, are attached to two springs with equal force-constants k in the arrangement shown. The natural length of each spring is ℓ. The left-hand end of spring 1 is held fixed, while the entire system travels in a circle about this point with angular velocity ω. (a) Use the isolaion diagram approach to find the stretch Δx_1 in spring 1 expressed in terms of ℓ and the dimensionless quantity $q = m\omega^2/k$. (b) Assuming the springs are infinitely stretchable as ω is gradually increased, what is the smallest value of q for which the length of spring 1 goes to infinity?

6P30. Two balls with masses m_1 and m_2 are bound together by a spring of force constant k. Another spring of force constant k' is attached to m_1. If the entire system is at rest at time $t = 0$, and for $t > 0$ the spring k' is pulled so as to keep its stretch $\Delta x'$ constant, find the total momentum P of the system as a function of

time, and find the acceleration of the CM of the system. Assume the springs have negligible mass compared with m_1 or m_2.

6P31. A cloud of N electrons, each of mass m and electric charge q, are located in a section of a linear accelerator in which the electric field is $E = E_0 t/\tau$, where E_0 and τ are constants, and t is the time. The force on each electron is $F = qE$. If the center-of-mass of the cloud is initially at rest, how far does it move by time $t = \tau$?

6P32. A globular cluster is a spherical assembly of 10^5 stars, moving in the gravity of a galaxy. During some period of time the force on the ith star is $m_i\mathbf{g} + \mathbf{F}_{ic}$, where m_i is its mass, $\mathbf{g}$ is a constant vector, and $\mathbf{F}_{ic}$ is the vector sum of the gravitational forces on the ith star due to all other stars in the cluster. Find the acceleration $\mathbf{a}_{CM}$ of the center of mass of the cluster in terms of given quantities.

6P33. In *Sample Problem* 7, (a) show that all the internal forces cancel out in the system of skaters. (b) Find the total external force and verify that $\mathbf{F}_{ext} = d\mathbf{P}/dt$, where $\mathbf{P}$ is the total momentum of the system.

6P34. Consider *Sample Problem* 7 in the case of an infinite number of skaters, each of infinitesimal mass. This corresponds to a rotating loop of rope of radius R and uniform mass/length λ. Consider a pie-shaped slice of the rope of angle $\Delta\theta$ and containing mass $\Delta m = \lambda\Delta\theta$; then according to *Sample Problem* 7, $\Delta m\, R\omega^2 = 2\, T\sin(\Delta\theta/2)$. Find an expression for the tension T in the rope as a function of the loop's total mass M, radius R, and angular speed ω.

Special Topic: Rockets

6P35. Ninety percent of the initial mass of a rocket is in fuel. If the rocket moves in gravity-free, empty space, starting from rest, find its final velocity v if the rocket's exhaust speed u is (a) 3.0 km/s (typical chemical burning), (b) 1000 km/s, (c) $c/10$ (where c is the speed of light.)

6P36. Starting from rest in gravity-free space, a rocket achieves speed $v = 2.0$ km/s by the time its mass has dropped to half its initial value. What is the rocket's exhaust speed u in km/s, assuming it is constant?

6P37. A rocket in gravity-free, empty space accelerates uniformly with $a = 29$ m/s^2 (about 3 "g's" of acceleration.) Its exhaust velocity is $u = 2.5$ km/s. (a) What fraction of its initial mass is left at the end of 3.0 minutes? (b) How fast is the rocket moving at that time relative to its initial rest-frame?

6P38. A rocket has the exhaust velocity $u = 3$ km/s, a typical value for chemical propulsion. How long can the rocket accelerate at $g_{\text{eff}} = 10$ m/s^2 if the fuel comprises 90% of the original rocket mass? Assume the rocket is in gravity-free, empty space.

6P39. We want to travel for time $t = 6$ months with acceleration $a = 10$ m/s^2, with a final rocket mass of $M = 1000$ tonnes, (1 tonne = 1000 kg.)
(a) Using a chemical rocket with exhaust speed $u = 3160$ m/s, what must have been the original ship mass M_0? (b) Compare M_0 with the mass of the observed universe. (Observed matter has a density $\rho \approx 10^{-30}$ g/cm^3, and the observed universe has a radius $R \approx 10^{10}$ light-years. One light-year is the distance light travels in one year at the speed $c = 3 \cdot 10^8$ m/s). (c) Suppose instead we find a fuel whose exhaust can be ejected at $u = 3.16 \cdot 10^7$ m/s, about 10% the speed of light. What must the original mass have been in this case? (d) How fast is the ship moving at the end of six months? (e) How far has the ship gone? Compare with the distance to the nearest star (not counting the sun), about four light-years.

6P40. Consider a ship in gravity-free empty space with the fueled mass M_0 and exhaust velocity u equal to those of a Saturn V rocket (first stage): $M_0 = 3100$ tons $= 2.8 \cdot 10^6$ kg and $u = 2500$ m/s. The ship's acceleration remains at $a = 10$ m/s^2.
(a) Find the initial rate of fuel ejection $|dM/dt|_{t=0}$. (b) After how many minutes has the mass of the ship decreased to $1/e$ of its original value? (c) Suppose the ship accelerates as described for time $t = 20$ minutes. What percent of its original mass is left? How many kilograms is this? What is the ship's velocity at this time?

6P41. A fueled rocket starts from rest in gravity-free space. It has initial mass M_0 and burns fuel at the constant rate $|dM/dt| = \lambda$. The exhaust has constant speed u relative to the rocket. (a) Find the rocket's velocity v as a function of time t.
(b) Sketch $v(t)$. Beyond what v (or t) does your sketch become invalid?

6P42. Two shotputters and a rocket scientist, each of mass M, are at rest relative to one another in space. (a) The first shotputter holds a shot of mass m, which he "puts" so that afterwards it moves at speed v relative to his *final* speed v_f. Show that $v_f = mv/(M + m)$. (b) The second shotputter holds two shots, each of mass $m/2$. He puts them one at a time in the same direction, so that *after each put* the relative speed between him and the shot is v. What is his final speed v_f in terms of m, M, and v? Is this speed equal to, greater than, or less than the final speed of the first shotputter? Explain. (c) The rocket scientist holds a rocket of negligible mass, containing fuel of mass m, which exhausts it fuel relative to the rocket at speed v. What is her final speed v_f ? (d) Evaluate the three answers for the case $M = 75$ kg, $m = 15$ kg, $v = 12$ m/s. Which method is most efficient for gaining velocity?

6P43. A gun of mass M sitting in gravity-free empty space contains 100 bullets, each of mass m ($m << M$). If the gun automatically starts firing bullets one at a time, each with speed u relative to the gun, find the approximate speed v of the gun after it has shot off all its bullets. Explain your reasoning.

6P44. A rocket of mass M is propelled by an on-board laser, which shoots n_γ photons per second out the rear of the rocket. Each photon has momentum $p_\gamma = h\nu/c$, where h is Planck's constant, ν is the frequency of the photon, and c is the speed of light. The mass of the rocket stays constant to a good approximation. (a) Find the acceleration a of the rocket in terms of M, n_γ, and p_γ, assuming p_γ = con. (b) Find the velocity v of the rocket as a function of time, if it starts at rest.

6P45. A small spaceship of mass M is propelled by light fired continuously from a laser on the moon. A mirror covers the rear of the rocket; light from the laser strikes it and bounces directly back. In the laser's frame of reference, n_γ photons are fired per second, each with momentum p_γ. (a) Show that in the small time interval Δt, the change in the rocket's momentum is $2\, n_\gamma' p_\gamma'$, where n_γ' is the number of photons striking the mirror per second, and p_γ' is the momentum of each photon in the ship's frame of reference. (b) The photons are Doppler-shifted in the ship's frame, so their frequency is only $\nu' = \nu(1 - v/c)$, where v is the velocity of the ship. The momentum of each photon is therefore shifted as well (see 6P44). (c) Show also that $n_\gamma' = n_\gamma(1 - v/c)$, and then show that the ship's acceleration is $dv/dt = \alpha\,(1 - v/c)^2$, where α is a constant. Express α in terms of M, n_γ, and p_γ. (d) Find the ship's velocity v as a function of time.

6P46. Consider a two-stage rocket as shown. The first stage of mass $M_1 = M_{1c} + M_{1f}$ consists of a casing with mass M_{1c} containing fuel of mass M_{1f}; the second stage of mass $M_p + M_{2c} + M_{2f}$ consists of a payload of mass M_p and a casing of mass M_{2c} containing fuel of mass M_{2f}. The entire initial mass of the rocket including fuel is M_o. (a) Show that at the end of the first-stage burn the rocket (*accelerating in gravity-free space*) achieves the velocity

$$v_1 = u\,\ell n \frac{M_o}{M_o - M_{1f}}.$$

(b) At the end of the first-stage burn the first-stage casing is disconnected and the second stage is ignited. Show that the second stage achieves the *additional* velocity

$$v_2 = u\,\ell n \frac{M_o - M_1}{M_o - M_{1c} - M_f}, \text{ where } M_f = M_{1f} + M_{2f}.$$

(c) Show that the final velocity $v_f' = v_1 + v_2$ obeys the condition

$$v_f' - v_f = u\,\ell n \frac{(M_o - M_1)(M_o - M_f)}{(M_o - M_{1f})(M_o - M_{1c} - M_f)}$$

where v_f is the velocity the rocket would have achieved if it were a single-stage rocket of total mass M_o, including fuel mass M_f.

(d) Prove that if $M_{1c} > 0$, as of course it is for a real two-stage rocket, that $v_f' - v_f > 0$. That is, prove that a two-stage rocket achieves a greater final velocity than a single-stage rocket with the same amount of fuel and with the same total mass.

(e) For the special case $M_{1f} = M_{2f} = 0.4\,M_o$, and $M_{1c} = M_{2c} + M_p = 0.1\,M_o$, find v_f'/v_f, the factor by which the two-stage velocity beats a single-stage velocity.

(f) What is the physical reason why a two-stage rocket is more efficient?

Day Seven

This week we have begun to understand how objects move and why they move as they do. The motion of objects is always relative to the observer, so we started out by identifying the arenas in which the game looks the simplest, the inertial frames of reference.

> *Inertial frames are those frames of reference in which isolated bodies stay at rest if they start at rest, or travel in a straight line at constant speed if they are set moving.*

This assumes of course that there is no net force on the bodies. We also found that

> *inertial frames do not accelerate (and in particular, they do not rotate) relative to the distant stars.*

Needless to say, we were not the first to discover these things! For example, the "law of inertia" was stated by Galileo Galilei (1564 - 1642) in *Two New Sciences*: "Furthermore we may remark that any velocity once imparted to a moving body will be rigidly maintained as long as the external causes of acceleration or retardation are removed ...". In *Principia Mathematica*, Isaac Newton (1642 - 1727) stated it this way, as his first law: "Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it."

The transformation $x' = x - Vt$, $y' = y$, $z' = z$ between inertial frames of reference, with relative translation at velocity V in the x direction, and with the primed frame moving to the right relative to the uinprimed frame, has been used implicitly or explicitly for centuries. It was named the "Galilean Transformation" by Albert Einstein (1879 - 1955) in the early twentieth century.

The corresponding transformation between the *velocity* components of objects viewed in the primed and unprimed frames is

$$v'_x = v_x - V \qquad v'_y = v_y \qquad v'_z = v_z .$$

In vector form the position and velocities transformations are simply

$$\mathbf{r}' = \mathbf{r} - \mathbf{V}t \qquad \text{and} \qquad \mathbf{v}' = \mathbf{v} - \mathbf{V} .$$

The German physicist and philosopher Ernst Mach (1838 - 1916) went beyond the simple observation that inertial frames do not rotate relative to the distant stars, by suggesting that "rotation" is meaningless except in relation to observed matter. According to a version of "Mach's Principle" (again a name due to Einstein) matter on the large scale causes inertia; some unidentified influence of the stars determines which local frames of reference are inertial, namely those frames which are unaccelerated relative to the stars. Is this true? We don't know.

We defined the mass m of an object by collision experiments.

We defined the momentum $\mathbf{p}$ of an object to be $\mathbf{p} = m\mathbf{v}$, the product of the object's mass and velocity.

> *We found that the total momentum of a system of objects is conserved in collisions.*

> *We found that the momentum of any isolated system (i. e., a system isolated from outside influences) is conserved.*

We argued that the total impulse $\mathbf{I}$ acting on an object is responsible for any change in the object's momentum, according to $\mathbf{I} = \Delta\mathbf{p}$.

> *We found that if object a* exerts *an impulse $\mathbf{I}_{ba}$ on object b, then object b simultaneously exerts an equal but opposite counterimpulse $\mathbf{I}_{ab}$ on object a; that is, $\mathbf{I}_{ab} = -\mathbf{I}_{ba}$.*

We also discovered that the concept of *force* is very useful.

> *We defined force F by spring stretch, using $F = k\Delta x$. The force exerted by a spring is opposite to its stretch direction, so the formula is often written $F = -k\Delta x$.*

We showed that the force **F** *on a body is equal to the time rate of change of the impulse* **I** *on the body; i. e.,* $\mathbf{F} = d\mathbf{I}/dt$.

We discovered the law $\mathbf{F} = d\mathbf{p}/dt = m\mathbf{a}$ *for the motion of a body, and showed that it works in several special cases;*

We discovered that the "**F**" in $\mathbf{F} = m\mathbf{a}$ *is the vector sum of the forces on m.*

We showed that for every force there is an equal but opposite counterforce. That is, if body a exerts a force on body b, body b exerts an equal but opposite force on body a.

We found that if a body moves in a circular path of radius R at speed v, it is accelerating toward the center of the circle with the centripetal (i.e., inward) acceleration

$$a_{\text{centripetal}} = v^2/R = R\omega^2,$$

where ω *is the angular velocity of the body,* $\omega = d\theta/dt$, *the rate at which it sweeps out its angle in the circle measured from the center. The speed* v *and angular speed* ω *of the body are related by* $v = R\omega$.

The laws of mechanics are often summarized by *Newton's Laws*:

(I) If no forces act on a body, the body continues in its state of rest, or of uniform motion in a straight line.

(II) The net force **F** is responsible for momentum change, through $\mathbf{F} = d\mathbf{p}/dt$. (or $\mathbf{F} = m\mathbf{a}$).

(III) For every action there is an equal but opposite reaction. (In the interaction between two bodies, the forces are equal but opposite.)

We have taken a particular point of view:

> *The "law of inertia" is primarily a definition of inertial frames. It is only a "law of nature" insofar as it implies the existence of inertial frames, frames in which bodies move in straight lines at constant speed if no forces act. The "law of inertia" is* not *just a special case of* $\mathbf{F} = m\mathbf{a}$ *corresponding to* $\mathbf{F} = 0$, *because we first have to use the "law of inertia" to be sure we are at rest in some inertial frame. Only then are we allowed to use* $\mathbf{F} = m\mathbf{a}$.
>
> *The relationship* $\mathbf{F} = m\mathbf{a}$ *is an experimental law, which means that it is falsifiable. That is, we believe that although it is true that in all of the experiments we have done so far the acceleration of a particle is given by* $\mathbf{a} = \mathbf{F}/m$, *where* $\mathbf{F}$ *is the net force acting, this law could always be shown to be false if we make extremely precise measurements, or if we apply it to some situation we haven't tried before.*
>
> *We showed that Newton's third law is equivalent to the conservation of momentum, because* $\mathbf{F} = d\mathbf{p}/dt$). *Therefore it is also falsifiable. If we find an interaction for which momentum is not conserved, then Newton III will be violated as well.*
>
> *So taken together, we think that Newton's first law is primarily a definition of inertial frames, and Newton's second and third laws are experimental laws based upon our experiments so far, but might not agree with all experiments. They are therefore provisional, only as good as the precision and generality of the experiments that support them.*

The method of *isolation diagrams* is an organized approach to finding the laws of motion for systems of objects. The steps in using the isolation diagram approach are as follows.

1. *Draw a physical picture of the system, labeling everything.*

2. *Draw a separate isolation diagram for each body of interest. That is, draw a picture of one of the bodies, and the forces acting on that body only, using an arrow to designate the direction of each force. Include a double-arrow to represent the positive acceleration direction of that body. Repeat for each body of interest.*

3. *Choose coordinate axes, and write out the component equations of* $\mathbf{F} = m\mathbf{a}$ *for each isolation diagram, where* $\mathbf{F}$ *is the vector sum of all the forces on the body.*

4. *Try to solve the equations for the unknowns.*

5. *Check that the solutions are reasonable.*

Suppose we have found the net force $\mathbf{F}$ on an object of mass m. How do we find the motion of the object?

The first step is to find the object's acceleration, which is simply $\mathbf{a} = \mathbf{F}/m$.

We can then try to find the velocity $\mathbf{v}$ of the object, by integrating the acceleration $\mathbf{a} = d\mathbf{v}/dt$. That is,

$$\mathbf{v}(t) = \mathbf{v}_0 + \int_0^t \mathbf{a}(t)\, dt\,.$$

However, this is only possible to carry out if $\mathbf{a}$ *depends only on time, i. e., if the net force* $\mathbf{F}$ *depends only on time, and not on the position or velocity of the object.*

If we have managed to find $\mathbf{v}(t)$, we can then in principle find the position of the object by integrating $\mathbf{v} = d\mathbf{r}/dt$. That is,

$$\mathbf{r}(t) = \mathbf{r}_0 + \int_0^t \mathbf{v}(t)\, dt\,.$$

If (*and only if*!) the net force $\mathbf{F}$ (and therefore the acceleration) is *constant*, the integrals are especially easy to do. In that case, for one-dimensional motion, we find that

$$v = v_0 + a_0 t, \qquad x = x_0 + v_0 t + (1/2)\, a_0 t^2$$

$$x = x_0 + [(v + v_0)/2]\, t \qquad v^2 = v_0^2 + 2a_0 (x - x_0)\,.$$

A week without gravity has helped us discover all of these things. Gravity is an important part of our world, however, so next week we will have to learn to deal with it!

Next week we will also discover techniques that can often be used to find the motion of an object even when the net force $\mathbf{F}$ depends upon the object's position or velocity.

SELF QUIZ 7

A review quiz, with answers at the end of the text

1. An airplane pilot wants to fly to an airfield a distance 80 km due south. There is a wind blowing from west to east at speed 60 m/s, and the plane flies at speed 100 m/s relative to the air. How long (in seconds) does it take the plane to reach the airfield?

2. Tsunamis are waves in the ocean caused by earthquakes. They are often called "tidal waves" when they reach the shore (although they have nothing to do with tides) and can be very destructive. The velocity v of these waves in the open ocean depends only upon the acceleration of gravity g and the depth of the ocean h. Suppose such a wave begins traveling horizontally with velocity v_0 at a location where the ocean has depth h_0. If the wave eventually reaches a location where the ocean depth has increased to $2h_0$, what now is the velocity of the wave in terms of v_0?

3. Two point particles with masses m_1 and $4m_1$ are separated by a distance L. How far is their center of mass from m_1 ?

4. A space traveler of mass m leaves her spaceship with relative velocity v_0. After drifting away for a time T, she is struck by a stray spacerock of mass $m/3$ moving directly toward her at velocity $v = 5v_0$ *relative to her*. She catches the rock and holds on to it. (a) What impulse does the rock exert upon her? (b) In terms of T, how long after catching the rock does it take her to drift back to the spaceship?

5. The position of a particle of mass m between times t_1 and t_2 (both > 0) is given by $x = c\,\ell n(t/t_0)$ where c and t_0 are constants. Find the net force on the particle as a function of time between t_1 and t_2, in terms of given constants.

6. A team of dogs pulls three sleds across a frozen lake in the Yukon. The sleds have masses $m_1 = 100$ kg, $m_2 = 150$ kg, and $m_3 = 250$ kg, and are separated by light cables as shown below. The surface is frictionless as far as the sleds are concerned, although the dogs are able to pull on the first sled with a force $F = 100$ N.

(a) What is the acceleration a of the sleds? (b) What is the tension T_1 between the first and second sleds? (c) What is the tension T_2 between the second and third sleds? (d) The dogs and sleds are at rest at time $t = 0$; the dogs then begin to pull as described above, until time $t = 1.0$ min. At this time the ice becomes so slick that the dogs also slide without experiencing any horizontal force, so there is no more acceleration. What is the total distance the dogs have moved from $t = 0$ to $t = 2.0$ min?

ICEWORLD

We wake up feeling different somehow. The difference becomes painfully obvious when one of us rolls out of bed and falls on the floor. Gravity is back in full force.

Light streams in through a window of our large rectangular room. We drag ourselves over to get a better look. Through the thick panes we see a rather small bluish-white sun not far above the horizon, and a black sky studded with thousands of stars, signifying an airless planet. The landscape outside is entirely *ice*, hills and valleys of smooth, white ice everywhere. It looks frigid and forbidding.

Day Eight

A Matter of Gravity

"I've gone where the hand of man has never set foot." - attributed to Samuel Goldwyn

8.1 The force of gravity and *weight*

There is a new force here. *Gravity* on this icy world is very different from the forces we studied in the Bubble: It somehow reaches out and grabs things without touching them. How does it do that?

At least we can measure its strength: We can hang one of the 1-kg balls on a spring and see how much the spring stretches. If the ball is in equilibrium the force $F_{\text{spring}} = k\Delta x$ *upward* due to the spring has to equal the gravitational force F_{grav} *downward.*

The standard springs we used in the Bubble are too fragile to use here: They stretch so much when we hold them vertically that they cannot support a ball. We have stiffer springs, however, which we can calibrate using the fragile springs we have used before. We calibrate stiffer and stiffer springs in turn until we find one with a force constant $k = 50$ N/m. This spring withstands the pull of gravity when we attach a 1-kg ball to one end and hang the other end from a hook on the ceiling. The spring stretches a distance $\Delta x = 20$ cm, so the gravitational force downward must be $F_{\text{grav}} = 10$ N, matching the force upward due to the stretched spring, $k\Delta x = (50\text{N/m})(0.20\text{m}) = 10$ N.

When we hang *two* 1 kg balls on the spring it stretches twice as much. Other experiments confirm that the gravitational force is directly proportional to the mass m of the hanging ball, so we can write the gravitational force in the form

$$\boxed{\mathbf{F}_{\text{grav}} = m\mathbf{g}} \tag{1}$$

where $\mathbf{g}$ is a proportionality constant, independent of the mass.* This constant $\mathbf{g}$ has to be a vector, because it has a direction (*downward*) as well as a magnitude.

We call $\mathbf{g}$ the *gravitational field.* It seems to exist throughout the space around us, and to have the same value everywhere. Wherever we hang up the spring, the stretch is the same.

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* It is interesting that the force of gravity is proportional to the mass m, the quantity we defined during *Day Three* using collision experiments, and the quantity in Newton's second law $\mathbf{a} = \mathbf{F}/m$ that is a measure of an object's "sluggishness", its reluctance to accelerate under a given force $\mathbf{F}$. Why should gravity pull on this quantity m? We return to this question during *Day Fifteen.*

Our experiments have found that the gravitational field $\mathbf{g}$ here on Iceworld has the magnitude

$$g = \frac{F_{\text{grav}}}{m} = \frac{10\ \text{N}}{1\ \text{kg}} = 10\ \text{N/kg}.^{*}$$

A scale stands in one corner. It registers a number when we step on it, our *weight* in Newtons. How does our weight compare with the gravitational force acting on us?

We remain at rest, so the gravitational force $\mathbf{F}_{\text{grav}} = m\mathbf{g}$ acting downward on us must be balanced by an upward normal force $\mathbf{N}$ due to the scale; that is, $\mathbf{N} = -m\mathbf{g}$. The scale pushes up on us, so we must be pushing down on the scale with an equal but opposite force $+\ m\mathbf{g}$, according to the force-counterforce rule (Newton's third law.)

How does the scale work? The top of the scale is supported by some springs as shown. When we press downward on the scale the springs are compressed by a distance proportional to mg. The scale has been calibrated, so it registers a number proportional to this spring compression.

We define the magnitude of our weight to be the reading of the calibrated scale that supports us. We also define the direction of weight to be "downward", in the direction in which we push on the scale when we stand on it.

As we reasoned above, when we are standing at rest in gravity g our weight must have the magnitude $W = mg$.**

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* The gravitational field at the surface of Earth is $g = 9.8$ N/kg, 2% weaker than on Iceworld.

** So what's the difference between weight and the force of gravity $\mathbf{F}_{\text{grav}} = m\mathbf{g}$? As we will see later, our weight is only $m\mathbf{g}$ when we are not accelerating up or down. If we are freely falling, for example, we are actually *weightless*! See *Sample Problem* 7.

8.2 Vertical motion

Does $\mathbf{F} = m\mathbf{a}$ apply if the only force is the force of gravity? We start by dropping a ball from rest. Gravity is the only important force on it because air resistance should be negligible for short times (for longer times the speed of the ball might get so large that air resistance would become important.) Interestingly, the "m's" cancel out in $\boldsymbol{F} = m\boldsymbol{g} = m\boldsymbol{a}$, so the law predicts that $\boldsymbol{a} = \boldsymbol{g}$. Therefore the ball should accelerate *downward*, with constant acceleration 10 N/kg = 10 m/s^2. To compare this prediction with experiment we can use the rules for uniformly-accelerated motion worked out last week to compare positions, velocities, and times.

Take the positive y-direction to be upward, as illustrated in Fig. 1. Then the acceleration is $a = -g$ where g is positive. The uniform acceleration equations 9a-d of *Day Five* then become

$$v = v_0 - gt \tag{2a}$$

$$y = y_0 + v_0 t - \frac{1}{2}gt^2 \tag{2b}$$

$$y = y_0 + \left(\frac{v + v_0}{2}\right)t \tag{2c}$$

$$v^2 = v_0^2 - 2g(y - y_0) \tag{2d}$$

Equation 2b, for example, predicts that if we drop a ball from rest at height $y_0 = h$ its altitude at time t will be $y = h - (1/2)gt^2$, as illustrated in Fig. 1.

Fig. 1. The altitude of a dropped ball as a function of time.

The time it should take to reach the floor is $t = \sqrt{\frac{2h}{g}}$, which we can test experimentally. Repeated measurements agree with the prediction so $\mathbf{F} = m\mathbf{a}$ works in these experiments for the gravitational force $\mathbf{F} = m\mathbf{g}$ acting alone.

Note that if gravity is the only force acting, that is, if $\mathbf{F} = \mathbf{F}_{grav} = m\mathbf{g} = m\mathbf{a}$, *then every object we drop, regardless of its mass, falls at the same rate, a uniform "acceleration of gravity"* $\mathbf{a} = \mathbf{g}$.

Sample Problem 1. Drop three items simultaneously, a large 1-kg ball, a small ball bearing, and a feather, from the same altitude in the room. In what order will they reach the floor?

Solution: If only gravity acts, the acceleration of all objects is the same, regardless of size, shape, or composition. There is air in our room, however, so gravity is not the only force. Air resistance is not very important for the two balls during the short time they fall, since for low speeds, the gravitational force on them is much stronger than air resistance. The two balls, of very different mass and size, should therefore reach the floor almost simultaneously. Air resistance *is* important for feathers, however, because they can be quite large even though their masses are small, so air resistance is appreciable while the gravitational force on them is weak. Inside the air-filled room the two balls will therefore reach the floor at about the same time and the feather will arrive later. Outside, on this airless planet, they should all reach the ground simultaneously.

Sample Problem 2. Throw a ball upward with initial velocity v_0 from an altitude y_0 above the floor. (a) At what altitude does the ball come instantaneously to rest? (b) What is its speed just before it hits the floor?

Solution: (a) Equation 2(d), $v^2 = v_0^2 - 2g(y - y_0)$, shows that $v = 0$ when $y - y_0 = v_0^2/2g$. This is the altitude above the starting point at which the ball comes to rest. (b) When it reaches the floor the altitude of the ball is $y = 0$; the speed at this point is therefore the square root of $v^2 = v_0^2 + 2gy_0$. [For example, if $g = 10$ m/s^2 and we throw the ball up with speed $v_0 = 10$ m/s from an altitude $y_0 = 15$ m, then the altitude where the ball comes to rest is $y = y_0 + 5$ m $= 20$ m, and it has speed $v = \sqrt{10^2 + 2(10)15}$ m/s $=$ 20 m/s when it reaches the floor.]

8.3 Projectile motion

Now we should test the full vector nature of the law $\mathbf{F} = m\mathbf{a}$. We can do that by throwing a ball horizontally, because the ball subsequently moves both horizontally and vertically. If gravity is the *only* force then the acceleration should be purely vertical if $\mathbf{F} = m\mathbf{a}$ is right. Let the y direction be vertical, positive upward, and let the x direction be the direction in which we throw the ball. Then $\mathbf{F} = m\mathbf{a}$ splits up into the component equations

$$F_x = ma_x = 0 \qquad F_y = ma_y = -mg. \tag{3}$$

In the horizontal (x) direction there is no acceleration, so the velocity in this direction stays constant. (In the other horizontal direction (z) there is no acceleration either, so if the ball starts with no velocity in the z direction there will never *be* any velocity in the z direction. The trajectory takes place in a *plane*, the x, y plane.) The velocity component equations are

$$v_x = v_{ox} \qquad v_y = v_{oy} - gt, \tag{4}$$

and the corresponding position equations are

$$x = x_o + v_{ox}\,t \qquad y = y_o + v_{oy}t - \frac{1}{2}gt^2. \tag{5}$$

We can also use the uniform-acceleration result

$$v^2 = v_o^2 - 2g(y - y_o) \tag{6}$$

where the velocities v and v_o can be either the y-components alone, or the total velocities of the ball. It doesn't matter, because the total velocity squared is the sum of squares $v^2 = v_x^2 + v_y^2$, and the x-components alone obey $v_x^2 = v_{ox}^2$ according to the first of Eqs. 4.

Equations 4 and 5 for v_y and y would be exactly the same if we just *dropped* the ball, without giving it any sideways velocity at all. In other words, the motion in the y direction is completely independent of the motion in the x direction. *The y motion doesn't know what the x motion is doing!*

The time it takes the ball to fall from height h, assuming that it has no initial velocity in the vertical direction, is the same whether the ball is given an initial horizontal velocity or not.

Elimination of the time t between Eqs. 5 in the special case $x_o = 0$, $v_{oy} = 0$, (i.e., if we throw the ball horizontally) yields

$$y = h - \frac{gx^2}{2v^2_{ox}} \tag{7}$$

where y_o is the height h; this equation shows the *shape* of the trajectory. One coordinate (y) is to the first power and the other (x) is to the second power, which means the shape is a *parabola* as shown in Fig. 2(a). The angle of the trajectory at any point can be found by comparing v_x with v_y at that point, as shown in Fig. 2(b). The angle of the path from the horizontal satisfies $\tan\theta = v_y/v_x$, where v_y and v_x are the velocity components at that point.

Fig. 2. (a) The shape of a trajectory. (b) The angle from the horizontal, using velocity components v_x and v_y. Note that $\tan\theta = v_y / v_x$.

Instead of throwing the ball horizontally, we can also throw it upward at angle θ_o relative to the ground. Then the initial velocity components are $v_{ox} = v_o\cos\theta_o$ and $v_{oy} = v_o\sin\theta_o$. Choosing $x_o = 0$, the first Eq. 5 gives

$$x = v_{ox}\, t = v_o\cos\theta_o\, t \;; \tag{8}$$

$$x = v_{0x}\, t = v_0 \cos\theta_0\, t \;; \tag{8}$$

the ball moves at constant speed $v_0 \cos\theta_0$ in the x-direction. Choosing also $y_0 = 0$, the second Eq. 5 gives

$$y = v_{0y}\, t - (1/2)gt^2 \,. \tag{9}$$

If we solve Eq. 8 for t and substitute it into Eq. 9, we get a shape equation

$$y = (\tan\theta_0)x - \left(\frac{g}{2v_0^2\cos^2\theta_0}\right)x^2 \tag{10}$$

which is still a parabola, now with a linear term in x as well as a quadratic term. If we could turn off gravity g the quadratic term would go away, leaving only the linear term, which gives the straight path shown in dotted lines in Fig. 3. The quadratic term represents how far the ball has fallen. This distance $\frac{1}{2}gt^2$ it falls relative to the straight-line path is the same distance a ball falls from rest in time t.

Fig. 3. A ball thrown upward at angle θ_0. The distance it falls below the straight dotted line is the distance it would fall from rest in the same time.

Sample Problem 3: The Range of a Projectile

(a) What is the range R of a projectile launched with speed v_0 and elevation angle θ_0?

(b) For what angle θ_0 is this range maximized?

Solution: (a) The range of a projectile is the horizontal distance it travels before it strikes the ground. The only parameters this distance can depend upon are the initial speed v_o, gravity g, and the elevation angle θ_o. Using dimensional reasoning, the only way we can find a distance in terms of a velocity (with dimensions L/T), gravity (with dimensions L/T^2), and a dimensionless angle, is to let

$$R = (v_o^2/g)\,f(\theta_o),$$

where $f(\theta_o)$ is some unknown function of the elevation angle. This unique combination of v_o and g eliminates the dimension of time on the right, while providing the dimension of length needed.

To find $f(\theta_o)$, we need to work harder. From Eq. 9, the height y of the projectile is *zero* if x obeys the equation

$$0 = (\tan\theta_o)x - \left(\frac{g}{2v_o^2\cos^2\theta_o}\right)x^2 = x\left[\tan\theta_o - \left(\frac{g}{2v_o^2\cos^2\theta_o}\right)x\right].$$

Therefore, either $x = 0$, or else

$$x = \tan\theta_o\,\frac{2v_o^2\cos^2\theta_o}{g} = \frac{v_o^2}{g}\,2\sin\theta_o\cos\theta_o = \frac{v_o^2}{g}\sin 2\theta_o,$$

using the trig identity $\sin 2\theta_o = 2\sin\theta_o\cos\theta_o$.

The solution $x = 0$ corresponds to the launching point, and the other solution is the point where the projectile returns to Earth. The range is therefore

$$\boxed{R = \frac{v_o^2}{g}\sin\left(2\theta_o\right).}$$

This makes sense: It depends upon v_o and g as expected from dimensional reasoning, Also, the range *increases* as v_o *increases* or as g *decreases*. The range of a projectile *should* be greater if the muzzle velocity is higher, and on Mars the range should be greater than on Earth for given v_o and θ_o, since g is smaller on Mars.

(b) For given v_o and g, for what launch angle θ_o is R maximized? The answer is clearly $\theta_o = 45^0$, because $\sin(2\cdot 45^0) = \sin 90^0 = 1$, the largest value the sine function can have. In the absence of air resistance, the elevation angle for maximizing the range of a projectile is $\theta_o = 45^0$, and the corresponding range is $R = v_o^2/g$.

Sample Problem 4. A pitcher throws a fastball with horizontal velocity $v = 100$ mi/hr toward home plate, a distance $d = 60$ ft away. (a) How long does it take the ball to travel halfway to the batter? (b) How far does the ball fall during this time? (c) How far does the ball fall during the second half of its flight? (d) Why aren't the answers to (b) and (c) the same? Pretend there is no air resistance, and use $g = 32$ ft/s^2.

Solution: (a) The horizontal velocity of the ball is

$$100\ \frac{\text{mi}}{\text{hr}} = \left(100\ \frac{\text{mi}}{\text{hr}}\right)\left(\frac{5280\ \text{ft}}{1\ \text{mi}}\right)\left(\frac{1\ \text{hr}}{3600\ \text{s}}\right) = 147\ \frac{\text{ft}}{\text{s}},$$

so the ball travels 30 ft in time $t = \dfrac{30\ \text{ft}}{147\ \text{ft/s}} = 0.20$ s.

(b) The vertical (y) motion undergoes constant acceleration $a = g$, if we choose the positive direction to be downward; the ball therefore falls a distance

$$y = (1/2)gt^2 = (1/2)\,(32\ \text{ft/s}^2)\,(0.20\ \text{s})^2 = 0.64\ \text{ft} \qquad \text{(about 7.7 inches)}$$

during the first half of its flight.

(c) During the entire flight to home plate, which takes time $t = 0.40$ s, the ball falls

$$y = (1/2)\,gt^2 = (1/2)\,(32\ \text{ft/s}^2)\,(0.40\ \text{s})^2 = 2.56\ \text{ft},$$

four times as far, because y varies as the *square* of t. It therefore falls a distance 2.56 ft - .64 ft = 1.9 ft during the second half of its flight.

(d) The ball falls farther during the last half of its motion than during the first half, because it has a downward velocity

$$v = gt = (32\ \text{ft/s}^2)\,(.20\ \text{s}) = 6.4\ \text{ft/s}$$

already after the first half. This initial velocity for the second half makes the ball travel, during the second half, the additional vertical distance

$$y = v_0 t + (1/2)\,gt^2 = (6.4\ \text{ft/s})\,(0.20\ \text{s}) + (1/2)\,(32\ \text{ft/s}^2)\,(0.20\ \text{s})^2 = 1.9\ \text{ft}.$$

Sample Problem 5. Two artillery pieces, a distance d = 25 km apart, are aimed horizontally directly at one another, and then each gun barrel is raised to angle $\theta = 45^\circ$ as shown. At time $t = 0$, each fires a shell of mass m at speed v_0 = 500 m/s. Let g = 10 m/s^2, and neglect air resistance.

(a) With improbable good luck, the shells hit one another in flight. At what time is this, and what is their altitude when they meet?

(b) The shells melt and stick together upon impact. Describe their subsequent motion, and find the time at which they strike the ground.

(c) Describe the motion of the CM of the two shells throughout the experiment.

Solution: (a) The shells meet halfway between the guns, a horizontal distance of 12.5 km from the one at the left, which we will call the origin. Using the equation $x = v_{0x}t$, the time t at which they meet can be found from

$$x = v_{0x}t = v_0 \cos 45^\circ\, t = (v_0/\sqrt{2})\, t$$

$$\text{so} \quad t = \frac{x\sqrt{2}}{v_0} = \frac{12{,}500 \text{ m } \sqrt{2}}{500 \text{ m/s}} = \underline{25\sqrt{2} \text{ s}}.$$

Their altitude may be found from $y = v_{0y}t - (1/2)gt^2 =$

$$(500\sin 45^\circ \text{ m/s})\,(25\sqrt{2}\text{ s}) - (1/2)\,(10 \text{ m/s}^2)(25\sqrt{2}\text{ s})^2 = 12{,}500 \text{ m} - 6{,}250 \text{ m} = 6{,}250 \text{ m}.$$

(b) When the shells meet their vertical velocity is

$$v_y = v_{oy} - gt = \frac{500}{\sqrt{2}}\ \text{m/s} - 10\ \text{m/s}^2 \cdot 25\sqrt{2}\ \text{s} = (250\sqrt{2} - 250\sqrt{2})\ \text{m/s} = 0,$$

so they are moving horizontally, at the top of their trajectories. From momentum conservation, they come to rest when they stick together; they then drop vertically to the ground. Their vertical position upon hitting the ground is

$$y = 0 = y_o + v_{oy}t - (1/2)\, gt^2 = 6{,}250\ \text{m} + 0 - (1/2)(10\ \text{m/s}^2)\, t^2\ ,$$

so the time to reach the ground is $t = \sqrt{\dfrac{12{,}500\ \text{m}}{10\ \text{m/s}^2}} = \underline{25\sqrt{2}\ \text{s}}$, after impact. This is the same length of time it took the shells to rise after being fired.

(c) The CM of equal-mass shells is always halfway between them. It begins on the ground, and rises vertically according to $y_{CM} = v_{oy}t - (1/2)\, gt^2$ after the shells are fired, where $v_{oy} = 500/\sqrt{2}$ m/s. At the moment of impact and thereafter the CM coincides with the shells themselves, and falls back with them to the ground. So the CM rises and falls much as though an object were shot vertically upward from the midway point, with initial speed $500/\sqrt{2}$ m/s.

Note also that since $F_{net} = Ma_{CM} = 2ma_{CM}$ for the system, and since also $F_{net} = -2mg$, the acceleration of the CM is $a_{CM} = -g$, showing again that the CM behaves as though it were an object in free-fall.

8.4 Isolation diagrams with gravity

To find the motion of an object in the presence of gravity, we have to modify our isolation diagram approach slightly, by including the gravitational force $\mathbf{F}_{grav} = m\mathbf{g}$ acting on each massive object. In addition to the usual forces exerted by things *touching* the object, such as springs, surfaces, or strings, we have to add the gravitational force, which somehow acts at a distance, without the need for contact.

Take for example the problem of a block of mass m initially at rest, sliding down a frictionless plane of height h inclined at angle θ to the horizontal, under the influence of gravity. We would like to find the acceleration of the block, the force exerted on the block by the plane, and the speed of the block when it reaches the bottom. The steps are:

1. **Draw a picture of the physical situation, labeling everything.**

2. **Draw a separate isolation diagram for each body of interest. That is, draw a picture of one of the bodies, and the forces acting on that body only. Then repeat for each body in turn.**

The forces on the block include the force of gravity and the normal force due to the plane. Note that the gravitational force is oriented at angle θ relative to the normal to the plane. With a double-arrow we indicate a choice for the positive direction of acceleration. It is convenient to let a point *down* the plane, because that is the direction in which we expect the block to accelerate.

3. **Choose a coordinate system, and write out the component equations of $F = ma$ for each isolation diagram, where F is the sum of all the forces acting on the body.**

We expect that the acceleration of the block is parallel to the plane, because the block doesn't jump off the plane or burrow into it. So we orient our coordinates with the x axis down the plane, and the y axis off the plane, as shown. (*It is usually best to orient one of the coordinate axes along the direction of acceleration, because then we do not have to resolve the acceleration vector into all of its components.*)

The sum of the forces in the y direction is

$$F_{\text{net y}} = N - mg\cos\theta = ma_y = 0,$$

since the block stays on the plane, so it cannot have any acceleration in the y-direction. The sum of the forces in the x direction is

$$F_{\text{net x}} = mg\sin\theta = ma_x.$$

4. **Try to solve the equations!**

We find that the acceleration in the x direction is $a = a_x = g\sin\theta$ and the force exerted by the block on the plane is $N = mg\cos\theta$. The acceleration is uniform so we can use the "kinematic equations" 9a,b,c,d, from *Day Five*. In particular, we can use the equation $v^2 = v_0{}^2 + 2a(x - x_0)$. The block starts at rest and slides down along the plane a distance $h / \sin\theta$, giving

$$v^2 = 2(g\sin\theta)\frac{h}{\sin\theta} = 2gh$$

so $v = \sqrt{2gh}$ at the bottom.

5. **Check that the results are reasonable.**

It is certainly reasonable that the acceleration down along the plane is $a = g\sin\theta$, because then $a = 0$ if $\theta = 0$ when the plane is horizontal, and $a = g$ if $\theta = 90^\circ$ when the plane is vertical. The result $N = mg\cos\theta$ also makes sense, because it has the right dimensions and because it gives $N = mg$ for a horizontal plane and $N = 0$ for a vertical plane. The final result for the speed seems reasonable as well: $v = \sqrt{2gh}$ has the dimensions of velocity, and is zero if the plane is horizontal, with $h = 0$. It is also correct for a vertical plane, since we know that an object reaches speed $v = \sqrt{2gh}$ if it falls vertically. *It is interesting* that the block reaches the same speed no matter what the tilt-angle θ is, as long as the block ends up falling a vertical distance h.

Sample Problem 6. Atwood's Machine

Weights with masses m_1 and m_2 $(m_2 > m_1)$ are attached to the two ends of a string, and the string is hung over a pulley of negligible mass which can turn about an essentially frictionless bearing, as shown. The string does not stretch, and its mass is also negligible. Find the acceleration of the weight m_2.

Solution: The isolation diagrams of the two weights are as shown. We expect m_2 to accelerate downward, so m_1 must accelerate upward so the string will keep constant length. In fact, at any time the two weights must have the same speed (since the string doesn't stretch), and their accelerations must also always have the same magnitude as one another (otherwise one weight would get going faster than the other, which can't happen for a string that doesn't stretch or compress.)

We have taken the tension on the left and the tension on the right to be equal to one another. Why are they the same? The reason has to do with the (essentially) frictionless, massless pulley. If the tensions on each side were different, a massless, frictionless pulley would gain an infinite angular acceleration (that is, it would spin up infinitely fast). It can't spin up infinitely fast however, because that would mean that each of the two weights would have infinite acceleration as well, which cannot happen for finite forces. In *Motion I* all pulley problems involve frictionless, massless pulleys so we can always assume that the tension is the same on both sides. In *Motion II* we will introduce pulleys with mass and friction. The tension is then generally different on each side.

From the isolation diagram for m_2 we learn that $m_2 g - T = m_2 a$, and from the isolation diagram for m_1 we learn that $T - m_1 g = m_1 a$. The tension T can be eliminated by adding the two equations; we can then solve for a to find

$$a = \frac{m_2 - m_1}{m_2 + m_1} g.$$

Note that $a = 0$ if the masses are equal, which is reasonable, and that the magnitude of a is equal to g if either one of the masses is zero, which is also reasonable.

[Atwood's Machine is named after the English mathematician and physicist George Atwood, who in 1784 used it to demonstrate uniformly accelerated motion. Note that the acceleration a is small if m_1 and m_2 are nearly equal, so it is easy to observe the uniform acceleration.]

Sample Problem 7. *Weightless on Iceworld.*

An explorer enters an elevator on Iceworld. What is her weight
(a) if the elevator has a constant *upward* acceleration a_0?
(b) if the elevator has a constant *downward* acceleration a_0?

Solution: In Section 8.1 we defined the weight of an object as the force it exerts upon the scale that supports it. We can picture the explorer standing on a scale inside the elevator. Then the scale reads the magnitude of her weight.

There are only two forces on the her: The gravitational force $m\mathbf{g}$, and the normal force $\mathbf{N}$ due to the scale, as shown in the isolation diagram. If she is not accelerating, the net force on her must be zero. Her weight is therefore $\mathbf{W} = -\mathbf{N} = m\mathbf{g}$, equal but opposite to the force of the scale on her.

(a) If the elevator has an acceleration $\mathbf{a}_0$ *upward* the explorer has that upward acceleration as well, so the net force on her is upward, $F_{net} = N - mg = ma_0$. The normal force therefore has magnitude $N = m(g + a_0)$, which is also the scale reading (since scales measure the normal force exerted upon them.) Therefore her weight is directed downward, with magnitude

$$W = m(g + a_0).$$

She is heavier than she would be if the elevator were not accelerating. From her point of view, the effective gravity inside the elevator is $g + a_0$. Note from the formula that it *looks* as though ma_0 is another force, like mg. It isn't really, however; ma_0 is the *result* of the force, and not a force itself. We've simply placed ma_0 and mg together in solving for W.

(b) If the elevator has an acceleration $\mathbf{a}_0$ *downward* the explorer is also accelerating downward (in the negative direction), so all we have to do is change the sign of the a_0 term. Her weight is still directed downward, but with magnitude

$$W = m(g - a_0)\ .$$

She is now lighter than if the elevator were not accelerating. This will show up as a reduced reading on the scale. If in particular the elevator is in free-fall, with $a_0 = g$, then her weight is *zero*. She can then float around in the elevator without requiring any normal force of support. From her point of view, it is as though gravity has been turned off.

[The National Aeronautics and Space Administration (NASA) sometimes trains astronauts for weightless conditions by flying them inside a large room within a KC 135 cargo plane. The plane (called by passengers the "vomit comet") is flown by the pilot at high altitudes in a parabolic trajectory with downward acceleration g. The astronauts float around inside for quite a few seconds, in a weightless condition. Pilots sometimes fly the trajectory by setting a pencil in the air within their view, and adjusting the plane's controls so the pencil stays floating in about the same position. *Question:* Why can't the pilot simply turn off the engines and move naturally in a freely-falling parabolic trajectory?]

Sample Problem 8. A carnival Ferris Wheel has a radius $R = 50$ m and rotates with constant angular speed $\omega = 0.2$ rad/s. What is the *weight* of a rider of mass $m = 50$ kg when the rider is at
(a) the top of the wheel?
(b) the bottom of the wheel?
Use $g = 9.8$ m/s^2, its value on Earth.

Solution: We can picture the rider sitting on a scale. Then the scale reads the magnitude of the rider's weight.

The forces on the rider are the gravitational force $m\mathbf{g}$ and the normal force $\mathbf{N}$ of the scale, as shown. The normal force is equal but opposite to the rider's weight, because a scale reads the normal force that acts upon it. If the wheel is at *rest* the rider does not accelerate, so the two forces must balance. In that case $N = mg = 50 \text{ kg} \cdot 9.8 \text{ m/s}^2 = 490$ N. The rider's weight is then 490 N (about 110 lbs) downward toward the earth. If the wheel *rotates*, the rider accelerates toward the center of the wheel with $a = v^2/R = R\omega^2$.

$F_{grav} = mg$

(a) If the rider is at the top of the Ferris wheel, $F = ma$ becomes $mg - N = mR\omega^2$. We have chosen the positive direction downward for convenience, because we know the rider's acceleration is downward in this position. The rider's weight is therefore only

$$N = m(g - R\omega^2) = 50\text{kg}\ [\ 9.8\text{m/s}^2 - 50\text{m}\ (0.04\ \text{rad}^2/\text{s}^2)\ ] = 390\ \text{N}$$

(about 88 lbs) at the top of the wheel.

(b) If the rider is at the bottom of the Ferris Wheel, $F = ma$ gives $N - mg = mR\omega^2$. Now we have chosen the positive direction to be upward, because we know the rider's acceleration is upward in this position. The rider's weight is much larger in this case:

$$N = m(g + R\omega^2) = 50\ \text{kg}\ [\ 9.8\ \text{m/s}^2 + 50\text{m}\ (\ 0.04\ \text{rad}^2/\text{s}^2)] = 590\ \text{N},$$

about 133 lbs.

If the wheel were to spin fast enough, the rider would be weightless at the top, freely falling for an instant. But the rider's weight would then become very large at the bottom of the wheel. How large would it be?

To be weightless at the top, ω would have to be so large that $R\omega^2 = g$. Then the rider's weight at the bottom would be $m(g + g) = 50\ \text{kg}\ [\ 2 \cdot 9.8\ \text{m/s}^2\ ] = 980$ N, twice the rider's weight when at rest.

8.5 Inertial frames

We've forgotten something. We have been using $\mathbf{F} = m\mathbf{a}$ in this room on *Iceworld*, but we should first have checked that our frame is inertial! We can't use $\boldsymbol{F} = m\boldsymbol{a}$ unless our frame is inertial.

We have two tests for inertial frames, a *local* test and a *global* test. The local test is: Do objects in our frame move in a straight line at constant speed if there is no force on them, or stay put if we set them at rest? If so, the frame is inertial. The global test is: Is our frame unaccelerated relative to the distant stars?

Is the law of inertia valid inside this room on Iceworld? We are supposed to remove all forces from a ball at rest and then see if it stays put. Back in the Bubble all we had to do was make sure nothing *touched* the ball. But gravity exerts a force on the ball even when nothing touches it, so we *can't* remove all forces here.

However, we can *balance* gravity by hanging the ball from a spring. Then at least the net force on the ball is zero. If we do this with a ball at rest, the ball stays at rest. If we do it with a ball initially moving at velocity $\boldsymbol{v}$(and with the spring carried along with it), the ball should keep moving at velocity $\boldsymbol{v}$. So if we generalize our definition of inertial frame a bit, not by requiring that all forces be removed, but by requiring instead that the *net* force be zero, then objects obey the law of inertia in this room. It is consistent to let our room be an inertial frame, if we add the gravitational force on an object to forces caused by things in contact with the object, like springs and strings.

Our room may also pass the global test. We've been here already for several hours, and the stars and even our dim sun haven't budged in the sky. So at least this world is not rotating relative to the sun and stars.*

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*We see no evidence that it is accelerating in a straight line either. One way to test for straight-line acceleration is to observe the wavelengths of light from a star. According to the *Doppler effect*, light from the star will be shifted to shorter wavelengths (i.e., toward the blue end of the visible spectrum) if we are moving *toward* the star, and to longer wavelengths (i.e., toward the red end of the spectrum) if we are moving away. If *Iceworld* were undergoing constant linear acceleration, the Doppler shift of wavelengths would increase with time as the velocity increased, so the blue shift or red shift would change with time. This might in principle be detected by a telescope equipped with a spectrograph, which can be used to measure wavelengths precisely.

State of the Search

1. In the reference frame of Iceworld's surface, there is a gravitational force $\mathbf{F}_{grav} = m\mathbf{g}$ on an object of mass m, where $\mathbf{g}$ is a constant vector field, called the gravitational field.* Gravity acts on objects without anything having to touch them.

2. If gravity is the only force acting on an object of mass m, it follows from $\mathbf{F} = m\mathbf{a}$ that the object will accelerate with $\mathbf{a} = \mathbf{g}$, whatever its mass.

   If constant-$\mathbf{g}$ gravity is the *only* force on an object, the object moves in a parabolic path. The horizontal (x) motion is at constant velocity

$$v_x = v_{ox}, \quad \text{so} \quad x = x_o + v_{ox}t.$$

   The vertical (y) motion is at constant acceleration $a = -g$ (if the positive y-direction is upward), so

$$v_y = v_{oy} - gt \qquad\qquad y = y_o + v_{oy}t - \frac{1}{2}gt^2$$

$$y = y_o + \left(\frac{v_y + v_{oy}}{2}\right)t \qquad\qquad v^2 = v_o^2 - 2g(y - y_o)$$

   In the last equation v and v_o can be either the total velocities or the y-components alone.

3. The frame of our room can be considered an inertial frame, in which a gravitational force acts on an object as well as forces due to agents which touch the object. Then we can use $\mathbf{F} = m\mathbf{a}$ to find the motion of the object, where $\mathbf{F}_{grav} = m\mathbf{g}$ is included with any other forces acting on the object.

*The gravitational force $\mathbf{F}_{grav} = m\mathbf{g}$ on an object of mass m is sometimes said to be the "weight" of the object. Using this definition, a passenger living (temporarily) within a freely-falling elevator near the Earth's surface, or an astronaut living in an orbiting spacecraft, both have "weight", even though they are commonly said to experience "weightlessness" since they float freely around in their living quarters.

We have instead defined the weight of an object to be the force that object exerts on a scale that supports it against falling in the local environment. This definition matches our perceptions much better, because now the passenger in the freely-falling elevator and the astronaut in the orbiting spacecraft are truly weightless, even though gravity acts on both of them.

In the special case of a person standing at rest on the surface of a planet with gravitational field $\mathbf{g}$, the two definitions of weight are equivalent, because in equilibrium a person standing on a scale exerts a downward force $\mathbf{F}_{grav} = m\mathbf{g}$ upon it.

PICTURE QUIZ 8

Take the quiz. Answers are at the end of the text.

1. Two cannons simultaneously shoot cannonballs from a flat plain on the airless Iceworld. One shoots its ball straight upward with initial speed v_o and the other shoots its ball at angle 45° to the horizontal. Both land simultaneously. It follows that the initial speed v of the second cannonball was

(a) $v_o/2$

(b) $v_o/\sqrt{2}$

(c) v_o

(d) $\sqrt{2}\ v_o$

(e) $2\ v_o$.

2. Two cannons simultaneously shoot cannonballs vertically upward from a flat plain of the airless Iceworld. One shoots its ball with initial speed v_o; this ball reaches maximum altitude h. The other shoots its ball with initial speed $2v_o$; this ball reaches maximum altitude .

(a) $2h$

(b) $(5/2)h$

(c) $3h$

(d) $(7/2)h$

(e) $4h$

You can get the answer using dimensional reasoning alone!

Three balls A, B, and C, each of mass 2.0 kg, are hung by light strings 1, 2, and 3 from the ceiling of an elevator on Iceworld, where the gravitational field is $g = 10.0$ m/s^2.

3. What is the tension in string 2 if the elevator is stationary?

(a) 20 N

(b) 30 N

(c) 40 N

(d) 50 N

(e) 60 N

4. What is the tension in string 1 if the elevator is moving downward with constant speed 2.0 m/s?

(a) 1 N

(b) 10 N

(c) 16 N

(d) 20 N

(e) 24 N

5. What is the tension in string 3 if the elevator is accelerating downward with $a = 4.0$ m/s^2?

(a) 18 N

(b) 36 N

(c) 54 N

(d) 60 N

(e) 84 N

PROBLEM QUIZ 8

Take the quiz. Answers are at the end of the text.

1. The gravitational field on Earth is $g = 32$ ft/s^2. How many feet does a dropped rock fall during the first second? During the *second* second? How can you find the depth of a vertical mineshaft by dropping a rock into it?

2. An Apollo astronaut once dropped a hammer and a falcon feather simultaneously from the same height above the surface of the Moon. If the Moon's gravity is g_m and the speed of the hammer was v_h just before striking the surface, how long did it take the feather to fall to the surface?

3. A tennis racquet is tossed in the air, tumbling as it goes. There is a single spot on the racquet that moves along a simple curve. What is the spot, and what is the shape of the curve it follows?

4. What is the weight of a person of mass m on a planet with gravity g downward, who is inside an elevator accelerating upward with $a = g/2$?

5. A field goal attempt is made in a football game on Earth from the 50 yard line. The goal post is a distance $d = 60$ yards away, and the cross bar is a height $h = 10$ ft above the ground. If the football is kicked with an initial velocity $v = 80$ ft/s at an angle $\theta = 45^\circ$ from the horizontal, does it make it over the bar? Use $g = 32$ ft/s^2, and pretend there is no air drag.

6. Two blocks of mass m_1 and m_2 are tied to opposite ends of a light cord, and the cord is slipped over a frictionless peg, as shown. There is a gravitational field **g** directed downward. Identify which of the following is a correct set of isolation diagrams for m_1 and m_2, and state what is wrong with each of the other sets of diagrams. Then using the correct set of diagrams, write out the $F = ma$ equations, and find the acceleration of each block and the tension in the cord.

Special Topic 1: Ballistic Trajectories and the V2 Rocket

During World War II Germans designed, built, and deployed a new weapon of terror, the V-2. The V-2 was a rocket (named the "A4") with an explosive payload, developed by Werner von Braun and other engineers on an island named Peenemünde off the north coast of Germany. It was test-fired over the ocean beginning in October 1942, subsequently approximately 6400 V-2s were launched operationally against London and locations in continental Europe. The "V" stands for "Vergeltungswaffen" (Vengeance Weapon). The rockets were 47 feet long, had a diameter of 5.4 feet, and carried a warhead weighing 2300 pounds, just over one metric tonne, as shown in Fig. 2. A diagram of the rocket and its initial test trajectory is shown in Fig. 1. V-2 rocket shells can be seen at the National Air and Space Museum in Washington, D. C. and at the Imperial War Museum in London.

Fig. 1. A4 rockets set up for firing on the foreshore at Peenemunde (from "Most Secret War" by R. V. Jones. Coronet Books - 1978)

Fig. 2. German diagram from a Peenemünde album showing the trajectory of the first successful V-2 firing, 3 October 1942 (from "Most Secret War" by R. V. Jones, Coronet Books - Hodder and Stoughton, 1978)

The "V-1" weapons were the so-called "Buzz bombs", pilotless aircraft which would travel a certain number of propeller rotations before the engine automatically shut off and the aircraft fell to the ground, with its payload exploding on contact. The V-1 aircraft traveled much slower than the speed of sound, so people on the ground could hear them coming with a "buzzing" sound that gave warning to enter bomb shelters. The V-2 rockets, however, descended through the atmosphere faster than sound so people on the ground had no warning; the payload would explode and only afterwards the sounds of the rocket moving through the atmosphere might be heard.

British air reconnaissance took photographs of Peenemünde and discovered the existence of the rockets on their launching pads. British military intelligence then worked out the probable trajectory of a rocket, as shown in Fig. 3. Note the similarity to the German diagram (The laws of physics and efficient design were the same in England as they were in Germany!) The rocket starts by rising vertically, spending the shortest time possible within the atmosphere, to minimize the effects of air resistance. When the atmosphere has effectively been left behind the rocket is aimed at about 45° to the horizontal and the engines shut off, so the missile moves in a parabolic path as expected for projectile motion in uniform gravity. The unpowered (parabolic) part of the trajectory is called the "ballistic" or free-fall part of the motion. Ultimately the rocket comes crashing back through the atmosphere at supersonic speeds and explodes upon reaching the ground.

Modern ballistic missiles generally travel much higher and farther than the V-2s. ICBMs (Intercontinental Ballistic Missiles), used primarily for the delivery of atomic warheads, travel so high and far than the gravitational field varies considerably in both magnitude and direction. Therefore it turns out that the ballistic portions of the trajectories are not parabolic in shape, but correspond to a portion of an ellipse.

Fig. 3. Diagram from the Scientific Intelligence report of 26 August 1944, showing the guessed trajectory.

Special Topic II: The Sun's Gravity and Dark Matter in the Galaxy

The following table lists the mean radii r and periods T of planetary orbits around the Sun.

planet	mean orbital radius r	period T
Mercury	0.387	.241
Venus	.723	.615
Earth	1.000	1.000
Mars	1.523	1.881
Jupiter	5.203	11.860
Saturn	9.541	29.460
Uranus	19.190	84.010
Neptune	30.090	164.800
Pluto	39.500	248.000

The mean radii and periods of the planets, in Earth units

The mean orbital speeds $v = 2\pi r/T$ can be calculated from the data and plotted in terms of r, as shown in Fig. S1. Note that the orbital speed of Mercury is about ten times that of Pluto, while the orbital radius of Pluto is about one hundred times that of Mercury. This suggests that $v \propto 1/\sqrt{r}$: The speed of orbits falls off as the square root of their radii. This relationship is easily confirmed for all of the planets.

Fig. S1. The orbital speeds of the planets vs their mean orbital radii.

It follows from the behavior of the orbital speeds v that the centripetal acceleration $a = v^2/r$ of the various orbits is proportional to $1/r^2$. Therefore there must be a central force $F = ma \propto m/r^2$ acting on the planets, presumably due to the Sun. The universal law of gravitation, confirmed by studying the orbit of the Moon around the Earth, the orbits of other moons around other planets, and from measurements of gravitational forces in the laboratory, is

$$F = \frac{GMm}{r^2}$$

where "Newton's gravitational constant" $G = 6.67 \times 10^{-11}\ m^3/kg \cdot s^2$. This is Newton's law of gravity, in which the force is proportional to the product of the two masses, and inversely proportional to the square of the distance between them. It is sometimes called the "inverse square law" for short.

For a planet of mass m, the most important force is that due to the mass M of the Sun, because the Sun has over 99.8% of the entire mass of the solar system. There are also much smaller gravitational forces on each planet from any moons it has, and from all of the other planets.

The stars in the disk of a spiral galaxy like our own Milky Way galaxy also travel in roughly circular orbits, around the central nucleus of the galaxy. Our solar system orbits with a period of about 220 million years, for example, at a distance of about 8.5 kiloparsecs from the center. (1 parsec = 3.084×10^{13} km = 3.26 light-years). Fig. S2 shows the Andromeda Nebula (M31), a nearby spiral galaxy much like the Milky Way in size and structure. If this were a picture of the Milky Way, the sun would be a point of light toward the outskirts of the galaxy.

Fig. S2. The Andromeda Nebula, a spiral galaxy much like the Milky Way.

Nearly all the solar system's mass in contained in the Sun, but only a small fraction of the galaxy's mass is contained in its nucleus. There is therefore no reason to expect that the orbital speed of stars around the nucleus should follow the law $v \propto 1/\sqrt{r}$. How *should* v depend on r in this case?

To answer this question, we need to know how the mass is distributed in the galaxy. Suppose $M(r)$ is the mass within a sphere of radius r, centered at the galactic nucleus. One can then show that the centripetal gravitational force on a star of mass m located at r is $F = \dfrac{GM(r)m}{r^2}$. In effect, only the mass $M(r)$ within radius r pulls on m; the gravitational attractions due to masses at larger radii all cancel out. (This is only exactly true if the mass distribution is spherically symmetric, but it is a pretty good approximation here for spiral galaxies as well.) If we can find $M(r)$, we can then predict the orbital speed $v(r)$ using $F = ma = mv^2/r$.

Astronomers can measure the luminosity (i.e., the brightness) of stars as a function of position within the galaxy. Some of this light comes from young, massive, hot and bright stars; some comes from average stars like our Sun; and some comes from small, dim, red dwarfs. From observational and theoretical analyses, astronomers are able to assign a mass due to stars in a particular region, based upon how much light they see. The mass of hydrogen clouds and dust clouds can also be estimated, based on observational data. Using all this information, the total mass $M(r)$ is calculated and from it $v(r)$ is predicted. Fig. S3 shows the predicted $v(r)$ for the spiral galaxy NGC 3198, in the lower curve.

Fig. S3. Velocity of stars in the spiral galaxy NGC 3198, as a function of their distance from the galactic center. The bottom curve is the prediction based on mass estimates using observed luminosities, and the top curve shows the observations, based on Doppler shifts.

There is a way to check the predictions, to see whether the velocities of stars really are as predicted. Using spectrometers mounted on telescopes, astronomers measure the

Doppler shift of light coming from various regions of the galaxy. The light from stars approaching us is shifted toward the blue end of the spectrum: The frequency of spectral lines from these stars is higher than the frequency of the same lines produced in the laboratory. Similarly, spectral lines from receding stars are red-shifted, their frequencies reduced. A simple formula then allows the determination of the velocity toward us or away from us. These observed velocities for NGC 3198 are plotted in the upper curve of Fig. S3. Stars are observed to orbit much faster than predicted. Instead of tailing off at large radii, velocities are roughly independent of radius for a substantial range of radii. Why the discrepancy? There are at least four possibilities:

(1) $F \neq ma$ over large distances. Or perhaps $F \neq ma$ for the very small accelerations experienced by stars in their 200 million-year orbits.
(2) The law of gravitation $F = GMm/r^2$ fails for such large distances. What works in the solar system may not work in the galaxy.
(3) $F = ma$ is correct, and the law of gravity is correct, but there is a (so-far) unidentified force which is important at such large distances.
(4) $F = ma$ is correct, and gravity is the only important force, but there is some mass missing from astronomer's observations, called *dark matter*.

Each of these possibilities has been seriously considered. The great majority of researchers believe that (4) is the correct answer. From $F = ma = mv^2/r$ and from the gravitational law $F = \dfrac{GM(r)m}{r^2}$ it follows that $v(r) \propto \sqrt{M(r)/r}$. In the outer regions of spiral galaxies the mass $M(r)$ must increase linearly with r, to match the observation that $v(r)$ is nearly constant. When compared with observations, the mass needed is about ten times the observed mass. If true, 90% of the mass of a galaxy is in the form of unseen dark matter! Is this mass contained in very dim stars? In black holes? In planet-type objects (so-called "Jupiters"), too small to have formed stars? In massive neutrinos? In some so-far undiscovered fundamental particle, like axions, photinos, ...? In the past few years some of this mass has been found by astronomers using *gravitational lensing* techniques in objects called MACHOS (Massive Compact Halo Objects), which are low-mass dim stars or "Jupiters" in our galaxy. It is not yet clear how much of the dark matter is in this form. As of 2001, it is quite possible that WIMPS (Weakly Interacting Massive Particles) like photinos also exist; experiments are underway around the world to try to find them. Many researchers believe that such particles could account for a substantial fraction of the dark matter needed to understand the velocities of stars around galactic nuclei.

Questions and Problems from Day Eight

8Q1. Gravity on the Moon is one-sixth that on Earth. How does the weight of an astronaut on the Moon compare with her weight on Earth? What changes would you have to make in the game of baseball to make it playable on the Moon? How about basketball? Various track and field events?

8Q2. Suppose a particular six-foot tall high-jumper can clear a six-foot high bar on Earth. About how high could the high-jumper jump on the Moon? (It is *not* six times as high!)

8Q3. A monkey hanging from a tree limb notices a hunter on the ground aiming a rifle at him. The monkey watches the hunter's trigger-finger like a hawk, reasoning that if he

drops from the limb at the moment the hunter pulls the trigger, he will fall below the bullet's path by the time it arrives. Is the monkey exercising good judgment?

8Q4. A ball bearing can fall from point *A* to point *B* by either of two paths through the tube arrangement shown. If the ball starts from rest, would it take less time to fall by the upper path or the lower path? Assume the corners are gently rounded.

8Q5. On the Moon, a 2-lb steel ball is dropped from the same height, and at the same instant, as a 30 g wooden ball is projected horizontally with speed 40 cm/s. Which, if either, will strike the Moon's surface first, assuming the surface is flat?

8Q6. Some Michael Jordan fans claim the former basketball star could remain airborne for two seconds. If this is true, how high would he have to leap? [In fact, he typically leaps about three feet up with an airborne time of .87 sec. "What helps make this brief time look so much longer is Jordan's remarkable ability to release his shots on the way down. But holding the ball until he has passed the peak of his leap, he gives himself an extra .2 or .3 seconds to get the ball over, under, or around the defenders." - In *Discover*, March 1991, by Peter J. Brancazio.]

8Q7. Long-range artillery guns are typically set to fire shells at elevation angles greater than 45°. Why? Civil War cannons were typically fired at elevation angles less than 15°. Why were they fired at such shallow angles?

8Q8. Neglecting air resistance, a projectile moves in a parabolic trajectory above Earth's surface. If the projectile explodes in flight, how does the CM of the fragment move? Does your answer depend upon whether some of the fragments have already struck the ground?

8Q9. A man walks along a level surface, holding a ball just outside his left shoulder. If he drops the ball and continues walking at the same pace, will the ball strike the ground ahead of his feet, beside his feet, or behind his feet?

8Q10. A four-wheeled cart can roll without friction on a pair of rails. The cart supports a spring-loaded gun with a ball sitting in a cup, as shown. If the cart is at rest and the gun triggered, the ball is projected vertically upward, and is subsequently caught by the cup. Now the cart is caused to roll horizontally on the rails at constant speed; at some point the gun is automatically triggered. Will the cup later catch the ball? Why or why not? Finally, the rails are tipped at some angle θ, as shown. As the loaded cart accelerates down the slope, the gun is again automatically triggered. Will the cup later catch the ball? Why or why not?

Section 1. The force of gravity and *weight*

8P1. What is the gravitational force on a person of mass 55 kg on (a) Iceworld? (b) Earth? (c) Mars (g = 3.7 N/kg) ? What is her weight at each location if she is at rest on the surface?

8P2. A spring scale measures the gravitational force on objects placed in a pan. The force constant of the spring is k = 12 N/m. (a) If a cabbage is placed on a scale on the Earth, the spring stretches an additional 25 cm. What is the gravitational force on the cabbage? (b) What is the mass of the cabbage? (c) If the same cabbage is weighed on the same scale on the planet Mercury (g = 3.70 N/kg), how much will the spring stretch (assuming k is unchanged)?

Section 2. Vertical motion

8P3. A ball is thrown upward from the floor on Iceworld. It comes to rest just before reaching the ceiling, and then falls back to the floor. If the entire trip takes time t = 1.8 s, find (a) the height of the ceiling; (b) the speed of the ball when it hits the floor.

8P4. (a) Find the gravitational force $F_g = mg$ of an astronaut of mass m = 65.0 kg on Earth (g = 9.81 m/s^2). (b) Find the ratios of her gravitational force on the Moon (g = 1.62 m/s^2), on Mars (g = 3.71 m/s^2), and on the asteroid Vesta (g = 0.18 m/s^2), to that she has on Earth. (c) If she can jump upwards the vertical distance h on Earth, how high can she jump up with the same initial velocity on the Moon, Mars, and Vesta ?

8P5. While walking past a dormitory, the Dean of Students hears a cry of "bombs away" and observes a water balloon fall vertically past her nose. With trained reflexes, she notes that the water balloon hits her shoes at time Δt = 0.20 s after passing her nose, a distance Δy = 2.0 m. Assuming the balloon was released from rest, the Dean asks you to calculate from what height h above the ground the guilty party released the bomb, thus enabling her to apprehend the culprit. Use g_{Earth} = 9.8 m/s^2.

8P6. A rocket is launched upward from rest with acceleration a = 25.0 m/s^2. At altitude h = 400 m the engines are shut off, but the rocket continues to coast upward. If there were no air resistance, find the total time t_T from lift-off until the rocket crashes. Use g_{Earth} = 9.8 m/s^2.

8P7. A rifle bullet is fired straight upward from the ground on Earth with muzzle velocity $v_0 = 450$ m/s. (a) Neglecting air resistance, what maximum altitude should the bullet achieve? (b) What maximum altitude would it achieve on the Moon, with $g_{Moon} = (1/6)g_{Earth}$? How long does it take the bullet to fall back to the ground from the time it is fired, (c) on Earth? (d) on the Moon?

8P8. You are in an elevator at rest at the top of the Sears Tower in Chicago, 1454 ft above street level. Suddenly the cables break and the elevator plunges downward in free fall (pretend there is no air resistance or any other drag forces on the elevator.) (a) How long would it take you to reach street level? (b) How fast would you be traveling at that time, in miles/hour? (Note that $g_{Earth} = 32$ ft/s^2 in the English system of units.)

8P9. Without noticing two Pisans standing below, Galileo drops a lemon off the Leaning Tower of Pisa with no initial velocity. The taller of the two Pisans sees the lemon pass by her eyes, 2.500 m above the ground, and notes the time; the shorter Pisan, whose eyes are 1.700 m above the ground, sees the lemon pass by her eyes 0.026 s later. From what height above the ground did Galileo drop the lemon? Pretend there is no air resistance and use $g = 9.81$ m/s^2.

8P10. An astronaut standing on the surface of a large asteroid throws a wrench straight upwards from height 5 ft above the surface, with initial velocity 6.00 ft/s. The wrench falls to the surface 2.00 minutes later. What is the gravitational field g on the asteroid, expressed as a fraction of g_{Earth}?

Section 1.3. Projectile motion (Neglect air resistance in all of the following problems)

8P11. A baseball is batted with initial speed $v_0 = 45$ m/s, at elevation angle $\theta = 30^o$ from the bat, a distance $y_0 = 1.4$ m above the ground. Find (a) the maximum altitude h_{max} achieved by the baseball, (b) the distance D from the batter at which the ball strikes the ground. Use $g_{Earth} = 9.8$ m/s^2.

8P12. A projectile of mass m is launched with initial speed v_0, at angle θ relative to the horizontal. Gravity is the only force acting. In terms of m, g, v_0, and θ,

(a) Find the maximum altitude h_{max} of the projectile.
(b) Find the speed of the projectile when it reaches maximum altitude.
(c) Find the range of the projectile, R. (d) Evaluate (a), (b), and (c) numerically, for a baseball hit on *Iceworld* with $v_0 = 68$ m/s at angle $\theta = 30^o$.

8P13. A baseball pitcher on Earth throws a curve ball with velocity $v = 88$ ft/s (= 60 mi/hr) toward home plate, whose center is a distance $D = 60.0$ ft away. He gives the ball sufficient spin so that the air exerts a sideways force F on it equal to one-third of the gravitational force mg on it. *Estimate* the sideways deflection d of the ball (in feet) by the time it reaches home plate. (Note that $g_{Earth} = 32$ ft/s^2.)

8P14. A shotputter stands on top of the Eiffel Tower, a height $h = 310$ m above Paris. He puts a shot of mass $m = 12$ kg purely horizontally, at velocity $v_0 = 9.0$ m/s. When the shot hits the ground, (a) how far is it from the center of the tower's base? (b) what is its speed? (c) With what constant speed v_0 would a sprinter with a wastebasket have to run to "catch" the shot in the basket, if she starts from the center of the tower's base at the instant the shot is put?

8P15. At the end of a movie chase scene a car moving at 35 m/s (78 mi/hr) runs off a horizontal dock, 9.0 m above the surface of the ocean. Just before the car hits the water, (a) what is its horizontal distance from the dock? (b) what is the angle of its velocity vector relative to the horizontal?

8P16. The bullets from a typical target rifle have muzzle velocity 1000 m/s. Suppose the target is 300 m away and is level with such a rifle. At what elevation angle should the rifle barrel be aimed so the bullet will strike the center of the target?

8P17. The maximum height of a particular football's trajectory is half its range (i.e, half its horizontal displacement by the time it strikes the ground.). At what elevation angle was the football thrown by the quarterback?

8P18. The standard firehose in the U.S. delivers 250 gallons of water per minute with initial velocity 25 m/s. How high will the water rise if such a hose is aimed (a) straight up? (b) with elevation angle θ_0? (c) If each story in a multistory building is 10 ft high, what is the highest story reachable by a standard firehose from the ground? (Some people take this into account when they reserve hotel rooms. In fact water won't reach this high because air resistance is important.)

8P19. Show that for angles of projection which are greater than or less than 45° by equal amounts, the ranges are equal for the same initial speed, as illustrated. (a) Prove this, if there is no air resistance. (It is not generally true otherwise.) (b) If a baseball is thrown with speed $v =$ 25 m/s to reach a catcher a distance $d = 42$ m away, what are the two possible elevation angles?

8P20. Following the 1991 Gulf War, United Nations inspectors discovered that Iraq had built a "supergun" capable of firing 16 inch shells great distances. The gun was a prototype for a still larger gun meant to fire 40 inch shells with a range of 1000 km or more. (a) What would be the minimum muzzle velocity required for a shell to be projected 1000 km, neglecting air resistance and any variation in g? (b) What would be the maximum altitude reached by such a shell?

8P21. A cannonball with initial speed $v_0 = 55$ m/s is launched at angle $\theta = 65^0$ to the horizontal, as shown. After a time $t = 6.3$ s, the cannonball strikes the top of a mesa. Using $g = 9.8$ m/s^2, find (a) the altitude h of the mesa above the plain; (b) the maximum altitude h_{max} reached by the cannonball; (c) the speed of the ball just before impact; (d) the angle (relative to the horizontal) of the ball just before impact.

8P22. A daredevil motorcyclist on Earth parks a number of buses, each of width $w = 4.0$ m, side-by-side between two ramps angled at $\theta = 45^0$, as shown. If the maximum speed the motorcyclist can muster at the top of the ramp is $v = 30.0$ m/s, what is the maximum number (N) of buses the motorcyclist can leap over?

8P23. A projectile is fired at speed v_0 with elevation angle θ_0 in uniform gravity g. (a) Use dimensional arguments alone to find the dependence of h_{max}, the maximum height reached, upon v_0 and g. (b) If v_0 were doubled, keeping θ_0 and g fixed, by what factor would h_{max} increase? (c) Find a complete formula for h_{max} in terms of given variables. (d) Find h_{max} as a fraction of the projectile's range R, if $\theta_0 = 45^o$.

8P24. A thick castle wall contains a single horizontal window at altitude h above the ground. Attackers launch balls from a catapult with fixed speed v_0 and an adjustable launch angle θ relative to the horizontal. (a) What should be the launch angle θ (in terms of h, v_0, and g) so that launched projectiles will have a purely horizontal velocity when they reach the wall, so they will be able to enter and traverse the horizontal window? (b) In terms of the same parameters what must be the distance d of the catapult from the base of the wall? (c) What is the maximum altitude h_{max} the window can have, such that for given v_0 a ball can reach the window in this way?

8P25. A field goal attempt is made from the 40 yard line. The goal post is a distance $d =$ 50 yards away and the cross bar is at height $h = 10$ ft above the ground. If the football is kicked from the ground in the correct direction with an initial velocity $v_0 = 65$ ft/s at angle $\theta = 45^o$ from the horizontal, does it make it over the bar? Use $g = 32$ ft/s^2.

8P26. According to the San Diego Wild Animal Park, fleas can jump as far as 13 inches, humans 29 feet, and kangaroos 44 ft. Find in each case the minimum initial speed (in ft/s) required to make the jump if air resistance is neglected.

8P27. In U. S. basketball the free-throw line is a horizontal distance 15 ft from the backboard. The basket, with a diameter of 18 inches, is a height $y = 10$ ft above the court, attached to the backboard so the center of the basket is 12 inches from the backboard, and therefore a distance $d = 14$ ft from the free-throw line. A player executes a free throw from height $y_0 = 7.0$ ft so the ball has an initial angle of 45° to the horizontal. (a) What is the initial speed of the ball if the player makes a basket directly through the hoop? (b) At what angle to the vertical does the ball enter the basket? (Work the problem symbolically before substituting in numbers.)

8P28. Two astronauts are playing a ballgame on the Moon, where gravity g_M is one-sixth that on the Earth. Each is standing on the edge of a vertical rock face. The rock faces are a distance d apart and each is the same distance h above the plain. At time $t = 0$, one astronaut throws a ball horizontally in the direction of the other astronaut. At what time t should the other astronaut step off the rock face so she can catch the ball in flight while she is falling?

8P29. An artillery officer wants to fire as far up a hill of slope-angle ϕ as possible, as shown. (a) If the initial speed of the shell is v_0, at what angle θ to the horizontal should the gun be aimed? (b) What is the range R in this case? *Parts (c) and (d) are optional:*(c) At what angle θ to the horizontal should the gun be aimed to fire *down* from the *top* of a hill of slope-angle ϕ, to achieve the greatest range? (d) What is this range?

8P30. News item from Friday, May 21, 1999 (Los Angeles Times):

From Associated Press

GRAND CANYON, Ariz.— Motorcycle daredevil Robbie Knievel cleared a sliver of the Grand Canyon on Thursday, breaking his world record of 223 feet with room to spare.

Fireworks erupted and a crowd of about 500 cheered as Knievel soared 55 feet into the air over the 200-foot-wide gorge at 90 mph on an ordinary 500-cc motorcycle.

Officials said Knievel traveled 228 feet, eclipsing the record he set in Panama City, Fla., in the early 1990s.

If he had failed, Knievel risked plunging 2,500 feet to the canyon floor on the Hualapai Indian Reservation, west of Grand Canyon National Park.

Knievel tumbled off the bike after clearing the landing ramp, coming to rest in a cloud of dust in hay bales set up to cushion a fall.

"I'm wiped out in the head a little," Knievel said.

He was examined by paramedics, who applied a neck brace and placed him on a gurney. He was flown by helicopter to University Medical Center in Las Vegas for an examination.

The extent of any injuries was unclear.

Assume the altitude of the plateau is the same on both sides of the canyon, that air resistance is negligible, and that the numbers reported are accurate to two significant figures. (a) Find Knievel's launch angle θ in degrees (relative to the horizontal), based on his maximum trajectory height (55 ft). (b) Find the *two* possible launch angles θ, based instead on his range (228 ft). Note that neither of the answers in (b) is consistent with the launch angle found in (a); the discrepancy may be due mostly to air resistance. (c) Neglecting air resistance and using the launch angle found in (a), can we get the correct range by supposing that the plateau has a different altitude on the landing side than on the launching side? If so, is the landing side higher or lower than the launching side?

8P31. A velociraptor leaps up onto the top of the back of a tarbosaurus standing a horizontal distance d away. If the back of the tarbosaurus is at height $h = d/\sqrt{3}$ above the ground, what is the minimum speed v_0 the 'raptor must have when it takes off from the ground, in terms of g and d? What is the take-off angle θ_0 of the 'raptor relative to the horizontal? *Hint:* Derive an expression for the initial speed v_0 as a function of g, d, and the take-off angle θ_0; then vary θ_0 to minimize v_0.

8P32. A Chief Executive Officer holds the string of a yo-yo of mass m, and lets the yo-yo go, so it spins in the usual way as it rolls down the string. The acceleration of its center of mass is constant, $a = g/3$. (a) Draw an isolation diagram of the yo-yo. (b) What force F_0 does the string exert on the CEO's hand?

8P33. A gallery worker hangs a new painting by attaching a length of piano wire tautly between two points on opposite sides of the back of the frame a horizontal distance $D = 24$ inches apart, and then setting the wire's center on a screw protruding from the wall. If the painting weighs $W = 12$ pounds and the support-points on the frame sink a distance $d = 2.0$ inches below the position of the screw when the painting is hung, what is the tension T in the wire, in pounds?

8P34. A prisoner on Earth wants to lower himself from his jailhouse window using a rope barely long enough to reach the ground, a distance $d = 20.0$ m below. However, his mass is $m = 110$ kg and the rope will break if its tension exceeds $T = 900$ N. (a) How can he descend without breaking the rope, so he arrives at the ground with the minimum possible speed? (b) Will he be able to run away, if hitting the ground with a speed greater than $v = 10$ m/s would break his legs? Use $g = 9.8$ m/s^2.

8P35. A four-link chain is raised with acceleration $a = 3.2$ m/s^2 on Earth, by a force F_0 on the top link, as shown. Each link has mass $m = 0.050$ kg. (a) What is F_0? (b) What is the net force F_{net} on each link? What is the force of (c) link 2 on link 3? (d) link 4 on link 3? (e) link 2 on link 1?

8P36. The ends of a massless string are attached to two weights; one has mass 1.0 kg and is suspended in space, and the other has mass 2.0 kg and is resting on the ground. The string is hung over a massless, frictionless pulley, which is held up by a force F_0 exerted on the pulley axis, as shown. The gravitational field is $g = 9.8$ m/s^2. (a) Draw isolation diagrams of each weight and of the pulley. (b) What is F_0 (in Newtons) if nothing moves? (c) What force N does the ground exert on the 2.0 kg weight, in this case? (d) What is the force F_0 required to just barely lift the 2.0 kg weight off the ground? (The weight remains at rest, just above the ground.) (e) What is the acceleration of the 1.0 kg weight in this case, valid until the 1.0 kg weight hits the pulley? (f) Repeat part (d), assuming the suspended weight has a mass 0.5 kg instead of 1.0 kg.

8P37. The ends of a massless string are attached to two weights with masses m_1 and m_2; m_1 is suspended in space, and m_2 is resting on the ground, as in the figure for 8P36. The string is hung over a massless, frictionless pulley, which is held up by a force F_0 exerted on the pulley axis. (a) Draw isolation diagrams of each weight and of the pulley. (b) What is F_0 if nothing moves, in terms of any or all of m_1, m_2, and g? (c) What force N does the ground exert on m_2 in this case? (d) What is the force F_0 required to just barely lift m_2 off the ground? (m_2 remains at rest, just above the floor) (e) What is the acceleration of m_1, valid until m_1 hits the pulley? (f) What is the acceleration of the pulley in this case?

8P38. A block of mass m is connected by a massless string that runs over a frictionless peg to a block of mass $2m$ as shown. Both blocks rest on frictionless inclined planes in uniform gravity g. (a) Draw an isolation diagram for each block. (b) Write out $\mathbf{F} = m\mathbf{a}$ equations for each block. (c) Solve them to find the acceleration a of the system. (d) Find the tension T in the string.

8P39. A small ball of mass m is attached to the lower end of a string of length ℓ whose upper end is held fixed. The ball moves in a circle while the string maintains a constant angle θ with respect to the vertical. (The string sweeps out a cone of angle θ; the arrangement is known as a *conical pendulum.*) (a) Draw the isolation diagram. (b) Determine T, the tension of the string, in terms of m; g, and θ. (c) Is your result reasonable as $\theta \to 0$ and as $\theta \to \pi/2$? (d) Determine P, the period of motion, in terms of ℓ, g, and θ.

8P40. (a) A previously calibrated spring scale is attached to two strings, as shown on Earth; the strings are placed over frictionless pulleys, and 10.0 lb weights are attached to the other ends of the strings. What does the spring scale read? Suppose the scale and strings have negligible weight. (b) The same spring scale is attached as shown, with one of the strings fastened to a wall instead of a 10.0 lb weight. What does the scale read in this case?

8P41. The same spring scale described in the preceding problem is again attached to two weights on the Earth, this time a 10.0 lb weight and a 20.0 lb weight, as shown. (a) Find the acceleration of the spring scale, while it is between the pulleys. Assume that the stretch of the spring inside the scale remains constant during the acceleration. (b) What does the scale read in this case?

8P42. Suppose the spring scale described in the preceding problem weighs 1.0 lb. (a) Find the acceleration a of the scale, while it is between the two pulleys. Assume the stretch of the spring inside the scale remains constant during the acceleration. (b) What does the scale read in this case? *Hint:* In the accelerating frame of the scale, in addition to real forces there is an effective "pseudoforce" $m_{\text{scale}}a$ acting on the scale, directed opposite to the acceleration direction. (We prove this later in the text.)

8P43. A man of mass M stands on a platform of mass m; the man's shoes are strapped to the platform. The man raises himself, along with the platform, with upward acceleration a, by pulling on a rope, as shown. The rope passes over a frictionless, massless pulley, which means that the tension is constant throughout the rope. Find F_O, the force with which he pulls on the rope, in terms of M, m, a, and the gravitational field g.

8P44. A small ball of mass m is attached to the lower end of a string of length ℓ whose upper end is attached to one end of a horizontal bar of length L. The other end of the bar can turn on a frictionless bearing about the vertical axis, as shown. Both the bar and the string have negligible mass. The ball moves in a circle while the string maintains a constant angle θ with respect to the vertical. The bar turns with the ball, so the bar, string, and ball are always coplanar. (a) Draw an isolation diagram of the ball. (b) Find T, the tension of the string, in terms of any or all of m, g, and θ. (c) Is your result reasonable as $\theta \to 0$ and as $\theta \to \pi/2$? (d) Determine P, the period of motion, in terms of any or all of ℓ, L, g, and θ.

8P45. Three blocks of masses $3m$, m, and $2m$ are connected by massless strings over a frictionless peg, as shown All surfaces are frictionless, and gravity g is uniform. (a) Draw an isolation diagram for each mass. (b) Find the acceleration a of the system. (c) Find the tension T_1 in the cord attached to the $3m$ mass. (d) Find the tension T_2 in the cord connecting m with $2m$.

8P46. A carnival ride consists of a number of cars arranged in a circle in a vertical plane, as shown; each car has mass M, and is supported by a metal spoke of length r and negligible mass. The entire assembly rotates with constant angular velocity ω. The gravity g is uniform. (a) Find the tension T_b in the spoke that happens to be supporting the bottom car (car A) at some instant. Use an isolation diagram. b) Find the tension T_t in the spoke that happens to be supporting the top car B at some instant. Again, be sure to use an isolation diagram. (c) For what critical angular velocity ω_{crit} would a passenger in the top car tend to float up off her seat? (d) What is her weight at the instant described in (c)?

8P47. A steel ball is suspended by two cords within a box as shown, each making an angle $\theta = 60^o$ with the horizontal. In terms of g, what horizontal acceleration a must be given to the box so that the tension in cord A is twice that in cord B?

8P48. A "Double Atwood's Machine" features three masses, three massless cords, and two frictionless, massless pulleys, as shown. (a) Show why, under the assumed conditions, the tension in the uppermost cord (#1) is twice that in the cord (#2) supporting m_1 and the lower pulley. (b) Find the accelerations a_1, a_2, and a_3 of masses m_1, m_2, and m_3, relative to the inertial frame in which the center of the upper pulley is at rest, and (c) the tension in cords 1, 2, and 3, in terms of g and the masses.

8P49. A truck driver wants to tow a car up a slope angled at $\theta = 15^0$, as shown. The cable from the truck makes an angle of $\phi = 30^0$ angle with the slope. The car has mass m = 1200 kg. What is the minimum time t t for them to reach the top of the slope, a distance $d = 350$ m away, if they start from rest and the cable has a breaking strength of $T = 3{,}900$ N? Ignore any air resistance or friction between the car and road.

8P50. A massless rope is hung over a frictionless peg as shown. One end of the rope is attached to a box of mass m_1. Resting on the floor of the box is a block of mass m_2. A mountaineer of mass m_3 climbs up the other side of the rope, with uniform acceleration a' relative to the ground (which isn't touching either the box or mountaineer.) Gravity g is uniform.(a) Draw isolation diagrams for the mountaineer, for the box plus block, and for the block by itself. (b) Write the $F = ma$ equations for each of the systems drawn in (a). (c) Find a formula for the normal force N of the box up on the block, in terms of the masses, a', and g.

8P51. The high-speed Amtrak train *Acela* has been designed to make the passenger compartments tilt as the train rounds a curve so the passengers will not be as likely to slide sideways on their seats. (a) What should be the tilt-angle for a train moving at $v = 220$ km/hr rounding a section of curve whose radius $R = 880$ m, so the passengers would be pressed down *perpendicular* to the train seats? (b) By what factor would the weight of a passenger increase during such a turn?

8P52. A physicist stands on a weighing scale inside an elevator, as shown. The elevator and scale together have mass M; the physicist has mass $M/2$. The elevator is suspended by a (massless) rope that passes over a (frictionless) peg, and is then attached to a block, also of mass M. The block is free to slide along a frictionless plane, inclined at angle 30° to the horizontal. The gravity is g.

(a) Draw an isolation diagram for the block, and find a formula for the tension T in the rope, in terms of M, g, and the block's acceleration a. (b) Draw an isolation diagram for the elevator plus contents (i.e. the scale and physicist.) (c) Find a formula for the acceleration a of the elevator, in terms of g only. (d) What does the scale read, expressed as a fraction of what it would read if the elevator were not accelerating? This is her weight in the elevator as a fraction of her weight at rest.

8P53. A small bead of mass m slides on a hoop of wire of radius R, as shown. There is no friction between the bead and wire, so the wire can only exert a force perpendicular to itself, inward or outward from the center. The hoop rotates with constant angular velocity ω about a vertical axis through its center. There is uniform gravity g. (a) Draw an isolation diagram for the bead when it is at some arbitrary angle θ, showing all of the forces on it.

(b) Define a critical angular velocity by $\omega_{\text{crit}} = \sqrt{g/R}$. Show that if $\omega > \omega_{\text{crit}}$, there are four different locations on the hoop for which the bead is in equilibrium, i.e., where it will stay in the same place on the rotating wire. Find the angles for these four locations. (Note: each point on the hoop is specified by a unique angle if $-\pi < \theta \leq \pi$.) (c) Show that if $\omega \leq \omega_{\text{crit}}$, there are only two equilibrium positions. What are they?

8P54. A block of mass m_1 on a frictionless table is attached to string #1 wound around the massless pulley P_1, as shown. A second block of mass m_2 is attached to string # 2 draped over the stationary massless pulley P_2; the other end of string #2 is attached to the axle of P_1. (a) Draw 3 isolation diagrams, for m_1, m_2, and P_1 . (b) What is the relationship between the accelerations a_1 and a_2 of m_1 and m_2 ? (*Hint:* If m_2 falls a certain distance, how far does m_1 travel to the right?) (c) Find a_1 , a_2 , T_1 , and T_2 in terms of the masses and g.

Special Topic: Gravity in the solar system and in galaxies

8P55. Show that the Sun pulls more than twice as hard on the Moon as Earth does. So why doesn't the Moon escape from Earth? (The Sun has mass $3.3 \cdot 10^5$ times that of Earth, and is about 400 times as far from the Moon as Earth is from the Moon.)

8P56. (a) Show from Newton's inverse-square law of gravity that the periods of planetary orbits T are related to their orbital radii r according to $T^2 \propto r^3$, i.e., the periods squared are proportional to the orbital radii cubed. (The orbits of the planets are actually somewhat elliptical. In that case r has to be replaced by the *semimajor axis* of the orbit. This relationship is called *Kepler's Third Law*, and was discovered by observation before Newton proposed his laws.)

8P57. Planet Earth has average radius $R = 6371$ km and gravity $g = 9.81$ m/s^2. (a) Find the period T of a hypothetical Earth orbit at tree-top level (hypothetical because air resistance would slow down and burn up anything trying to orbit near the ground). (b) Find also the periods of hypothetical near-surface orbits around the following planets:

planet	average radius (km)	g (m/s^2)
Mercury	2,433	3.58
Venus	6,053	8.87
Mars	3,380	3.74
Jupiter	69,758	26.01

8P58. Some orbital radii and rotation periods are given in the table for the four Galilean satellites of Jupiter. Fill in the missing information.

Moon	orbital radius (X 10^3 km)	period (days)
Io		1.77
Europa	671	3.55
Ganymede	1070	
Callisto		16.69

8P59. Communications satellites are often placed in geosynchronous orbits, which are circular orbits in the equatorial plane of Earth which rotate once around every twenty-four hours so they always stay above the same spot on the equator. Find the altitude above Earth's surface that such a satellite must have, and compare it with Earth's radius.

8P60. Using the graph of observed rotation speeds of stars as a function of their distances from the center of galaxy NGC 3198 (shown in Fig. S3), estimate the mass of the galaxy located within a radius of (a) 15 kiloparsecs (b) 30 kiloparsecs from the galactic center. Express the results in terms of solar masses, where 1 solar mass = $2 \cdot 10^{30}$ kg.

8P61. Suppose that a uniform mass density ρ_{DM} of *dark matter* suffuses the solar system, and that each planet feels a gravitational force $F = GM(r)m/r^2$ due to this dark matter which is directed toward the Sun, where $M(r)$ is the mass of dark matter *inside* the orbital radius r of the planet. (That is, only the dark matter whose distance from the Sun is less than the planet's orbital radius has a net gravitational attraction on the planet; the gravitational forces of dark matter sources at larger radii all cancel out, assuming spherical symmetry. The planet's mass is m.
(a) Assume the dark-matter mass inside Mercury is already much larger than the mass of the Sun. Show in that case that all planets would have the same orbital period!
(b) In fact, we know that if there is dark matter in the solar system, its mass is very small compared with that of the Sun. Show that the periods T of planetary orbits are related in general to their orbital radii by $T^2 = K\left(\frac{r^3}{1 + \alpha r^3}\right)$ where K and α are constant. Find K and α in terms of G, M_{sun}, and ρ_{DM}.

8P62. Apollo astronauts and Moon vehicles kicked up Moon dust and sand as they roamed around on the Moon's surface, while video cameras recorded the action. Gravity on the Moon is only one-sixth as great as on Earth, so the motion of astronauts and the trajectory of sand grains looked as though the motion had been filmed on Earth, and replayed in slow motion. If a simulated moon expedition were actually filmed on Earth, could the film be shown in slow motion so a careful viewer would be fooled, and if so by what factor should the film be slowed? Consider how various moon scenarios might be faked on a sound stage on Earth. (Some people have claimed the Apollo moon landings were a NASA hoax, filmed on a sound stage.)

Day Nine

Fact or Friction?

On the day of Oliver Cromwell's death a great storm swept over England; fifteen-year old Isaac Newton carried out his first scientific experiment, jumping upwind and downwind to compare how far he could leap in either direction.

We're still not accustomed to gravity. We push ourselves out of bed as we did last week, but instead of sailing across the room (which is how we moved from place to place in the Bubble), we find ourselves face down on the floor. In this meditative position we reflect on the uses of friction in normal locomotion.

We hadn't thought about this before, but it must be the frictional force of the carpet on our shoes that gets us moving as we begin to walk, and slows us down again when we quit walking. As we try to walk forward, for example, we push backward on the carpet. By Newton's third law the carpet exerts an equal but opposite "reaction" force on us, which propels us forward.

The floor exerts *two* forces on us:

(1) a *normal* force, perpendicular to the floor, which supports us against gravity;

(2) a *frictional* force, parallel to the floor, which changes our horizontal velocity.

We would get nowhere if we tried to walk on frictionless ice. Without friction, cars couldn't speed up or slow down. Trains would spin their wheels on the rails. Knots would not stay tied. Violin strings could be plucked but not bowed. Down-hill skiing would be nearly uncontrolled.

9.1 Static friction

Against one wall stands a table covered with patches of various woods, metals, leather, cloth, and rubber. Several blocks of similar materials sit on the table, each with a hook screwed into one end.

We attach one of our springs to the hook on a wooden block, and set the block on a patch of carpet. We pull the spring sideways and measure its stretch. The block stays put if we don't pull too hard, so the frictional force must be equal but opposite to the spring force $F_{\text{spring}} = k\Delta x$.

As we stretch the spring more and more, the block finally gives way and starts to slip. We conclude that *static* friction, the friction between two surfaces which are mutually at rest, obeys the rule $F_S \leq F_{S\,max}$, where $F_{S\,max}$ is the maximum static friction possible between the block and the carpet patch. Static friction can resist the applied force of the spring up to a certain point, beyond which the block breaks free and starts to accelerate in the direction of the applied force. When we try the same experiment over and over, we get more or less the same result, although there is a fair amount of variation in the spring stretch at breakaway.

Our block looks like a sawed-off piece of two-by-four. So far we have set it on its "flat" side, the wider 4" side. We now try putting it on its narrower, 2" side. To our surprise we get the same results as before. We had thought the flat side would have greater friction, because there is more of it in contact with the surface than if it rests on its narrower side. But it doesn't matter *which* side rests on the surface beneath; friction seems to be independent of the area of contact.

When we set the block on a wood patch instead of the carpet patch, it starts to move with a substantially *smaller* spring stretch, so $F_{S\,max}$ is less in this case. Friction does depend on the properties of the two surfaces in contact.

We also attach a second block of the same size on top of the first one and try pulling along the carpet patch again, as shown in Fig. 1. The friction is stronger than before. We have to stretch the spring *twice* as much to start the double-block moving along. So friction *does* depend on the mass of the object being pulled. Doing many experiments we

Fig. 1. The maximum friction is twice as much for a double block.

find that the maximum friction between the same two surfaces increases linearly with mass. The reason must be that larger masses require larger supporting contact forces N between the two surfaces. The more the surfaces are pressed together, the larger the friction should be between them. In the cases we have tried, with the block sitting on a level surface with only the gravitational force $F_g = mg$ pressing the block downward, the normal force is $N = mg$. If we could turn off gravity, friction should vanish because the surfaces would not be pressed together. We conclude that

$$\boxed{F_S \leq F_{S\,max} = \mu_S N,} \tag{1}$$

where N is the normal force and μ_s is some constant of proportionality that depends on the board and the surface it sits on. We do multiple experiments to measure this "coefficient of static friction" μ_s for the wooden block pulled along a carpet patch, and for the same block pulled along a wood patch. We also try pulling a leather-covered block along the same two surfaces. Our results are shown in Table 1. Shown are averaged values; errors are typically 10% or so.

	wooden block	leather block
carpet patch	0.30	0.81
wood patch	0.23	0.37

Table 1. The coefficient of static friction μ_s between various surfaces

There is a way to test more generally that the maximum frictional force really *is* proportional to the normal force between two surfaces. We set the wooden block on a sloped wooden board, as shown in Fig. 2 (a). We gradually tilt the board up more and more,

Fig. 2(a) Block on a sloping surface (b) Isolation diagram of the block

and find an angle θ_{max} beyond which the block slides down. If Eq. 1 is correct, then

$$F_{s\,max} = \mu_s N = \mu_s mg\cos\theta_{max}\,, \tag{2}$$

from the isolation diagram of Fig. 2(b). Note that now N is less than mg in general, going to zero as $\theta_{max} \rightarrow 90^o$. But if the block does not slide, it must also be true that

$$F_{s\,max} = mg\sin\theta_{max} \tag{3}$$

because the two force components along the slope must cancel. Therefore we have $F_{s\,max} = mg\sin\theta_{max} = \mu_s mg\cos\theta_{max}$, from which we predict that the block will start to slide when θ exceeds θ_{max}, where

$$\tan\theta_{max} = \mu_s. \tag{4}$$

This rule works pretty well when we try it out.* The block breaks free and starts to slide when the tilt-angle $\theta_{max} \approx 13^o$, giving $\mu_s = \tan\theta_{max} \approx 0.23$, the value we had measured before for wood on wood, as given in Table 1. However, the block sometimes breaks free at a different angle; results from one experiment to another are not exactly consistent. Perhaps the coefficient of friction is quite sensitive to the surface, which may be slightly different from one place to another, and even the same spot may be damper or greasier in one experiment than in another, so we do not get absolutely repeatable data.

Sample Problem 1. A steel block of mass $m = 2.0$ kg is placed on a wooden board, and the board is slowly tilted up, as shown. When the slope angle with the horizontal reaches $\theta = 30^o$, the block starts to slip down the board.

(a) What is the coefficient of static friction μ_s between the block and board?

(b) If a block of mass $m = 4.0$ kg made of the same material is placed on the board instead, at what angle would it start to slip?

(c) If the whole apparatus were taken to Mars, at what angle would the two-kilogram block start to slip, assuming μ_s is not changed by the lower temperature ?

Solution: (a). According to Eq. 4, the coefficient of static friction must be $\mu_s = \tan 30^o = 1/\sqrt{3} = 0.58$.

(b) From Eq. 4 the tilt angle at slip depends only on μ_s, and not on the mass of the block, for example. So the angle at which slip starts to occur will still be $\theta = 30^o$.

(c) Again, the tilt angle at slip depends only on μ_s, and not (for example) on gravity g. So the angle at slip will still be $\theta = 30^o$ on Mars.

Summarizing our results for static friction, we have found that the frictional force is

(1) less than or equal to a maximum value $F_{s\ max}$, where $F_{s\ max}$ is directly proportional to the normal force of contact N, i.e. $F_{s\ max} = \mu_s N$;

(2) independent of the area of contact;

(3) dependent on the nature of the materials in contact. That is, μ_s depends on which materials are touching one another.**

There is also lots of scatter in our data; it seems that the rules we have found are only *approximate* ones.

* Note that since $\tan 45^o = 1$, if an object begins to slip when the surface is tipped up to 45°, then the coefficient of static friction between the object and surface is $\mu_s = 1$. There are materials which start to slip for considerably larger angles than 45°; that is, it is possible for coefficients of static friction to exceed unity (See Table 2).

** The static friction rules (1)-(3) were first enunciated by Leonardo da Vinci in the late fifteenth century.

Sample Problem 2. Two blocks, with masses M and m, are pushed along a frictionless table as shown, by an external horizontal force F_O. The coefficient of static friction between the blocks is μ_s. What is the minimum force F_O needed to keep m from falling down?

Solution: Isolation diagrams of the entire system, of m, and of M, are shown. Here N is the force of contact between the two blocks, F_S is the frictional force between them, and N' is the normal force of the table on M.

The net horizontal force on the entire system is just F_O, as seen in the first diagram, so $F_O = (M + m)a$. As long as the blocks keep moving together, the acceleration of each block must be the same, $a = \dfrac{F_O}{M + m}$ to the right.

From the isolation diagram of the small block we learn that $F_S = mg$ if m doesn't fall down, and also that $N = ma$, since N is the net horizontal force on m. We therefore have

$$mg = F_S \leq \mu_s N = \mu_s ma = \mu_s m \left[\frac{F_O}{M + m} \right].$$

Solving for F_O, we find that $F_O \geq (M + m)\, g / \mu_s$, so the minimum outside force needed to keep the small block from falling is

$$F_{O\,\min} = \frac{(M+m)g}{\mu_S}.$$

The final isolation diagram, of the large block by itself, could be used to check our results.

Box I: Friction in Geological Faults

Set a block on a tabletop and attach a spring to the block, as shown. Begin pulling on the spring, slowly increasing its stretch. At first the block does not slip, because static friction between the block and table is sufficient to cancel the applied spring force.

When the stretch in the spring becomes large enough, static friction can no longer keep the block in place, so it slips along the table. That takes place when the spring force exceeds the maximum force of static friction $F_{s\ max} = \mu_s N$, where μ_s is the coefficient of static friction and N is the normal force between the block and table. After the slip, the spring is less stretched than before, so as we keep pulling steadily on the spring the block again remains at rest until the stretch reaches a critical value where the maximum frictional force is again overcome and the block slips once again. The figure illustrates the spring force as a function of the block's displacement. The vertical portions represent the static positions of the block and the sloped portions occur during the slips.

This type of motion is very similar to that taking place in geological faults, particularly in so-called *strike-slip faults.* California's San Andreas Fault, for example, is the boundary between the Pacific Plate to the west and the North American Plate to the east. The fault slips from time to time in a quite noticeable way. The Pacific Plate is moving northwest relative to the North American Plate at about 4 cm/year on average*; this is called the *strain*, analogous to a steady stretch in a spring. Los Angeles on the Pacific Plate is moving (on the average) toward San Francisco on the North American Plate at about 3 cm/year. (Not all of the strain is accommodated on the San Andreas Fault alone.)

*About the rate at which fingernails grow.

As a result of the strain in a strike-slip fault, a *shear stress* τ develops. This stress is the force per unit area of the fault, pushing the Pacific Plate northwesterly relative to the North American Plate in the case of the San Andreas. It is analogous to the force of a stretched spring; that is, the strain Δx in the spring produces the stress $F = k\,\Delta x$.

In addition to the shear stress, there is also a *normal compressive stress* σ_n between the two plates pushing them together. This stress is the force per unit area of the fault pushing the two sides together; it is analogous to the normal force between a block and tabletop.

As with a block on the table, if the shear stress is relatively small (perhaps because a slip has recently occurred), the fault will not slip. But as the shear stress gradually increases the fault will suddenly give way; this takes place when the shear stress starts to exceed the coefficient of static friction times the normal compressive stress; that is, when

$$\tau > \mu_s \sigma_n$$

where μ_s is the static coefficient of friction between the rock on the two sides of the fault.

The sudden slip along a fault generates an earthquake. Many years later the shear stresses may again grow large enough (or for some reason the normal compression may grow small enough) for another slip to occur. The photograph shows the results of the 3 m slip of the San Andreas Fault during the great 1906 earthquake (the fence offset was on the ranch of E. R. Strain (!) in Marin County.)*

The figure on the following page shows the results of experiments measuring the maximum shear stress to initiate sliding as a function of normal stress for a variety of rock types. The linear fit defines a maximum coefficient of static friction equal to $\mu_s = 0.85$. It is interesting that the coefficient of friction seems to be very similar from one type of rock to another.

* "Although the above description of fault friction works well in laboratory conditions, it remains something of a mystery how earthquakes can occur at great depths in the Earth, since the normal stress due to the weight of overlying rocks is enormous. Rare earthquakes have been observed at depths greater than 600 km. Under such conditions, some novel "lubricating" process must be acting to reduce the effective coefficient of friction. Sudden chemical phase changes and melting have been proposed, but experiments under such extreme conditions are difficult to perform." - G. A. Lyzenga

Maximum shear stress to initiate sliding as a function of normal stress for a variety of rock types. The linear fit defines a maximum coefficient of static friction max f_s equal to 0.85. Data from J. D. Byerlee, Friction of rocks, in *Experimental Studies of Rock Friction with Application to Earthquake Prediction*, ed. J. F. Evernden, pp. 55–77, U.S. Geological Survey, Menlo Park, Calif., 1977.

[The graph above and the photo on the preceding page are from *Geodynamics* by Donald L. Turcotte and Gerald Shubert, John Wiley and Sons, Inc. 1982]

9.2 Sliding friction

When we pull a block along our level table, it accelerates quickly when the spring force exceeds $F_{s\,max}$. That is, when the block breaks free from its static position it jerks away, accelerating more rapidly than we would expect if the spring force just barely exceeded the frictional force. The same thing happens on the sloping surface. As the angle is slowly increased, the block suddenly breaks loose and slides down. Its acceleration is not as gradual as we had expected.

We conclude that although the spring force and friction force were balanced before the block breaks free, they are not at all balanced just afterwards. The spring force has increased only slightly, so it must be that the frictional force has *decreased* once the block gets moving. The coefficient of friction we have been measuring must apply only to static friction, for two surfaces that are not moving relative to one another.

Now we have to find out how friction behaves when the two surfaces are sliding past each other. We know from experience there is still plenty of friction when we slide along the ground, and on skis sliding along snow, especially if they are not waxed. There is obviously friction when we strike a match. It seems likely that this *sliding* or *kinetic* friction still depends on the normal force, because that is the force of contact between the two surfaces. It is reasonable that if the block is pressed harder onto the table, the sliding friction should increase. But we have to do some experiments to see how it depends on the normal force N, how it depends on the area of contact, and also how it depends on the relative velocity between the surfaces.

Beside the table is a long continuous belt made of some sort of rubberized cloth, made to turn by a motor. We can learn about the kinetic friction between various blocks and this rubberized cloth, by setting a block down on it with a spring attached.

For a given belt speed we measure the spring stretch, which gives us the spring force $F_{spring} = k\Delta x$; when the block stays at rest we therefore know the equal-but-opposite force of kinetic friction. Our experiments again show that the kinetic friction is proportional to the normal force and is independent of contact area.

There is a dial on the wall that allows us to control the speed of the belt. It is interesting that the frictional force seems to be roughly independent of velocity. Whatever the belt speed is, the spring stretch stays about the same. To a good approximation kinetic friction therefore depends only on the normal force and (we suspect) the nature of the two surfaces, so

$$\boxed{F_k = \mu_k N} \tag{5}$$

where μ_k is the *coefficient of kinetic friction*. As with the coefficient of static friction μ_s, the coefficient μ_k depends on which block we are using.

Our experiments again are not completely reproducible. Sometimes the block alternately sticks and slips on the belt instead of maintaining a constant spring stretch. Even when the block doesn't stick and slip, our conclusions that kinetic friction

(1) is proportional to the normal force

(2) is independent of surface area

(3) is independent of velocity

are only approximate. There is quite a lot of variation from one experiment to another.

Some published coefficients of static and kinetic friction are given in Table 2.

Materials, lubricant	static	sliding
Hard steel on hard steel		
Dry	0.78	0.42
Oleic acid	0.11	0.12
Castor oil	0.15	0.081
Stearic acid	0.0052	0.029
Aluminum on mild steel, dry	0.61	0.47
Aluminum on aluminum, dry	1.05	1.4
Teflon on teflon, dry	0.04	--

Table 2. Some published coefficients of static and sliding friction, from Theodore Baumeister, Mark's Mechanical Engineer's Handbook, 7th ed. (McGraw-Hill, New York, 1967). Note that the coefficient of sliding friction is not always less than the coefficient of static friction! There is nothing simple about friction.

Sample Problem 3. (a) The light turns green, and a car of mass m accelerates forward. What force propels it?

(b) If the car brakes, what force slows it down?

(c) If the car brakes too quickly, it may start to skid. What force slows it down?

Solution: (a) The isolation diagram shows that the only forces acting on the car are those due to gravity and to the road. There are two forces caused by the road: the normal force N, which supports the car, and friction. Friction is the only horizontal force, so it has to be to the *right* (on the front wheels, if we assume a front-wheel drive car), since the car accelerates to the right. At first this seems counterintuitive: Aren't frictional forces suppose to resist motion instead of cause it?

front-wheel drive

Here it is actually *static friction* between the tires and road that propels the car. When the driver steps on the gas, the engine tries to turn the front wheels clockwise, as shown in the figure; this makes the tires push on the road toward the left. But by the action-reaction law (Newton's third law), the road must then push on the car toward the right, which causes the car to accelerate. So it is not the engine *directly* that accelerates the car, but the engine through the front wheels that makes the road exert a frictional force on the tires. This is *static* friction, because if the tires don't skid, the rubber is instantaneously at rest relative to the road.

accelerating

(Incidentally, the friction force F_s acting to the left on the road is of course not included in the isolation diagram of the car in the first picture above, because only forces *acting on the car* should be included in its isolation diagram.)

decelerating

(b) When the driver steps on the brake, the brake drums try to make the wheels turn more slowly. Therefore the tires produce a force on the road toward the right, as shown. This in turn produces a force to the left on the tires by the road according to Newton's third law, slowing the car. Again the force on the car is *static* friction, as long as the tires do not skid.

(c) If the car skids, *kinetic* friction acts on the car. Since kinetic friction is often smaller than static friction, it will not slow down the car as quickly as static friction. If a skidding car has mass M, the force of kinetic friction is

$$F_k = -\mu_k N = -\mu_k Mg$$

if the car is on a horizontal surface (minus because the force is to the left). Friction is the only horizontal force, so from $F = Ma$ it follows that the car decelerates with $a = -\mu_k g$.

Note: The formulas $F_s \le \mu_s N$ and $F_k = \mu_k N$ are only approximate relationships.

Sample Problem 4. Three mountaineers, roped together, with masses m_1, m_2, and m_3 (where $m_1 < m_2 < m_3$) slide down a glacier of slope-angle θ in a *sitzglissade* position, as shown. They want to slide so the distance between them stays constant. Neglect air resistance and assume that the coefficient of kinetic friction μ_k with the ice is the same for each mountaineer. In what order should they slide (*e. g.*, m_1 first, m_2 in the middle, and m_3 in the rear) so neither rope bunches up, allowing one mountaineer to slide into the person in front?

Solution: It is often interesting to try to *guess* an answer first, using physical intuition. *What do you think is the correct order?* What would tend to prevent a mountaineer from running into the person in front?

Let the mountaineers be in the order a, b, c, with a in front. The isolation diagrams then are shown, where T_{ab} is the tension in the rope between a and b, and T_{bc} is the tension in the rope between b and c.

There is no acceleration perpendicular to the slope, so $N_a = m_a g \cos\theta$, $N_b = m_b g \cos\theta$, and $N_c = m_c g \cos\theta$. *Parallel* to the slope, the three $F = ma$ equations that follow from the isolation diagrams are

$$\text{(a)} \quad m_a g \sin\theta - T_{ab} - \mu_k N_a = m_a a_a$$

$$\text{(b)} \quad T_{ab} + m_b g \sin\theta - T_{bc} - \mu_k N_c = m_b a_b$$

$$\text{(c)} \quad T_{bc} + m_c g \sin\theta - \mu_k N_c = m_c a_c$$

If their distances apart stay constant, they must all have the same acceleration, $a_a = a_b = a_c = a$. If we add the three equations all the tensions cancel out, leaving

$$(m_a + m_b + m_c)\, a = (m_a + m_b + m_c) g \sin\theta - \mu_k (m_a + m_b + m_c)\, g \cos\theta.$$

The masses cancel out, giving an acceleration $a = g(\sin\theta - \mu_k \cos\theta)$, the same for each mountaineer, regardless of their masses. Substituting this expression for a back into Eq. (a), we find that the equation is satisfied only if $T_{ab} = 0$. Similarly, Eq. (c) is satisfied only if $T_{bc} = 0$. Then Eq. (b) is satisfied as well. In other words, under the circumstances stated, neither rope is sufficiently taut to have a tension, so the mountaineers are in effect sliding down independently, all with the same acceleration. The order of their descent doesn't matter! The heaviest mountaineer could be in front or at the rear, it doesn't matter.

Note: The formulas $F_s \le \mu_s N$ and $F_k = \mu_k N$ are only approximate relationships. Careful measurements show that friction is *not* exactly linear in N, nor is it exactly independent of surface area, nor is kinetic friction exactly independent of velocity. Surface friction is complicated, the result of several kinds of interactions, some at the level of individual molecules. Each surface is typically bumpy; the larger bumps have smaller bumps, etc. Therefore usually only a small proportion of two surfaces are really in contact. Only the bumps actually touch, so the real surface area in contact may be only about 1/1000 of the total area that *appears* to be in contact.

When the normal force between surfaces is increased, the bigger bumps deform to allow smaller bumps to touch. So it is reasonable that friction should increase, because there is more surface actually in contact. If the surface area is changed without changing the normal force, say by setting a block on a different side, there may be no more bumps hitting each other than before. Therefore friction should be largely independent of the apparent area of contact.

At the molecular level the surfaces are apt to be contaminated with various surface films. So what may appear to be contact between a wooden surface and a metal surface may really be contact between two layers of grease or a layer of grease and a layer of metal oxide, for example. Unknown and varying contaminants are one source of irreproducibility in friction measurements.

When the surfaces slide past each other, the block vibrates up and down slightly, because the bumps crashing into each other cause vertical as well as horizontal forces. So on the *average* there may be fewer bumps touching than when the surfaces are at rest, which means the friction may decrease. So it may happen that $\mu_k < \mu_s$ for the same two materials.

As bumps smash into each other, the molecules in a bump on one surface may bond with the molecules in a bump on the other surface. This can cause the surfaces to stick together. Then the applied force may break the bonds and the surfaces once again slip past each other until new bonds are formed and different bumps stick together. Well-cleaned and polished surfaces may establish so many bonds that the surfaces become permanently welded together. This is contrary to the usual intuition that rough surfaces have more friction than smooth ones. If the surfaces are very smooth, friction can be enormous.

[See "Friction at the Atomic Scale" by Jacqueline Krim, *Scientific American*, October 1996, p. 74 . See also the references listed at the end of this article. The study of friction is called *tribology*.]

the dimension of time t, and the dimensions of mass m cancel in taking the ratio k/m. Therefore letting $v = v_0 V$ and $t = m\tau/k$, the $F = ma$ equation $-kv = mdv/dt$ becomes simply

$$-V = dV/d\tau .$$

The parameters m and k have been absorbed into τ and the parameter v_0 has been absorbed into V. The $F = ma$ equation has been simplified, and because V and τ are dimensionless, only numbers are required, not units.

[Note that in this problem the equation can be solved exactly (this is very unusual!) By the properties of exponentials, the solution is $V = V_0 e^{-\tau}$, and we can then set the arbitrary constant $V_0 = 1$, since the initial velocity of the particle is v_0, so $V = v/v_0 = 1$ initially. Therefore the *exact* solution is

$$V = e^{-\tau},$$

valid for all times $\tau \geq 0$. The velocity of the particle decays exponentially.

We can integrate once more to find the dimensionless position X of the particle, where the X is related to V and τ by $V = dX/d\tau$. Separating variables, we have

$$X = \int_0^X dX = \int_0^\tau V\,d\tau = \int_0^\tau e^{-\tau}\,d\tau = -e^{-\tau}\Big|_0^\tau = 1 - e^{-\tau}.$$

This gives $X = 0$ at the initial time $\tau = 0$, and also $X \to 1$ as $\tau \to \infty$. Note that the differential $dX = Vdt = (v/v_0)\,(k/m)dt = (k/mv_0)\,dx$, since $dx = v\,dt$. Therefore the physical distance x is related to the dimensionless distance X by $x = (mv_0/k)\,X$.]

(2) *Second Step: Integrate the dimensionless $F = ma$ equation using finite differences.*

As described in *Special Topic* II of *Day One*, for smooth functions $y(x)$ we can evaluate the derivative $y'(x)$ using

$$y'(x) \cong \frac{y(x+\Delta x) - y(x)}{\Delta x}$$

to some given approximation so long as the finite quantity Δx is sufficiently small. For the equation $F = ma$, the necessary finite-difference equations are

$$a(t) \cong \frac{v(t+\Delta t) - v(t)}{\Delta t} \qquad \text{and} \qquad v(t) \cong \frac{x(t+\Delta t) - x(t)}{\Delta t}.$$

If we use constant time-steps Δt, these can be rewritten in the form

$$a_i \cong \frac{v_{i+1} - v_i}{\Delta t} \qquad \text{and} \qquad v_i \cong \frac{x_{i+1} - x_i}{\Delta t},$$

where a_i is the acceleration at the ith time-step, v_i is the velocity at the ith time-step, and v_{i+1} is the velocity at the following time-step, for example. Rewriting these equations,

$$v_{i+1} \cong v_i + a_i \Delta t \qquad \text{and} \qquad x_{i+1} \cong x_i + v_i \Delta t.$$

Therefore starting with some given initial velocity v_0 and position x_0, choosing some small time-step Δt, and knowing each acceleration $a_i = F_i/m$ from the known force and mass, we can solve these equations one step at a time to find both x and v indefinitely into the future.

In our particular problem $-V = dV/d\tau$, for example, the dimensionless acceleration $A \equiv dV/d\tau = -V$, the initial position is $X = 0$, and the initial dimensionless velocity is $V_0 = v_0/v_0 = 1$. Therefore after a single time-step, $X_1 = X_0 + V_0\Delta\tau = 0 + \Delta\tau = \Delta\tau$, and $V_1 = V_0 + A_0\Delta\tau = 1 - V_0\Delta\tau = 1 - \Delta\tau$. Knowing X_1 *and* V_1 we can find X_2 and V_2, etc. This simple algorithm is easily programmed. The Table below shows a few steps in this calculation for the case $\Delta\tau = 0.01$; because this particular problem is exactly solvable, the numerical solution can be compared with the exact solution.

Numerical (and exact) solution of V and X for the equation $-V = dV/d\tau$

time τ	V (numerical)	$V = e^{-\tau}$	X (numerical)	$X = 1 - e^{-\tau}$
0.00	1.00000	1.00000	0.00000	0.00000
0.01	0.99000	0.99005	0.01000	0.00995
0.02	0.98010	0.98020	0.01990	0.01980
0.03	0.97030	0.97045	0.02970	0.02955
0.04	0.96060	0.96079	0.03940	0.03921

The numerical solution is fairly good so far, but it tends to diverge farther and farther from the correct answer as time goes on. The numerical solution can be improved by using a smaller time-step. Better still, for serious calculations a better numerical algorithm (4th order Runge-Kutta, for example) should be used.

PICTURE QUIZ 9

Answers are given at the end of the text

1. In which direction is the frictional force of the sidewalk on the wheel of a unicycle when the unicyclist

(i) accelerates forward?

(A) forward (B) backward

(ii) slows down?

(A) forward (B) backward

2. A penny is laid flat on a large rotating horizontal turntable a distance of 1.6 m from the turntable's center, as shown in a top view. The coefficient of static friction between the penny and turntable is 0.11. Which of the following is closest to the maximum speed v the penny can move along its circular path without slipping off the turntable?

(a) 0.8 m/s (b) 1.3 m/s (c) 1.8 m/s (d) 2.3 m/s

3. Two skiers descend a hill side-by-side, one a linebacker for the Dallas Cowboys and the other a jockey at the Kentucky Derby. They reach a level snowfield with the same speed and then coast to a stop because of friction with the snow. If the coefficient of friction between snow and skis is the same for each skier and if air resistance is negligible, which skier comes to rest first?

(a) the linebacker

(b) the jockey

(c) they come to rest at the same time

PROBLEM QUIZ 9

Answers are given at the end of the text.

1. *Spoilers* are curved surfaces placed at the rear of race-cars, which cause air flowing past to exert a downward force on the rear of the car. Why is this important?

2. A theme park features a large cylindrical drum with vertical sides, as shown. The radius of the cylinder is r. A rider gets into the drum, and then it starts to turn around the vertical axis. If the coefficient of static friction between the drum wall and a rider is μ_s, what is the minimum angular velocity ω of rotation such that the rider will be able to raise her feet off the floor and stick to the cylinder wall without falling?

3. A bullet of mass m_1 is fired into a wooden block of mass m_2 initially at rest on a horizontal surface. The block then slides a distance d before coming to rest. If the coefficient of kinetic friction between block and surface is μ_K, what was the speed of the bullet just before impact?

4. Two blocks are attached to opposite ends of a massless cord, and the cord is placed over a frictionless peg so the blocks sit on a wedge as shown. Block #1, of mass m_1, sits on a 30° slope; there are coefficients of static friction $\mu_S = 2/3$ and kinetic friction $\mu_K = 1/3$ between that block and the surface. Block #2, of mass m_2, sits on the 60° slope; there is no friction between that block and the slope. Gravity g acts downward.

(a) In terms of m_1, find the *maximum* mass m_2 block #2 can have, so the system remains in equilibrium. Note that if m_2 is large enough, the frictional force on m_1 will have to be *downhill* to the left, to keep m_2 from falling. The isolation diagrams for each block are as shown.

(b) Suppose that $m_2 = 2m_1$. Show that the blocks will slide, and draw an isolation diagram for each block.

(c) Find the acceleration of the blocks as a multiple of g, and the tension in the cords as a multiple of $m_1 g$, in the situation described in part (b).

Special Topic I: Drag Forces and Reynolds Numbers

Two dimensionless ratios describe the motion of objects within fluids, including raindrops falling through air, cars racing along a track, and submarines gliding through the ocean. One is the *drag coefficient* (or *pressure coefficient*)

$$C_D = F_D/\rho\ell^2 v^2 \tag{S1}$$

where F_D is the drag force, ρ is the density of the fluid, ℓ is a typical linear dimension of the object, such as its width or length, and v is the velocity of the object relative to the fluid. Automobile designers go to considerable trouble to minimize the drag coefficient for at least some of their cars, because this will increase gas mileage. The other dimensionless ratio is *Reynolds number*

$$R = \rho\,\ell v/\eta\,, \tag{S2}$$

where ρ, ℓ, and v have already been described, and η is the viscosity of the fluid. The viscosity is a measure of a fluid's frictional effect: Molasses has relatively high viscosity, for example, and lubricating oil has relatively low viscosity. The type of flow past an object depends upon the Reynolds number, as illustrated by the sketches below for fluid flow past a cylinder. (From the *Feynman Lectures in Physics*, Fig 41-6)

Note that smooth "laminar" flow for very small Reynolds numbers (a) gives way to the formation of vortices in (b) and a "Karman vortex street" in (c). By (d) the motion has become more turbulent and chaotic, and then by (e) more flow lines have become regular, there is a steadier wake, and the drag force has dropped below that in (d). These fluid behaviors and the resulting drag forces are not well understood.

For sufficiently small velocities v, that is, for sufficiently small Reynolds numbers, the drag force on a ball of radius r moving through a fluid (air, water, molasses) with *viscosity* η is given by *Stokes's Law* $F_D = -6\pi\eta r v$. It follows that the viscosity η of a fluid has the dimensions $[\eta] = \left[\dfrac{F}{L \cdot L/T}\right] = \dfrac{M}{LT}$, so it can be expressed in units of kg/(m · s) or $\mathrm{N \cdot s / m^2}$.

As long as the only dimensional parameters involved in the motion of an object through a fluid are F_D, ρ, ℓ, v, and η, the drag coefficient C_D has to be some function of the Reynolds number R alone. That is, we must have $C_D = f(R)$ where f is some unknown function. (We cannot have some more general equation like $C_D = f(R, v)$, for example, in which C_D depends not only on the Reynolds number but also the object's velocity v all by itself, because there would then be no way to make the equation dimensionally consistent.)

The function $f(R)$ is generally too hard to calculate from first principles, so it is found experimentally for objects of a given shape.

The drag force on spheres has been studied extensively. In this case the characteristic length ℓ can be taken to be the radius r of the sphere. When the drag coefficient is measured as a function of the Reynolds number, results are as shown in the graph.

Drag (Pressure) Coefficient C_D for a sphere, in terms of the Reynolds Number R

[AIP Handbook of Physics 2nd Edition (1963) Sec II, p. 253]

The advantage of plotting C_D vs R is that this plot is independent of the sphere radius or of the fluid's density ρ or viscosity η. That is, the curve applies to *any* sphere within *any* fluid, as long as the parameters contained in C_D and R are the only ones that affect the motion. It does not apply to a ship on the ocean, not only because a ship is not a sphere, but also because surface water waves, which affect the ship's motion, depend upon the gravitational field g in addition to the other parameters.

Sample Problem S1. Use the graph on the preceding page to find out how the drag force on a sphere depends upon the sphere's velocity.

Solution: Suppose that the drag coefficient C_D is proportional to the Reynolds number taken to some power; that is, $C_D = \alpha R^n$ for some constants α and n. Taking the logarithm of both sides,

$$\log C_D = \log(\alpha R^n) = \log\alpha + n \log R,$$

so if $\log C_D$ is plotted *vs* $\log R$, the result should be a straight line whose slope is n. In fact, the above graph is *already* a plot of $\log C_D$ *vs* $\log R$, because each axis is linear in the exponent x in 10^x. For very low Reynolds numbers, the slope of the graph is about -1, so for small R, $C_D \propto 1/R$, i.e.,

$$F_D/\rho r^2 v^2 \propto \eta/\rho r v .$$

Therefore $F_D \propto r \eta v$, *linear* in v, and consistent with Stokes's Law. This is the so-called *viscous drag regime*.

Note from the graph that the drag coefficient is roughly constant for $10^3 < R < 10^5$, so then $F_D \propto \rho r^2 v^2$. In this range of Reynolds numbers, the drag force is roughly *quadratic* in v.

Sample Problem S2. Find the terminal velocity v_T of falling spherical hailstones as a function of their radii r.

Solution: A falling hailstone is subject to gravity and to the drag force F_D of air resistance, as shown in the isolation diagram at the right. The gravitational force is $F_{grav} = mg = \rho_{ice}(4/3)\pi r^3 g$, where

ρ_{ice} (= 0.92 g/cm^3 = 920 kg/m^3) is the density of ice, and r is the radius of the hailstone. The drag force is given by $F_D = C_D\, \rho_{air}\, r^2\, v^2$, where ρ_{air} (= 1.2 kg/m^3) is the density of air (which we take to be at sea level), and v is the speed of the hailstone. The weight of the hailstone stays constant, while the drag force increases as the hailstone speeds up. Eventually F_D approaches mg; i.e., the two forces nearly cancel. The hailstone then nearly stops accelerating, and approaches a terminal velocity v_T as shown, corresponding to $mg = F_D$.

Setting $mg = F_D$ with $v = v_T$, we find that $C_D v_T^2 = \dfrac{4\pi g}{3}\left(\dfrac{\rho_{ice}}{\rho_{air}}\right) r$, so

$$v_T = \sqrt{\frac{4\pi\left(9.8\ \text{m/s}^2\right)}{3\,C_D}\left(\frac{920}{1.2}\right)\left(\frac{r}{1\ \text{cm}}\right)10^{-2}\text{m}} = \sqrt{\frac{315}{C_D}\left(\frac{r}{1\ \text{cm}}\right)}\ \text{m/s},$$

where r has been expressed as a multiple of one centimeter. It is an interesting problem to find v_T, because the drag coefficient C_D is a function of the Reynolds number $R = \dfrac{\rho\,\ell\,v}{\eta}$, which obviously depends itself on the velocity. So we have a "Catch 22" situation: we *need* v to *get* v. What we will do is find a self-consistent solution, starting with a guess that the terminal velocity is somewhere between 10 m/s and 100 m/s (20 mi/hr and 200 mi/hr). Then

$$R = \rho\,\ell\,v/\eta = \frac{\left(1.2\ \text{kg/m}^3\right)(.01\ \text{m})(\text{from 10 m/s to 100 m/s})}{1.8\cdot 10^{-5}\,\text{kg/m}\cdot\text{s}},$$

giving $6{,}700 < R < 67{,}000$ for this range of velocities, for a hailstone of radius $r =$ 1 cm. From the experimental graph of drag (pressure) coefficient on the previous page, $C_D \approx 0.5$ for this entire range of Reynolds numbers. So with $C_D = 0.5$ in the above formula for v_T, we find that

$$v_T = \sqrt{\frac{315}{0.5}\left(\frac{r}{1\ \text{cm}}\right)} = 25\ \text{m/s}\ \ (\approx 56\ \text{mi/hr})$$

for a 1 - cm radius hailstone. This is within the range of assumed velocities, so the value $C_D \approx 0.5$ is consistent. For hailstone radii which are not too different from 1 cm, the terminal velocity is then proportional to the square root of the radius. Large hailstones fall faster than small ones.

Questions and Problems from Day Nine

9Q1. The vibrations of a violin string are excited by the bow, which alternately sticks and slips on the string. *Why* does the bow stick and slip as it is drawn over the string, in terms of properties of static and kinetic friction?

9Q2. One can walk normally on a level surface on Earth if the coefficient of static friction between shoes and surface is at least $\mu_s = 0.2$. Show that this value is roughly what you might expect. (If μ_s is as low as 0.1, it is still possible to walk if it is done slowly and carefully.) What μ_s would be needed to walk normally on the moon? ($g_{moon} = (1/6)\, g_{earth}$.)

9Q3. The maximum stable angle of the slope on a sand dune is called the *angle of repose*. If one tries to make the slope steeper, the sand slides downward until the angle with the horizontal is less than or equal to the angle of repose. This can be thought of roughly as a frictional effect; a layer of grains on the surface is prevented from sliding by friction with the layer of grains just beneath it. The angle of repose on Earth is about 33°. What is the approximate angle of repose on Phobos, one of the moons of Mars, where the gravitational field is about one thousand times smaller than on Earth?

9Q4. A body's center of mass moves only if an *external* force is applied. It is however possible for you to get to the other side of the room in a chair without letting your feet touch the floor. How is this possible, since the energetic manuevers required of you lead to *internal* forces between yourself and the chair?

9Q5. Why is no inequality $F_k \leq F_{k\ max}$ needed for kinetic friction, as it is for static friction?

9Q6. Support a stick (like a pencil, chopstick, or broomstick, with or without the broom) on top of your two extended forefingers, as shown. The fingers do not have to be equidistant from the CM of the stick. Now slowly slide your fingers toward one another, keeping them level. The stick will not tip off and fall. Use properties of the frictional forces between your fingers and the stick to explain why. (The CM of the stick is the point about which the stick is balanced in uniform gravity.)

9Q7. Many cars have anti-lock brakes. What is the advantage of such a braking system?

9Q8. Why does chalk sometimes make a squealing sound on a blackboard?

9Q9. Why do tires sometimes squeal as a car turns a sharp corner?

9Q10. A resident of Maine had trouble driving in the ice and snow, so she put a log in the trunk of her car to increase traction. The trunk was in the rear, and the car was front-wheel drive. How might the log affect driveability of the car? (A call-in question posed on the "Car Talk" program on National Public Radio)

9Q11. Two spheres of equal radius but different mass are dropped from the leaning tower of Pisa. Which reaches the ground first, if air resistance is taken into account?

9Q12. Two hailstones of different radius fall through air. Which (if either) falls faster?

9Q13. What are the dimensions of the coefficients of friction?

9Q14, The force of sliding friction is supposed to depend on the normal force, but not on the area in contact with the surface. Why then are long skis "faster" than short skis?

9Q15. Which of the following expressions *cannot* be the correct answer to a problem in all systems of units? Here m is a mass, and F and N are forces.

$F = \mu_s(N + mg\cos\theta)$ $\qquad$ $F = N + \mu_k g\sin\theta$

$F = [mg + \mu_s N\tan\theta] / N(1 + \mu_s)$ $\qquad$ $F = mg\sin\theta / (1 + \mu_k mg\cos\theta)$

9Q16. What are typical Reynolds numbers for (a) a car moving at highway speeds? (b) a falling snowflake? (c) A running pig? (At standard temperature and pressure, air has the density $\rho = 1.2$ kg/m^3 and viscosity $\eta = 1.8 \cdot 10^{-5}$ N·s/m^2 .)

9Q17. Are there other dimensionless ratios, which are not linear combinations of the drag coefficient C_D and Reynolds number R, that can be formed from the same parameters F_D, v, ℓ, ρ, and η found in the definitions of C_D and R?

9?18. The following is taken from Mark Twain's *Answers to Correspondents.*

If it would take a cannonball 3 1/4 *seconds to travel four miles, and* 3 3/8 *seconds to travel the next four, and* 3 5/8 *seconds to travel the next four, and if its rate of progress continued to diminish in the same ratio, how long would it take to go fifteen hundred million miles?*

Arithmeticus
Virginia, Nevada

I don't know.

Mark Twain

Section 9.1 Static friction

9P1. A cubical block is placed on the outside surface of the vertical windshield of a bus, as shown. If the coefficient of static friction between the block and windshield is μ_s , what is the minimum forward acceleration the bus must have so the block will stay at the same spot on the windshield? Begin with an isolation diagram of the block.

9P2. A block of mass M is pushed by a purely horizontal force F_0 against a plate fastened to one end of a horizontal spring of force constant k, compressing the spring an amount x_0, as shown. The other end of the spring is attached to the wall so that the spring will not fall or sag under gravity. The plate and spring have negligible mass. As F_0 is slowly reduced, the block first begins to slip downward along the plate when the spring is compressed by the amount x_1 $(x_1 < x_0)$. Find the coefficient of static friction between block and plate in terms of M, g, k, and x_1.

9P3. A wheelchair-accessible ramp is constructed at an angle of $\theta = 6^0$ to the horizontal. What is the minimum coefficient of static friction μ_s between the rubber tires and ramp so a wheelchair will not slip on the ramp? Assume that essentially all the weight of the wheelchair rests on its two large wheels.

9P4. The coefficient of static friction μ_s between a train's wheels and the rails is roughly 0.3 for dry rails and 0.2 for wet rails. Estimate the minimum stopping distance d in meters for a train moving at speed $v = 100$ km/hour (60 mi/hr) on (a) dry rails; (b) wet rails. (The relatively low coefficients of friction is one reason why trains can't stop as fast as some automobile drivers think when they find themselves stuck on the tracks. Typically train wheels roll rather than skid when brakes are applied. If they are locked in an emergency stop, they are flattened on the bottom side by skidding along the rail, which can ruin the wheel.)

9P5. *Amtrak* has published the diagram shown, of estimated minimum stopping distances for an 8-car train. Estimate the coefficient of static friction between the wheels and rails, assuming the wheels do not skid along the rails.

9P6. How large can one make the coefficient of static friction μ_s between car tires and pavement? One way to find out is to look at drag-race statistics. According to the 1995 *Guinness Book of World Records*, the fastest 1/4 mile covered by a piston-engined dragster from a standing start is 4.726 seconds, by Scott Valitta in Bigtown, Texas on 6 March, 1994. Assuming that the action of the rapidly accelerating rear wheels nearly lifted the car's front wheels off the pavement, what is the lowest value of μ_s possible between the tires and road? Assume uniform acceleration.

9P7. A mountaineer of mass m walks up a rock surface tilted at angle θ to the horizontal. What is the minimum coefficient of static friction μ_s required between the mountaineer's boots and the rock, to prevent slipping?

9P8. Starting from rest on Earth's surface, a dragster of mass m finishes a quarter-mile race in time t_f. If 9/10 of the car's weight rests upon its rear wheels, and only those wheels propel the car, (a) in terms of t_f, what is the minimum coefficient of static friction μ_s required between rear tires and road? (b) If $t_f = 6.0$ s, what is this minimum μ_s? (c) What then is the velocity of the racer at the end of the six-second quarter mile ? (d) What minimum μ_s would be required for the same dragster to go a quarter-mile in 6.0 s on the moon, with $(1/6)^{th}$ Earth's gravity? Assume constant acceleration.

9P9. A sleepy driver awakens to find his car moving at speed v_o directly towards a wall, as shown. The coefficient of static friction between the tires and road (appropriate if the tires don't slip) is μ_s. Assume in the following that the tires do *not* slip. (a) He jams on the brakes while continuing to move straight toward the wall. Find his minimum stopping distance, in terms of v_o, g, and μ_s. (b) Suppose instead he turns in a circular path without touching the brakes, trying to avoid a crash as shown. Find the minimum radius of a circular path he can execute without skidding if his speed remains constant, in terms of the same parameters. (c) Which technique is more likely to avoid a crash?

9P10. A freeway off-ramp is a quarter-circle of roadway of radius $R = 150$ ft, as shown. (a) What is the ideal banking angle θ of a section of this road, if the speed of the cars on it is $v = 30$ mi/hr? At this ideal angle no friction is required for the cars to keep from slipping off the road.

view from behind *view from above*

(b) What minimum coefficient of friction μ_s is necessary so that a car at rest on this section will not slip sideways? Use an isolation diagram!
(c) For the μ_s found in part (b), what is the maximum speed a car can drive on this section without slipping sideways?

9P11. A car of mass m travels with constant speed v along the road shown. The coefficient of static friction between the car and road is μ_s, and the gravitational field is g. The dip and crest in the road are both circular arcs of radius R. If the driver

were to suddenly step on the gas when the car is at (a) the bottom of the dip (b) the top of the crest, what is the maximum possible instantaneous forward acceleration of the car, in terms of m, g, v, R, and μ_s ?

9P12. Two blocks, of mass M and mass m, are attached together with a massless string strung over a frictionless pulley. Block M sits on a surface tilted at angle θ to the horizontal, as shown, and block m hangs free vertically. There is a coefficient of static friction μ_s between the sloped surface and block M. If the blocks are released from rest, in terms of M and μ_s, (a) what is the minimum mass m for which m will start to fall and M to rise? (b) what is the maximum mass m for which m will start to rise and M to fall?

9P13. A pig sitting inside a large metal cone slides to the bottom, because the static coefficient of friction μ_s between the pig and cone is too small to prevent it. But if the cone is made to spin about its vertical axis of symmetry, the pig may be able to remain at the same spot on the cone, rotating around with it. Suppose the cone has half-angle θ as shown, and the pig sits at distance r from the axis. For given μ_s and g, find (a) the minimum (b) the maximum angular velocity ω of rotation for which the pig will not slip on the cone. What happens if (c) $\omega < \omega_{min}$? (d) $\omega > \omega_{max}$?

Hint: Which way would the pig tend to slip if the cone spins very slowly? very fast? Draw an appropriate isolation diagram in each case, showing the acceleration direction.

9P14. A pair of trousers is hung on a coathanger. One notices that if less than one-third of the length of the trousers is on one side of the coathanger (so there is more than two-thirds on the other side), the trousers slip off the coathanger. Estimate the coefficient of static friction between the trousers and coathanger, modeling the pair of trousers as uniform along its length.

9P15. As "Indy Cars"(racecars used in the Indianapolis 500) travel around curves they develop as much as 4 "g's" of lateral acceleration. What would the minimum required coefficient of static friction between tires and track surface have to be to prevent the cars from slipping on curves, if the track were unbanked (i.e., if it remained horizontal)?

Section 9.2 Sliding friction

9P16. The Iceworld explorers could check the properties of kinetic friction by sliding a block down a plane inclined by angle θ, as shown. Find an expression for the time needed to fall a distance s down the plane, assuming $F_k = \mu_k N$. Begin by drawing an isolation diagram of the block.

9P17. A mountaineer descends a snowy 45° mountain slope in a "sitzglissade" position as shown. Starting from rest, she slides a distance $d = 80$ m in time $t = 8.0$ s at uniform acceleration. Gravity is vertically downward with constant value $g = 10$ m/s^2. (a) Draw an isolation diagram of the mountaineer. (b) Find the coefficient of kinetic friction μ_k between the mountaineer and snow.

9P18. The coefficient of kinetic friction μ_k between a car and road is roughly 0.8 for a dry road, 0.5 for a wet road, and 0.1 for an icy road. Estimate the stopping distance d in feet (or in meters) for a car going at speed $v =$ 60 mi/hr (100 km/hr) which locks its brakes on all four wheels (a) on a dry road; (b) on a wet road; (c) on an icy road.

9P19. A base runner of mass m = 80 kg slides into second base. If the coefficient of sliding friction between the runner and base path is $\mu_k = 0.2$, (a) what is the runner's deceleration a? (b) If the runner starts sliding a distance $d = 4$ m from the base, when he was moving at speed $v_0 = 5$ m/s, what is his speed v when he reaches the base?

9P20. A block of mass $M/2$ rests upon a block of mass M sitting on a frictionless table. If a horizontal force F_0 is exerted upon the top block, that block accelerates to the right with $a = a_0$, while the bottom block accelerates to the right with $a = a_0/2$. In terms of any or all of M, g, and a_0, find (a) F_0 ; (b) the coefficient of kinetic friction μ_k between the blocks.

9P21. An ox plans to drag a large box along the ground at constant velocity. It uses a light attached rope, as shown, oriented at angle θ to the horizontal. The box has mass M and there is a coefficient of sliding friction μ_k between the box and the ground. (a) Find the angle θ, in terms of given parameters, which minimizes the required force exerted by the ox on the box. (b) What is this minimum force F_{min}? (c) What is the ratio of this force to the force that would be required with $\theta = 0$?

9P22. Two blocks, each of mass M, are attached to the two ends of a string. The string is slung over a frictionless peg so that one block hangs vertically and the other moves at constant velocity on a plane inclined at angle θ to the horizontal. What must be the coefficient of kinetic friction between the inclined plane and the block moving upon it?

9P23. Two wooden blocks, with masses m and M, are attached by an inextendible massless string. The blocks are placed in Earth's gravity on an inclined plane of angle θ, as shown. The coefficients of kinetic friction between the blocks and the plane are μ_m and μ_M, respectively.

(a) Draw isolation diagrams for each block. Then assuming the string stays taut, (b) Find the acceleration of the blocks down the plane. (c) Find the tension in the string. (d) Find a condition ensuring that the string will remain taut.

9P24. In a race, downhill skiers typically reach speeds v= 70 mi/hr on winding runs inclined at angles of about 10 to 15 degrees. If the coefficient of kinetic friction between skis and slope is $\mu_k = 0.080$ for a skier of weight $W = 150$ pounds skiing at a steady speed of 70 mi/hr on a slope inclined at angle $\theta = 15$ degrees, how large is the force of air resistance on the skier?

9P25. Block 1 of mass m is connected by a massless string that runs over a frictionless peg to Block 2, also of mass m, as shown. Both blocks rest on inclined planes, one at angle $\theta = 60^o$ and one at angle $\phi = 30^o$, in gravity g. The coefficient of static friction between blocks and surfaces is μ_s. (a) Draw an isolation diagram for each block, assuming it is at rest. (b) Find the value of μ_s below which the blocks will start to slide.

9P26. Suppose the blocks described in 9P25 have begun to slide; μ_k is the coefficient of kinetic friction between both blocks and the surfaces. (a) Draw an isolation diagram for each block. (b) Write out the $F = ma$ equations for each block. (c) Solve them to find the acceleration of the system in terms of μ_k and g. Then find the tension in the string numerically if $\mu_k = 0.20$.

9P27. A car whose driver claimed to be moving no faster than the speed limit of 90 km/hr left skid marks 25 m long when the car was braked. Police radar found the car was still traveling at 50 km/hr after the skid. If the coefficient of friction between tires and road is less than or equal to 0.80, would investigators find the driver had exceeded the speed limit?

9P28. A car whose driver claimed to be moving no faster than the speed limit of 75 km/hr left skid marks 20 m long when the car was braked. Police radar found the car was still traveling at 45 km/hr after the skid. If the coefficient of friction between tires and road is at least 0.80, would investigators find the driver had exceeded the speed limit?

9P29. A snowmobile pulling three sleds is moving at a constant velocity up a grade of angle θ. Each sled has mass m and slides on the snow with coefficient of kinetic friction μ_k.

(a) With what force is the snowmobile pulling on the leading sled?
(b) With what force is the last sled pulling back on the middle sled?
(c) What is the direction and magnitude of the component of force of the snow on the snowmobile which is tangential (i.e., parallel) to the surface? Find the magnitude in terms of m, g, θ, μ_k, and M, the mass of the snowmobile plus rider.

9P30. The stopping distance for a car moving at speed v_o on a level road is

$s = v_o^2 / 2g\mu_k$ if it locks its brakes on all four wheels.

(a) Derive this equation. Derive a similar equation for a car travelling on a road
(b) sloped *upward* by angle θ; (c) sloped *downward* by angle θ.

9P31. Let μ_s and μ_k be the static and kinetic coefficients of friction between a pushbroom and a floor. The broom has mass m, and the handle (of negligible mass) makes an angle θ relative to the vertical.

(a) What is the magnitude of the force F_0, directed along the handle, needed to move the broom with constant velocity along the floor?
(b) What is the critical angle θ_c less than which the broom cannot be made to slide, no matter how large F_0 is?

9P32. A three-car subway train is traveling down a 3 percent grade when the individual car brakes are simultaneously applied. Each 2200-kg car can generate a braking force of one-third the normal force exerted on it by the tracks. Find the train deceleration a and the forces T_1 and T_2 in the couplings 1 (between the lead and middle car) and 2 (between the middle and last car) for the cases when (a) all brakes function normally; (b) the brakes of car A fail; (c) the brakes of car B fail; (d) the brakes of car C fail.

9P33. Mountaineer A of mass $m_A = 65$ kg is rappelling at constant speed down the side of a vertical, frictionless icewall using a light rope. The rope passes over a frictionless lip at the top of the icewall, and is then tied to mountaineer B of mass $m_B = 80$ kg. Mountaineer B is wearing a backpack of mass $m_P = 20$ kg, and is seated at rest on a horizontal surface with coefficients of friction $\mu_s = \mu_k = 0.7$. (a) What is the tension T in the rope? (b) Verify that B will not slip with the pack on. (c) B then removes his backpack and sets it beside him on the ground. What then is the acceleration a of both A and B if A holds firmly to the rope? (d) What then is the tension in the rope? (Use $g = 9.8$ m/s^2.)

Section 9.3. Air resistance

9P34. Very small oil droplets in air fall under gravity g according to Stokes's law. Find a formula for the velocity of such a droplet as a function of time, assuming it starts at rest, given the mass m and radius R of the droplet, and the viscosity η of air.

9P35. In *Sample Problem* 4, the velocity of a ball thrown in the gravity-free Bubble is found as a function of time, if it starts with speed v_0. Find the position x of the ball as a function of time, assuming it begins with $x = 0$ at time $t = 0$.

Appendix A: Solving $F = ma$ on the Computer

9P36. Fill in the Table at the bottom of p. 9-21 to three additional time-steps, *i. e.*, up to the dimensionless time $\tau = 0.07$, doing the calculations by hand, using a calculator. At time $\tau = 0.07$, what is the percentage error in the numerical calculation of the velocity V? What is the percentage error in the position X?

9P37. (a) Write a computer program, implementing Euler's method to solve the equation of motion $-V = dV/d\tau$ discussed in the text, for arbitrary time spacing $\Delta\tau$. (b) If the particle starts at rest with dimensionless velocity $V = 1$ (exactly), what is the maximum value of $\Delta\tau$ that leads to a numerical answer for the position X at time $\tau = 1$ (exactly) which is within 0.01% of the exact answer for X?

9P38. A ball of mass m experiences a quadratic drag force $F_D = -bv^2$ due to air resistance, where b is a constant; no other forces act on the ball. The ball starts at position $x = 0$ with initial velocity v_0. (a) Write the $F = ma$ equation in terms of dimensionless variables. (b) Solve the equation *exactly*, to find the ball's dimensionless velocity and dimensionless position as functions of a dimensionless time. (c) Choose a time spacing and then solve the equation numerically by hand (with the help of a calculator), using Euler's method, for five time-steps. (d) Compare the numerical results with the exact results at the final time-step.

9P39. Solve problem 9P38, parts (a) and (b). Then (c) write a computer program to do the numerical calculations by Euler's method, for an arbitrary time-step. (d) Find the maximum dimensionless time-step permitted to achieve 0.01% accuracy in the velocity of the ball when the dimensionless time has reached 1.00000.

9P40. A block of mass m moves on the end of a Hooke's-law spring of force-constant k. As the ball moves it also experiences a cubic drag force $F_D = -bv^3$, the minus sign referring to the fact that the drag force is at all times opposite to the ball's direction of motion. (a) Write out the $F = ma$ equation of motion including both forces. (b) Each of the three parameters m, k, and b has dimensions. Find the three combinations of these parameters that have the dimensions of time, of distance, and of velocity. (c) Using these combinations, define a dimensionless time T proportional to the time t, a dimensionless distance X proportional to x, and a dimensionless velocity proportional to v. Then show that the $F = ma$ equation can be rewritten in the dimensionless form $dV/dT = -X - V^3$. (c) Write a computer program using Euler's method to solve the equation. The input should allow a choice of initial position and initial velocity. (d) Use the program to find the motion of the block if the initial velocity of the block is zero and its initial position is X_0. (e) Graph the position of the block as a function of time.

9P41. *Miniproject*: Think of an interesting motion problem that requires a computer to solve. Make the $F = ma$ equation dimensionless, write a program using Euler's method to solve it, and find the numerical solution for a variety of parameters, initial conditions, and time-steps. Graph the results, and check that they make sense.

Special Topic: Drag Forces and Reynolds Numbers

9P42. The drag force on a ball of mass m due to air resistance may be roughly approximated over a range of speeds by $F_D = -k'v^2$, where k' is a constant which depends on the density of air and the size of the ball. The negative sign means that the force is opposite to the direction of motion. Find the speed of the ball as a function of time, if it is thrown with initial speed v_0 in the gravity-free Bubble.

9P43. Let the drag force on a ball of mass m moving through air in the gravity-free Bubble include both "viscous" drag $F_V = -kv$ and "quadratic" drag $F_i = -k'v^2$, where k and k' are constants. Find the velocity $v(t)$ of a ball thrown with initial speed v_0.

9P44. A steel ball bearing of radius $r = 1.0$ cm is dropped from the Sears Tower in Chicago, a height $h = 1454$ ft above the street. (a) Ignoring air resistance, find the time t it takes for the ball to fall to the street. (b) Still ignoring air resistance, find the Reynolds number R of the ball when it has fallen 1/4, 1/2, 3/4, and (nearly) the whole distance to the street. (See 9Q16) (c) Using the results of (b), air resistance is what fraction of the gravitational force on the ball at the four locations? Use the graph from the AIP Handbook. (This is a rough way to gauge the importance of air resistance. You will have to figure out the ball's weight mg.) Note that whatever your results may be, F_{drag}/mg cannot actually exceed unity. If you found $F_{drag}/mg > 1$, what is wrong with the calculation? (d) Estimate the terminal velocity of the ball.

9P45. Early one clear, windless morning, a bicyclist of mass $m_p = 75$ kg returns to Claremont on her bicycle of mass $m_b = 10$ kg after visiting Snowcrest Lodge near the Mt. Baldy ski lifts. On one straight, uniformly steep section of road she attains a terminal speed $v = 53$ mph (23.7 m/s) without peddling or applying the brakes. She experiences the wind-resistance force $F_D = kAv^2$, where k is a constant, A is the total frontal area of the bicyclist and her bicycle, and v is her speed. (a) Draw an isolation diagram of the system. (b) What are the dimensions of the constant k? (c) If $A = 0.35$ m^2 and $k = 0.21$ kg/m^3, what is the percent grade of the road? (A road with a 10% grade climbs or drops 10 m for every 100 m it extends horizontally.) Neglect rolling resistance.

9P46. Estimate typical Reynolds numbers R in the following situations: (a) a raindrop of radius 1 mm falling in air; (b) a typical car driving on the freeway; (c) a ball bearing of 1-cm radius dropped in water; (d) a submarine of length 100 m cruising in water.

9P47. When moving through a medium of *high* viscosity (like molasses), the drag force on a ball depends (to a good approximation) only upon the viscosity η and the radius r and velocity v of the ball. If the medium has a very *low* viscosity the drag force (for a range of velocities) depends only upon the density ρ of the medium as well as the ball's radius r and velocity v. In each of these two cases, if the ball's radius were doubled, keeping η, ρ, and v the same, by what factor would the drag force increase?

9P48. A cat of mass $m = 2.0$ kg cat jumps from a window on the 20th floor of an apartment building in New York City, a distance $d = 70$ m above the street.
(a) If there were no air drag, what would be the cat's speed v when it hits the ground? (b) Air resistance is in fact important in the fall. Model it with an inertial drag force kv^2 opposite to the direction of motion, where k is a constant. Show then that as the cat falls, it asymptotically approaches a terminal velocity v_T. What is v_T in terms of given parameters? (c) Assuming the cat nearly reaches v_T and that $k = .05$ kg/m, about how fast is the cat moving when it hits the ground? (d) Would it make an appreciable difference if the cat jumped from the 40th floor instead?

(Cats often do leap out of apartment windows trying to catch birds. Many of them survive, thanks to air resistance. There is even some evidence they are more likely to survive if they jump from higher windows. It is not clear why, although some people have speculated they may be more relaxed by the time they hit the ground.)

9P49. The volume/second of fluid traveling through a straight pipe of circular cross-section depends upon the inner radius r of the pipe, the viscosity η of the fluid, and the pressure difference per unit length of the pipe $\Delta p/L$. (Pressure is the force/area exerted by the fluid. The pressure difference is the difference in pressure between one end of the pipe and the other). If the radius of the pipe were doubled, keeping the other parameters unchanged, by what factor would the volume/second transferred through the pipe be increased?

9P50. A baseball pitcher throws a fastball at velocity $v_0 = 97$ mi/hr.
(a) What average force F (in pounds) is exerted by the pitcher on the ball (of weight 5.2 ounces) while it travels a distance $d = 8$ ft from the pitcher's wind-up to his delivery? (1 lb = 16 ounces; $g = 32$ ft/s^2. In the English system of units, the mass of an object is equal to its weight on the Earth (in pounds) divided by $g = 32$ ft/s^2, so the mass of a 1-lb object is $m = 1$ lb s^2 / 32 ft.) (b) The ball crosses home plate with speed $v = 90$ mi/hr, a time $t = 0.41$ s after being thrown. Estimate the average drag force F_D (in pounds) on the ball due to air resistance, assuming the ball moves a distance $D = 57$ ft from the pitcher's hand to the plate. (c) Estimate the Reynolds number R of the ball while it is moving, if the ball has circumference $C =$ 9 1/4 inches. [Note that at standard temperature and pressure, air has the density $\rho = 1.21$ kg/m^3 and viscosity $\eta = 1.8 \times 10^{-5}$ N· s/m^2] Let ℓ = radius of ball.
(d) Is the drag force found in part (b) roughly consistent with the graph of pressure (drag) coefficient *vs* Reynolds number, for the Reynolds number computed in part (c) ? Explain.

Day Ten

Energy on Ice

"Money is always there, but the pockets change" - Gertrude Stein

We've been cooped up now for two days. Even the icy landscape outside looks inviting. We find a couple of airtight spacesuits in a closet, along with boots, crampons, and ice axes - all we need for an expedition.

A turn of a latch opens a door, and we step into an airlock. The door closes behind us and air hisses out; an outer door opens and we tentatively step out into the vacuum and the cold.

We slowly make our way along the slippery surface, and after a few minutes begin a gradual climb. There are icy hills as far as we can see. We soon reach the beginning of a long, straight ridge. On either side are valleys, with other ridges rising on the far sides. In cross section, the ridges and valleys look as shown in Fig. 1. Except for the curvature at the very top of the ridges and the bottom of the valleys, the right side of our ridge looks perfectly straight, as does the ridge on our far right. The ridges on our left have gentle, rolling curves.

Fig. 1. Ridges and valleys of ice

We step carefully along the nearly frictionless ice. We have brought one of the small wooden blocks with us; we set it at rest near the top of the straight ridge on our right, and let it go. It plummets down the slope, across the valley, and shoots up the slope on the

other side, as shown in Fig. 2. It gradually slows down, coming to rest at an altitude that seems to be about the same as our own; then it slides back down to the valley, whips quickly around and then right back up our own ridge, coming to rest at just about the spot we released it. Why did it move like this? Why did it reach the same altitude on both sides?

Fig. 2. The block's journey on straight-sided slopes

Next we set the block down on the left-hand, curved side of our ridge and let it go. The acceleration of the block is now uneven; it speeds up quickly down the steeper portions and less quickly on the gentler slopes. When it reaches the valley it whips up the far ridge, decelerating more on the steeper portions than on the flatter parts. To our surprise, the block again coasts to rest at an altitude about the same as our own. And on the return trip it slides right back up our own curved ridge, coming to rest at nearly the same point from which we released it. It is interesting that the maximum height reached by the block on either side of the valley is the same as the height at which we released it, and this is as true for the curved path on our left as it is for the path with straight sides on our right. Why?

10.1 Sliding along straight slopes

We start by thinking about the straight sections. The block has two forces on it, gravity and the normal force, as shown in the isolation diagram. The block cannot accelerate perpendicular to a straight slope; if it did, it would either have to jump off the slope or burrow into it. So the net perpendicular force must be zero, i.e., $F_{\perp} = N - mg\cos\theta = 0$. There is, however, a net

force $F_{\parallel} = -mg\sin\theta$ *parallel* to the slope (negative because we take the positive direction to be up the slope). So using $F = ma$ parallel to the slope we find that

$$a = -g\sin\theta . \tag{1}$$

Both g and θ are constants, so the acceleration a is also constant. That means we can use any of the constant-acceleration formulas (given in Eqs. 9 in *Day Five*) to find the motion.

Suppose the block moves upward from some point s_0 to another point s along the far slope, as shown. Then its speed v at s is related to its speed v_0 at s_0 by Eq. 9d of *Day Five*,

$$v^2 = v_0^2 + 2a\,(s - s_0) = v_0^2 - 2g\sin\theta\,(s - s_0) .$$

The displacement $s - s_0$ is positive, so v is less than v_0, which makes sense. The block slows down as it slides uphill. We notice that if s grows large enough, v goes to zero. This is where the block stops and starts to slide back down again.

There is an interesting feature to the quantity $\sin\theta\,(s - s_0)$. If we draw a right triangle with angle θ and hypotenuse $s - s_0$, then $\sin\theta\,(s - s_0)$ is equal to $y - y_0$, the *difference in altitude* between the two points. Therefore

$$v^2 = v_0^2 - 2g(y - y_0). \tag{2}$$

If the block starts up the bottom of the straight section with speed v_0, it will reach a given maximum altitude y (where v goes to zero). And this maximum altitude doesn't depend on the slope-angle θ !

That explains why the block reached the same altitude on the far slope as it had when we released it, and why it came back to us on the return trip. If for example the block is at the bottom of the far slope at altitude $y = y_0$ and speed v_0 headed up the far slope, it will come to rest (with $v = 0$) at altitude $h = y - y_0 = v_0^2/2g$ above the bottom. Then when it slides back down it will again have $v = v_0$ when it reaches the bottom, since the same equation $v^2 = v_0^2 - 2g(y - y_0)$ holds for the descent as it did for the ascent.

When the block whips around the little valley at the bottom and heads back up our own slope, it should have this same v_0, since in the little curved valley the speed shouldn't change appreciably. The same equation $v^2 = v_0^2 - 2\,g(y - y_0)$ still works on our own slope, because it doesn't depend on the slope angle. So again the block will come to rest with $v = 0$ at altitude $h = y - y_0 = v_0^2/2g$ on our hill, the same altitude as the other hill.

In summary, we have found that the relationship $v^2 = v_0^2 - 2\,g(y - y_0)$ is valid throughout the journey: The speed of the block at given altitude $y - y_0$ is the same on one of the straight slopes as on the other straight slope. We can rewrite this relationship as

$$\frac{v^2}{2} + gy = \frac{v_0^2}{2} + gy_0 \,, \tag{3}$$

putting the v_0 and y_0 terms on the right and the other terms on the left.

The quantity $\dfrac{v^2}{2} + gy$ seems to be *conserved* – it always stays the same.

When the block leaves the top of the ridge, the term gy is large while $v^2/2$ is small; as the block slides down, $v^2/2$ increases while gy decreases; the sum stays the same.

10.2 Sliding along curved slopes

How general is the relationship given by Eq. 3? Could it also be valid on the curved slopes? We begin by picturing a series of straight sections, each tilted at a slightly different angle, to match, at least roughly, the curved slope shown in Fig. 3. If we can figure out what happens on these short straight sections, we can then take a larger and larger number of shorter and shorter sections, to try to match the curved slope as closely as possible.

Fig. 3. Approximating a curved slope by a series of straight sections

We already know (From Eq. 2) that if the block starts up the lowest section with $y = y_0$ and $v = v_0$, then its speed v at any height y along this section obeys

$$v^2 = v_0^2 - 2g\,(y - y_0). \tag{4a}$$

In particular, at the top of this first section, where $y = y_1$, the block's speed v_1 obeys $v_1^2 = v_0^2 - 2g(y_1 - y_0)$.

On the second section, just to the right of the first section, the block's speed obeys

$$v^2 = v_1^2 - 2g(y - y_1), \tag{4b}$$

since it starts up this section with altitude y_1 and speed v_1. We already found that $v_1^2 = v_0^2 - 2g(y_1 - y_0)$, however, so anywhere on the second section

$$v^2 = v_1^2 - 2g(y - y_1) = v_0^2 - 2g(y_1 - y_0) - 2g(y - y_1). \tag{4c}$$

Cancelling the $2gy_1$ terms, we find

$$v^2 = v_0^2 - 2g(y - y_0), \tag{4d}$$

the same equation (4a) we had on the first section!

We can obviously follow the same procedure for the third section, for the fourth, and so on, each time showing that Eq. 4a is valid for each section in turn, referring initial velocities and altitudes all the way back to the first section. The method works for any number of sections, for short ones or long ones, for sections tilted downward as well as those tilted upward. Eq. 4a is valid *everywhere* along the series of sections.

Finally, if we take the limit of a very large number of very short sections, each tilted only very slightly relative to one another, we can approximate a curved slope by the set of straight sections to any desired accuracy. So the conservation relationship

$$\frac{v^2}{2} + gy = \frac{v_0^2}{2} + gy_0$$

is as true for a block on a frictionless curved slope as it is on a frictionless straight slope. The conserved quantity is intriguing: The first term depends on the speed of the block but not on its altitude, while the second term depends on the block's altitude, but not its speed.

Sample Problem 1. A skier starts at rest at the top of a slope of height h and then skis down onto a plateau ending at point A, the beginning of a circular hill of radius R as shown. The entire run is on frictionless ice. (a) How fast does she move on the plateau? (b) At what angle θ on the circular hill does she leave the ice?

Solution: (a) The quantity $v^2/2 + gy$ is conserved. Since she starts at rest, initially there is only the altitude term $gy = g(h + R)$, if we set $y = 0$ at the base of the circular hill. On the plateau the altitude term gy is simply gR. Comparing the starting position with the plateau, the conservation law states that $0 + g(h + R) = v^2/2 + gR$. Solving for v, we find that the skier's speed on the plateau is $v = \sqrt{2gh}$.

(b) When the skier is on the circular hill, her isolation diagram is as shown. The only two forces acting on her are gravity $m\mathbf{g}$ and the normal force $\mathbf{N}$. At the instant she leaves the ice her normal force goes to zero, since she is no longer in contact with the surface. Gravity generally has both a radial component (toward the center of the circle) $mg\cos\theta$ and a tangential component (tangent to the circle) $mg\sin\theta$. Therefore from $\mathbf{F} = m\mathbf{a}$ there must be both radial and tangential components of acceleration as well. The inward (centripetal) radial component of acceleration we know to be $a_{\text{centripetal}} = v^2/R$, since the skier is moving in a circle. Therefore the radial component of $\mathbf{F} = m\mathbf{a}$ at the instant she leaves the ice is

$$mg\cos\theta = mv^2/R.$$

At the same time, the conservation law tells us that $0 + g(h + R) = v^2/2 + gR\cos\theta$, since her altitude is $R\cos\theta$ when she is at angle θ on the hill. The left-hand side of this equation corresponds to the starting point. Solving for v^2 when she is at angle θ,

$$v^2 = 2gh + 2gR(1 - \cos\theta).$$

Substituting this expression for v^2 into the $\mathbf{F} = m\mathbf{a}$ equation above, we find that $mg\cos\theta = mv^2/R = 2mgh/R + 2mg(1 - \cos\theta)$. Solving for the angle θ where she leaves the ice and is projected into space,

$$\theta = \cos^{-1}\left[\frac{2}{3}\left(1 + \frac{h}{R}\right)\right].$$

(It is often interesting to test limiting cases! The cosine of an angle can never exceed unity, so that the ratio h/R cannot exceed $1/2$ in the expression just derived. What in fact happens if $h/R > 1/2$?)

10.3 Two blocks tied together

The new conservation relationship

$$\frac{v^2}{2} + gy = \text{constant} \tag{5}$$

works for a single block sliding up and down frictionless surfaces, or simply falling or rising in a gravitational field g. Can the law be extended to the case of two or more blocks? For any system of two blocks, for example, is it true that the sum $(v_1{}^2/2 + gy_1) + (v_2{}^2/2 + gy_2)$ always stays the same as the blocks change their velocities and altitudes?

As a test we think about two blocks, with masses m_1 and m_2, attached to the ends of a light cord slung over a frictionless peg. We start with both blocks at rest at the same altitude, as shown in Fig. 4(a). Then if $m_2 > m_1$, the lighter block m_1 will rise some distance y_1 while gaining speed v_1, and the heavier block m_2 will sink the same distance, decreasing its altitude to $y_2 = -y_1$ while gaining the same speed $v_2 = v_1$, as shown in Fig. 4(b). The distances and speeds have to be the same, because we assume that the cord doesn't stretch. Can we extend our new conservation relation to this system?

Fig. 4. (a) The initial positions of blocks m_1 and m_2, hanging over a frictionless peg. (b) Some time later, m_1 has risen by y_1 and m_2 has fallen to altitude $y_2 = -y_1$. Each has gained the same speed $v_1 = v_2$.

It is already clear that the sum $(v_1{}^2/2 + gy_1) + (v_2{}^2/2 + gy_2)$ is *not* conserved in this case. Each term is zero initially, so the overall sum is also zero. Since $y_2 = -y_1$ as time goes on, the sum of the two terms $g(y_1 + y_2)$ remains equal to zero. The other two terms, $v_1{}^2/2 + v_2{}^2/2$, *grow* as the blocks accelerate. Therefore the overall sum cannot stay equal to zero.

We will find the actual motion of the two blocks using $F = ma$. Isolation diagrams for m_1 and m_2 are shown in Fig. 5. We take the positive acceleration direction to be *up* for m_1 and (to be consistent) *down* for m_2. From the diagram, the corresponding $F = ma$

Fig. 5. Isolation diagrams of the two blocks

equations are $T - m_1 g = m_1 a$ (for m_1) and $m_2 g - T = m_2 a$ (for m_2), where T is the tension in the cord. Adding these equations to eliminate T, and then solving for the acceleration, we find

$$a = \frac{(m_2 - m_1)}{(m_2 + m_1)} g \,, \tag{6}$$

the same for each block. This acceleration is constant, so we can use the constant-acceleration formula $v^2 = v_0{}^2 + 2as$, where $s = y_1$ is the distance each block travels, to find the speed of the blocks

$$v^2 = 2ay_1 = 2\,\frac{(m_2 - m_1)}{(m_2 + m_1)} g\, y_1 \,. \tag{7}$$

Multiplying by $m_1 + m_2$ and rearranging,

$$\frac{1}{2}(m_1 + m_2)v_1^2 + (m_1 - m_2)\, g y_1 = 0 . \tag{8}$$

We know that $v_1^2 = v_2^2$ and $y_2 = -y_1$, so we can separate terms with subscripts 2 from those with subscript 1:

$$\left(\frac{1}{2} m_1 v_1^2 + m_1 g y_1\right) + \left(\frac{1}{2} m_2 v_2^2 + m_2 g y_2\right) = 0 \tag{9}$$

This is a conservation law that *works* for two particles of different mass, at least in this special case.* We have had to weight each term $(v^2/2 + gy)$ by the corresponding particle mass, m_1 or m_2. In an experiment with only a single block, say m_1, then the mass factors out so the quantity $(v^2/2 + gy)$ stays constant, the conservation law of Eq. 5 we started with.

10.4 Energy: kinetic energy and gravitational potential energy

We name the quantity

$$\boxed{E = \frac{1}{2}mv^2 + mgy} \tag{10}$$

the *energy* of a particle of mass m subject to the gravitational field g. We have found that the energy E of a single particle sliding up and down frictionless slopes is conserved. For a system of two particles 1 and 2, the sum of their energies $E = E_1 + E_2$ is conserved, at least in a special case. We call the unit of energy the *Joule*: (1 J = 1 kg m^2/s^2).

The first term of Eq. 10 depends on the mass and speed of the particle, but not on its altitude. We call this term the particle's *kinetic energy*

$$\boxed{K = \frac{1}{2}mv^2} \tag{11}$$

*Other special cases are described in the problems for Section 10.3.

because it is related to the particle's *motion.* The second term is related to gravity and the altitude y of the particle, but does not depend on the block's speed. We call this term the particle's *gravitational potential energy*

$$\boxed{U_{\text{grav}} = mgy\,.} \tag{12}$$

The energy E of the particle is the sum of these two kinds of energy

$$\boxed{E = K + U_{\text{grav}}} \tag{13}$$

In the experiments we have carried out so far, the total energy of all of the particles is conserved.

Figure 6 is an *energy diagram.* Plotted is the gravitational potential energy $U_{\text{grav}} = mgy$ of a particle as a function of its altitude y. U_{grav} is a straight line, with slope mg. The total energy E is independent of y, so is shown as a straight horizontal line. The kinetic energy is the difference between E and U_{grav}, so where the two lines meet the particle's kinetic energy is zero. That occurs at the altitude where the particle is instantaneously at rest, at the top of its trajectory.

Imagine throwing the particle upward; its altitude is initially $y = 0$, so the potential energy is small and the kinetic energy is large, as at the left in the figure. As the particle rises, the potential energy increases while the kinetic energy decreases until it reaches zero as the lines cross. The particle then starts falling back, returning to the ground with its original kinetic energy.

Fig. 6 Energy diagram for gravity

Sample Problem 2. An explorer of mass m prepares to slide down a frictionless semi-circular valley of ice of radius R, as shown. She sits on the lip of the semicircle, where the slope is vertical, and then falls from rest. At the valley bottom, find (a) her kinetic energy K ; (b) her speed v ; (c) her weight W. (d) Also find the normal force N exerted upon her when she is at an arbitrary angle θ.

Solution: (a) Her kinetic energy at the bottom can be found from energy conservation. Her initial potential energy is $U = mgh = mgR$, if we measure potential energy from the bottom of the valley, and her initial kinetic energy is *zero*. At the bottom her potential energy is *zero*, so her kinetic energy there must be $K = mgR$, to conserve energy.

(b) Her speed at the bottom is $v = \sqrt{2gR}$, since her kinetic energy there is $K = (1/2)mv^2 = mgR$. Note that her speed at the bottom doesn't depend upon her mass.

(c) To find her weight at the bottom of the valley, we start by isolating her as shown. There is a normal force upward on her due to the ice, and gravity downward; she therefore has no horizontal acceleration, but she does have an acceleration upward, toward the center of the circle, which keeps her moving in the circular path. The magnitude of the acceleration is v^2/R. Newton's second law gives $F_{\text{net}} = N - mg = ma = mv^2/R$, so the normal force is

$$N = mg + mv^2/R = mg + m(2gR)/R = 3mg.$$

Her weight is three times what it would be if she were just sitting still or if the track were straight at that point!

(d) Her isolation diagram, drawn when she is at an arbitrary angle θ, is as shown. The net force toward the center of the circle is $F_{\text{net}} = N - mg\cos\theta = ma_{\text{centripetal}} = mv^2/R$.

When she is at angle θ her distance below the starting point is $R\cos\theta$, so using energy conservation as in part (a), her kinetic energy at angle θ is $(1/2)mv^2 = mgR\cos\theta$. This gives $v^2 = 2gR\cos\theta$, so the normal force on her is

$$N = mg\cos\theta + m(2gR\cos\theta)/R = 3mg\cos\theta.$$

The normal force N is zero as expected at $\theta = -\pi/2$ (her starting point) and at $\theta = +\pi/2$; also $N = 3mg$ at $\theta = 0$, consistent with the result of part (c).

Note that there is a tangential component to F_{net} and therefore to the total acceleration. The effect of the tangential component is to slow the explorer as she moves up the slope.

Sample Problem 3. Two mountaineers tied together with a light rope are standing on a slippery horizontal glacier, when suddenly a thin crust of ice breaks under the lead mountaineer, and he falls into a crevasse. His partner is pulled off her feet, and starts to slide without friction toward the crevasse, as shown. As functions of time, find the energy of the lead mountaineer, his partner, and both as a system. Assume the rope is unstretchable, and that the second mountaineer has not yet reached the crevasse.

Solution: Isolation diagrams of the lead mountaineer (m_1) and his partner (m_2) are shown at the right, where T is the tension in the rope. The corresponding $F = ma$ equations are

$$m_1g - T = m_1a \quad \text{and} \quad T = m_2a\,.$$

Eliminating T, we find that the acceleration of each mountaineer is $a = \dfrac{m_1g}{m_1 + m_2}$. If we choose to measure height from the horizontal surface of the glacier, then the second mountaineer has no gravitational potential energy, so her energy is

$$E_2 = K.E._2 = \frac{1}{2}m_2v^2 = \frac{1}{2}m_2(at)^2 = \frac{1}{2}m_2\left(\frac{m_1g}{m_1 + m_2}\right)^2 t^2,$$

increasing with time as she accelerates. We have used $v = at$, valid because she starts at rest and moves with constant acceleration. When the lead mountaineer has fallen a distance d below the glacier surface, his gravitational potential energy is $U_1 = -m_1gd$, negative because he has a negative altitude. He moves with constant acceleration, so d is related to his speed v by $v^2 = v_0^2 + 2ad = 2ad$. His energy is therefore

$$E_1 = \frac{1}{2}m_1v^2 - m_1gd = \frac{1}{2}m_1\left[1 - \frac{g}{a}\right]v^2 = \frac{1}{2}m_1\left(1 - \frac{m_1 + m_2}{m_1}\right)v^2 = -\frac{1}{2}m_2v^2,$$

using $d = v^2/2a$ where $a = m_1g/(m_1 + m_2)$. His energy is the negative of his partner's energy (negative because his negative potential energy dominates his positive kinetic energy.) Note that the energy of neither mountaineer is individually conserved, but that their total energy $E_1 + E_2$ is always zero, so *is* conserved.

10.5 Spring potential energy

Can conservation of energy be extended beyond the realm of particles moving in response to gravity? What about spring forces, for example? Can the kinetic energy of a particle be converted into some sort of spring potential energy and then back into kinetic energy?

Suppose a block of mass m and velocity v_0 slides into a spring of force-constant k, as shown in Fig. 7. Initially, before the block reaches the spring, the kinetic energy of the block is $K = \frac{1}{2}mv_0^2$. As the spring is compressed the block slows down, losing

Fig. 7. (a) A freely-moving block (b) The block strikes a spring and compresses it.

kinetic energy. Is there a spring potential energy U_{spring}, analogous to the gravitational potential energy $U_{\text{grav}} = mgy$, that builds up to make up for the lost kinetic energy? If so, what is U_{spring} ?

The only quantities that U_{spring} can depend upon are the force constant k of the spring and its stretch or compression x. The dimensions of k are those of force/length, i.e., $(ML/T^2)/L = M/T^2$. The only way to combine k and x (which of course has the dimensions of length) to give the dimensions of energy ML^2/T^2 is to multiply k by x^2. Therefore, from dimensional reasoning alone, we know that

$$U_{\text{spring}} = C\,kx^2 \tag{14}$$

where C is some dimensionless constant. What is this constant C?

Including both the kinetic energy of the block and the potential energy U_{spring}, the total energy of the system block + spring is $E = \frac{1}{2}mv^2 + C\,kx^2$. This energy is supposed to be conserved, which means that the time derivative of E should be zero:

$$\frac{dE}{dt} = \frac{1}{2}m\left(2v\frac{dv}{dt}\right) + \frac{d}{dx}\left(C\,kx^2\right)\frac{dx}{dt} = 0, \tag{15}$$

where we have differentiated the kinetic energy directly, and used the chain rule to differentiate the potential energy. The block's velocity $v = \frac{dx}{dt}$, so Eq. 15 can be written

$$v\,[\,m dv/dt + 2Ckx\,] = 0\,. \tag{16}$$

The velocity v is not generally zero, so the quantity in square brackets has to be zero if the energy of the system is conserved. The first term in the brackets is the mass times acceleration of the block, which is equal to the net force on the block by Newton's second law. The only horizontal force on the block is the spring force $F = -kx$, so the quantity in square brackets can only be zero if we choose the constant C to be $C = 1/2$.

We have succeeded in finding the spring potential energy: It is

$$\boxed{U_{\text{spring}} = \frac{1}{2}kx^2,} \tag{17}$$

proportional to the force constant and to the *square* of the spring stretch or compression.

Sample Problem 4. A Buick of mass m and initial speed v_o, sliding on a frictionless ice sheet, runs into a giant spring of force constant k, which is initially uncompressed. How much is the spring compressed by the time the Buick comes instantaneously to rest?

Solution: The total energy of the Buick plus spring is conserved, so $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 =$ constant. Before impact, the energy $E = \frac{1}{2}mv_o^2$ is all kinetic; when the Buick comes instantaneously to rest all its kinetic energy is gone, and the energy stored in the spring has grown to $U_{\text{spring}} = \frac{1}{2}kx_{\text{max}}^2$, where x_{max} is the maximum spring compression. Conserving energy, $E = \frac{1}{2}kx_{\text{max}}^2 = \frac{1}{2}mv_o^2$, so $x_{\text{max}} = \sqrt{\frac{m}{k}}\,v_o$.

As expected, the faster (or more massive) the Buick is, the more the spring becomes compressed; the stiffer the spring is, the less it is compressed.

The spring potential energy $U_{\text{spring}} = \frac{1}{2}kx^2$ is quadratic in x, so U_{spring} is a parabola, as shown in the energy diagram of Fig. 8. The potential energy stored in the spring is the same for negative x (i.e., compressions) as for positive x (stretches). The potential energy is zero if the spring is neither stretched nor compressed.

The total energy E of a block + spring system is represented in Fig. 8 by a line that is horizontal because E is a *constant*. All the energy is potential energy (i.e., $E = U$) at the two locations where the E line intersects the potential-energy parabola. There is no energy left for kinetic energy at these points, so the block must come instantaneously to rest when it reaches either position. We call these locations *turning points*, because the block stops at either position and then starts to move back in the opposite direction. For example, suppose at some instant the block is located at $x = 0$, and is moving to the right. Then its potential energy is zero (its minimum value), and its energy is all kinetic. As it travels toward positive x its potential energy increases, so its kinetic energy decreases; it slows down. When it reaches the right-hand turning point it stops and then starts moving back toward the left, speeding up as the potential energy decreases. If there is no friction to rob the system of its energy, the block oscillates back and forth forever between the left and right turning points.

Fig. 8. Energy diagram for a spring-mass system

The spring force is the negative slope of the potential energy function; that is,

$$F_{\text{spring}} = -kx = -\frac{d}{dx}\left(\frac{1}{2}kx^2\right) = -\frac{dU}{dx}, \tag{18}$$

so we can read off the direction and magnitude of the spring force at any point of the motion from the slope of $U(x)$ in the energy diagram. For $x > 0$ the slope of $U(x)$ is positive, so the spring force there is negative; for $x > 0$ the block is pulled toward the left. For $x < 0$ the slope is negative, so the spring force there is positive, pushing the block back toward $x = 0$. The energy diagram is very useful: From it, we can see where the potential energy is large and where it is small; where the kinetic energy $K = E - U$ is large and where it is small; where the block stops and changes direction for any particular energy; and where the force is toward the right and where it is toward the left.

State of the Search

1. If an object of mass m in a uniform gravitational field $\mathbf{g}$ touches nothing but frictionless surfaces, its total energy

$$E = \frac{1}{2}mv^2 + mgh$$

is conserved; i. e., E is constant in time. Here

$$K = \frac{1}{2}mv^2 \qquad \text{and} \qquad U_{\text{grav}} = mgh$$

are the kinetic energy and the gravitational potential energy of the object, where v is the speed of the object and h is its altitude above some given point. It is possible for the object to have a *negative* potential energy if it finds itself at an altitude lower than the arbitrary point chosen to have $h = 0$.

2. The kinetic energy increases as the object loses altitude, while the potential energy decreases by just enough to keep their sum constant. We can represent this on an *energy diagram*, drawing the particular (constant) value of the energy E_0 along with the potential energy U_{grav} as a function of altitude. The kinetic energy K is the difference between them, large where U_{grav} is small and small where U_{grav} is large. At some point E_0 and U_{grav} are equal, so there is no energy left for kinetic energy. This point is called a *turning point*, because the object has stopped moving at this point; an initially rising object comes to rest at this point, and then starts to fall.

3. Energy is also conserved for an object moving on the end of a spring, if we use the potential energy $U_{\text{spring}} = \frac{1}{2}kx^2$ for a spring with force-constant k and spring-stretch x. We can think of U_{spring} as the energy stored in the spring. The energy conservation equation becomes

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

in this case, where E is constant.

4. An object on the end of a spring has a maximum kinetic energy at $x = 0$, where the potential energy $U_{\text{spring}} = \frac{1}{2}kx^2$ is zero. As the spring stretch or compression increases, the potential energy increases so the kinetic energy decreases; the particle slows down. Again, we can represent this on an energy diagram with the parabolic shape of the potential energy and the constant total energy. The kinetic energy goes to zero at one of the two turning points, where the particle comes to rest and then begins to move in the opposite direction.

5. Conservation of energy is a central feature of mechanics, and a powerful technique for problem-solving. If a particle moves in a uniform gravitational field, or if it moves on the end of a spring, a knowledge of its current position is enough to tell us its current speed, as long as we know its initial position and velocity. In those cases we no longer have to start with the forces, then find the accelerastion, and finally integrate to obtain the velocity.

PICTURE QUIZ 10

1. A skier can descend from a ridge down to a valley by several possible routes, as shown. The net vertical descent is the same in each case. Assuming the descent is via frictionless ice, by which path will the skier have the maximum final speed upon reaching the valley?

(A)

(B)

(C)

(D)

(E) The speeds upon reaching the valley will be the same in each case.

2. Two blocks, of mass m and mass M, are attached together by a light cord. One block sits on a frictionless table top and the other hangs freely, as shown. The pulley is massless and frictionless. If the blocks start from rest, what is the speed of m after M has dropped a distance h?

(A) $\sqrt{\dfrac{mgh}{M+m}}$ (B) $\sqrt{\dfrac{Mgh}{m}}$ (C) $\sqrt{\dfrac{Mgh}{M+m}}$ (D) $\sqrt{\dfrac{(M+m)gh}{M}}$ (E) none of these

3. The gravitational potential energy U_{grav} of an object is represented by the graph shown. The constant energy E of the object is represented by the horizontal straight line. At which of the given points A - F (a) is the kinetic energy a maximum? (b) is the kinetic energy zero? (c) will the object never be found?

PROBLEM QUIZ 10

1. Starting from rest, a car of mass m rolls down a slope of height h into a valley and up to a crest, as shown. Both the valley and crest have the shape of circular arcs of radius R. (a) At what altitude h (in terms of R) should the car begin so that it will be weightless at the top of the crest? (b) In this case, what will be the car's weight at the bottom of the valley?

2. A block of mass m oscillates on the end of a spring. (a) If the maximum speed of the ball is v_m, and the maximum distance it reaches from its equilibrium position is $x = A$, what is the force constant k of the spring, in terms of m, v_m, and A? (b) What is the distance x of the block from equilibrium when its speed is $v_m/2$?

Special Topic: Roller Coasters

Design your own roller coaster. How do you do it?

The cars are hauled up to the starting point, the highest point of the trip, and then released with some initial speed. The rest of the trip is all coasting, down, up, over, around, down, until finally the cars are slowed to rest.

What do you need to know? You need to know the speed of the cars everywhere. You need to know the effective "g-forces" acting on the riders. Positive g-forces push people more firmly into their seats; negative g-forces tend to pull them out of their seats.

Designers often start by neglecting friction and air resistance, to get a rough idea of what to expect. Then energy is conserved so it's easy to calculate the speed at every point of the trip. In the dips the speed is high; at the crests the speed is lower. Friction and air resistance slow the cars, so they will move more slowly in both the dips and crests than as calculated from energy conservation.

For simplicity suppose there is only a single car, starting its run at altitude y_O with speed v_O, on a track with a dip at point A at altitude zero followed by a crest at point B, at altitude y_B. How fast is the car going at A and at B?

Energy conservation gives

$$E = \frac{1}{2}mv_O^2 + mgy_O = \frac{1}{2}mv_A^2 = \frac{1}{2}mv_B^2 + mgy_B ,$$

so the car's speeds (squared) at A and at B are

$$v_A^2 = v_0^2 + 2gy_0 \quad \text{and} \quad v_B^2 = v_0^2 + 2g(y_0 - y_B).$$

Suppose for example the car starts at altitude $y_0 = 88$ m with initial speed $v_0 = 5.5$ m/s, and that the crest at B is at altitude $y_B = 55$ m above the bottom. Then at the dip A the car reaches speed $v_A = 42$ m/s (94 mi/hr) according to energy conservation, and at the crest B the car slows to speed $v_B = 26$ m/s (58 mi/hr).

What effective gravity is felt by a rider at the dip and at the crest? So-called "g-forces" are really effective gravitational accelerations, expressed as a multiple of one Earth "gee". As the car coasts along a horizontal section of track, for example, a rider experiences "one gee".

Let the bottom of the dip at A be a circular arc of radius R, and the top of the crest at B be likewise a circular arc of radius R, as shown. Isolation diagrams of a rider at A and at B are shown, where N is the normal force of the seat acting on the rider. At point A,

$$F_{net} = N_A - mg = ma = mv_A^2/R,$$

where in the final equality we have used the fact that the rider is moving in a circle, so her centripetal acceleration is $a = v_A^2/R$. The normal force is therefore $N_A = m(g + v_A^2/R)$, so the effective gravitational acceleration of the rider at A is

$$g_{eff} = g + v_A^2/R.$$

At point B, the net force on the rider is

$$F_{net} = mg - N_B = ma = mv_B^2/R$$

taking the positive direction to be downward, since the centripetal acceleration direction is downward at B. The normal force is therefore $N_B = m(g - v_B^2/R)$, so the effective gravitational acceleration of the rider at B is

$$g_{\text{eff}} = g - v_B^2/R,$$

which can be either positive or negative. If v_B is not too great, the rider is still pressed into the seat, but if v_B is sufficiently large the rider will tend to fly up out of the seat unless contrained by a seat belt.

In the special case described on the previous page, suppose the radii of curvature of the dip and the crest are both $R = 55$ m, as shown. Then at A, the bottom of the dip, the effective gravitational acceleration is

$$g + v_A^2/R = 9.8 \text{ m/s}^2 + \frac{(42 \text{ m/s})^2}{(55 \text{ m})} = 42 \text{ m/s}^2 = 4.3\,(9.8 \text{ m/s}^2) = 4.3 \text{ "g's"}.$$

A rider is pressed into her seat as though gravity were 4.3 times as large as usual. This causes blood to drain from the brain into the body. If the acceleration is more than about 5 g's, many riders black out. At B, the top of the crest, the effective gravitational acceleration is

$$g - v_B^2/R = 9.8 \text{ m/s}^2 - \frac{(26 \text{ m/s})^2}{(55 \text{ m})} = 2.5 \text{ m/s}^2 = -0.25\,(9.8 \text{ m/s}^2) = -0.25 \text{ "g's"}$$

A rider at the crest experiences a small negative g, meaning that she would float up out of the car if it were not for her seat belt. If she experiences negative g's of about 3 or more she could have a "red out", where excess blood flows into her head increasing pressure on blood vessels, potentially causing strokes or aneurisms.

Obviously we have just begun to describe roller coaster design. There is also the effective gravity on curves, the design of wheels, what happens in "loops" when riders are upside down at the top (and yet can have positive g), etc. Also when several cars are attached together, the effective g's can be different in the rear cars than in the front ones.

Questions and Problems from Day Ten

10Q1. A theme-park ride is designed so that a car, starting from rest, slides without friction from point A to point B. (a) For which ramp shown will the block reach the bottom with the greatest speed? (b) For which ramp do you think the block will reach the bottom in the least time?

10Q2. Two identical ramps with the shape shown are constructed on Earth and on the Moon. Starting from rest at the top, a block sliding down the Earth-ramp reaches speed v_f at the bottom. What final speed is reached by the block sliding down the Moon-ramp? Assume negligible friction. (Gravity on the Moon is 1/6 that on Earth.)

10Q3. Identical small blocks are pushed the same distance against identical springs at the left of two frictionless ramps. One ramp is horizontal everywhere, and the other has a dip in the middle, rising again to its original elevation, as shown. Each ramp has the same horizontal length. If the blocks are released simultaneously, which (if either) reaches the end of its ramp first?

10Q4. If a rising object loses half its kinetic energy, what fraction of its speed does it lose?

10Q5. A ball has energy E in one inertial reference frame. Will its energy have the same numerical value in other inertial frames?

10Q6. Which of the following formulas cannot be the solution to a problem valid in all systems of units? Here K and U are kinetic and gravitational potential energies, h and R are lengths, m is a mass, v is a velocity, and g is a gravitational field.

$K = m(R/h)v^2$ $\qquad$ $K + gU = 2mvR$ $\qquad$ $R = (K + U) / mg$

$mg = K / (h + R)$ $\qquad$ $U - mg = (1/2)v^2\sin(h/R)$ $\qquad$ $K+U = ghv^2/ (R+h/2)$

10Q7. If x is the stretch of a Hooke's-law spring (of force-constant k) from equilibrium, and k' is a constant, could the expression $U(x) = \frac{1}{2}kx^2 + k'$ be used for the potential energy stored in the spring, instead of $\frac{1}{2}kx^2$? How about $\frac{1}{2}k(x - k')^2$?

Section 10.1 Motion along the straight-sided slopes

10P1. Starting from rest, an explorer slides a distance $d = 30$ m down an icy slope tilted at angle $\theta = 30^o$ to the horizontal. Gravity is $g = 10$ m/s^2. (a) What is her speed at the bottom? (b) If she then slides up a slope tilted at angle $\phi = 45^o$, what distance ℓ along this slope will she slide before coming instantaneously to rest?

10P2. A skier moving at speed v_0 along an icy horizontal surface comes to a hill that slopes upward at angle θ.
(a) What distance along this frictionless hill will the skier ski before coming to rest?
(b) Just as the skier has almost come to rest, the hill ends in a vertical cliff, and the skier falls down to the same altitude at which he began. What is his speed just before hitting the bottom?

10P3. An iceskater starts at rest at the top of the frictionless slope shown, and skates down a slope of length D. She then skates a distance $D/2$ up a frictionless slope inclined at angle θ to the horizontal before coasting to rest. (a) In terms of θ, what was the tilt-angle θ_0 of the first slope? (b) If $\theta = 30^o$, what was θ_0?

10P4. A skateboarder moving horizontally at speed $v_0 = 15$ m/s encounters a ramp tilted upward at angle $\theta = 5^o$. (a) Pretending there is no friction, what distance D along the ramp will the skateboarder coast before coming instantaneously to rest? (b) If at that point the ramp ends and the skateboarder falls vertically down, what will be his speed just before again reaching the ground? Use $g = 9.8$ m/s^2.

10P5. A semitrailer truck with burned-out brakes, rolling downhill on the interstate, reaches speed $v = 89$ mi/hr just before turning onto an emergency escape ramp.

The ramp is inclined at angle $\theta = 12^o$ to the horizontal, as shown. Assuming no friction, what is the minimum ramp length ℓ needed to be sure the truck will stop? (Escape ramps are typically shorter than this, because they use deep gravel to provide frictional drag.)

Section 10.2 Motion along curved slopes

10P6. A downhill racer on frictionless ice skis from the top of a run, starting at speed $v_0 = 1.5$ m/s, down a curved slope ending a vertical distance $h = 55$ m below. What is the skier's speed at the bottom, in m/s and in miles/hour? Neglect air resistance.

10P7. A roller-coaster car starts with speed v_0 at point A on a frictionless track, as shown. In terms of g and h (the initial height of the track), what will be the speed of the car at points (a) B? (b) C? (c) D? Assume the car never leaves the track.

10P8. A skateboarder moving horizontally with speed v_0 encounters a concrete ramp in the shape of a quarter-circle, as shown. (a) What is the radius R of this ramp, if the skateboarder comes instantaneously to rest at the top of the ramp? (b) If the initial speed of the skateboarder is $v_0/2$, at what angle θ (measured from the vertical) would she come instantaneously to rest?

10P9. A small steel ball of mass m is attached to one end of a string of length ℓ and negligible mass; the other end of the string is tied to a hook on the ceiling. The hook is a distance h ($h > \ell$) above the floor. The ball is pulled aside through an angle $\theta_0 = 45^\circ$ and then released, as shown, with the string taut. In terms of given parameters, (a) What is the speed v of the ball when the string becomes vertical? (b) When the ball reaches an angle $\theta = 30^\circ$ up the other side, the string suddenly breaks. What is the ball's speed when it hits the floor?

Section 10.3 Two blocks tied together

10P10. Two blocks, of masses m and M, are attached together by a light cord draped over two frictionless pegs, as shown. If the blocks start from rest, what is the speed v of m after M has dropped a distance h?

10P11. Two blocks, of mass m and M, are attached together by a light cord. One block sits on a frictionless slope tilted at angle $\theta = 30^\circ$ to the horizontal, and the other hangs freely, as shown. The pulley is massless and frictionless. If the blocks start from rest, what is the speed of m after M has dropped a distance h?

10P12. A double-gondola system transports people frictionlessly up and down the triangular shaped mountain shown. The gondolas are balanced, so if given a small speed v_0, they retain these speeds throughout the journey. If the mass of the left-hand gondola is m, what is the mass of the right-hand gondola?

10P13. Two elevators of equal mass M are attached to light cables strung over a pulley with negligible mass and friction, as shown. They are initially at rest. A passenger of mass m enters the right-hand elevator, headed for a floor a distance h below. (a) When the right-hand elevator has fallen a distance y, what is its speed $v(y)$? (b) At what time t will the passenger arrive at $y = h$? (*Hint:* $v = dy/dt$; separate the variables y and t and integrate.) (c) With what speed v will the passenger arrive?

10P14. Two blocks, with masses m_1 and m_2, are attached to opposite ends of a light cord slung over a frictionless peg. Assume $m_2 > m_1$.
(a) Draw isolation diagrams of each block.
(b) Write out the $F = ma$ equations, and find the acceleration of each block in terms of m_1, m_2, and g.
(c) The acceleration is constant, so we can use the constant-acceleration formula $v^2 = 2ay_1$ (where y_1 is the altitude gained by m_1) to find the speed of the blocks as a function of y_1. Rearrange the result to show that the energy

$$E = (\frac{1}{2}m_1 v_1^2 + m_1 g y_1) + (\frac{1}{2}m_2 v_2^2 + m_2 g y_2)$$

of the system is conserved.

10P15. A teeter-totter consists of a very light plank of length L with heavy weights of masses m_1 and m_2 on the two ends. The plank is supported by a fulcrum a distance x from the left end of the plank, as shown. (a) If the system is in equilbrium, find m_2 in terms of L, x and m_1. (Hint: in equilibrium the fulcrum is just beneath the center of mass of the system.) (b) If the system is in equilibrium and at rest, and is then given a small angular speed, the angular speed does not change for small tipping angles, so the kinetic energy of each mass stays constant. Show that even though the gravitational potential energy of each mass changes, their sum $U = m_1 g y_1 + m_2 g y_2$ stays constant for small tipping angles. Thus overall energy is conserved. (Here y_1, y_2 are the altitudes of the two masses.)

10P16. The picture tube in a color TV set contains three electron guns, one for each primary color. Electrons shot from a gun toward the screen typically have a kinetic energy $K = 30$ keV. (a) How far does an electron with this energy *fall* if aimed horizontally from a gun to the screen, 50 cm away? Use $g = 9.8$ m/s^2. Note that 1 keV = 1000 eV, and 1 eV = $1.60 \cdot 10^{-19}$ J. (eV stands for *electron volt*)

(b) What is the distance fallen in units of *Fermis* (F) where 1 F = 10^{-15} m? (The radius of a proton is roughly 1 F.)

10P17. Cosmic rays are high-energy particles (protons and other atomic nuclei), originating at so-far unidentified sources in outer space, which strike Earth's atmosphere. The highest-energy cosmic ray observed to date had a kinetic energy $K = 3 \cdot 10^{20}$ eV (electron volts), about 10^8 times as much energy as protons have achieved in particle accelerators on Earth. (a) Find this kinetic energy in Joules. (See the table at the back of the book.) (b) How far would a 1-kg cateloupe have to fall near Earth's surface to achieve the same kinetic energy?

10P18. A small block of mass m is attached to one end of a rod of length ℓ and negligible mass. The other end of the rod is pivoted on a frictionless axle so the block can move in a vertical circle. The rod is pulled aside to the horizontal and given a downward push, so the rod swings down and then up the other side, just reaching the vertically upward position. (a) What initial speed v_0 was given to the block? (b) What is the tension T in the rod when the block reaches the top?

10P19. A small block of mass m is placed at the left-hand side of the track shown, and given an initial downward speed v_0. It then slides around the inside surface of the frictionless track of radius ℓ, and reaches the top of the track. (a) Find the magnitude and direction of the normal force N of the track on the block when it is at the top, in terms of m, g, v_0, and ℓ. (b) What is the minimum speed v_0 needed so the block will reach the top of the track?

10P20. A block of mass m slides down a frictionless ramp, starting from rest a distance h above the floor. When it reaches the floor, the block slides around the inside of a frictionless circular track of radius R as shown. (a) Find the minimum value of h such that the block will stay on the circular track all the way around. (b) What happens to the block if h is less than this minimum value?

10P21. A pendulum consists of a bob of mass m attached to a cord of length ℓ. The upper end of the cord is held in place, while the pendulum swings back and forth in a circular arc, making an angle θ with the vertical. (a) If the bob is released from rest at an angle θ_0, find its speed as a function of θ. (b) Find the tension T in function of θ. Where is the cord most likely to break?

10P22. A pig slides without friction down from the top of a hemispherical dome. (a) If the pig starts with only a very small speed v_0, at what angle θ will it leave the dome's surface? Use an isolation diagram. (b) If the pig starts instead with an arbitrary initial horizontal speed v_0, at what angle θ will it *then* leave the dome's surface? Is this a smaller or larger angle than before? (c) Find the minimum value of v_0 such that the pig will leave the surface immediately.

10P23. An iceskater, starting with initial speed v_0, coasts up a frictionless hill composed of circular arcs of radius R connected at angle θ_0, as shown. (a) Find the maximum value (v_{max}) of v_0 such that the skater will not leave the ice immediately after reaching A. (b) Show that if the skater does not leave the ice at A, he won't leave the ice farther up either. (c) If $v_0 = v_{max}$, will he reach B?

10P24. One end of a string of length ℓ is attached to a small ball, and the other end is tied to a hook on the ceiling. A nail juts out from the wall, a distance d below the hook, as shown. The ball is released when the string is horizontal. (a) With what speed v will the ball be moving at point A, its highest point after swinging about the nail? (b) Find the minimum value of d, as a fraction of ℓ, such that the ball can reach point A.

10P25. A toboggan slides on frictionless ice, starting from rest at height h above the top of the round hill shown. The radius of the circular portion of the hill is R, and the angle θ is measured clockwise from the top of the hill.
(a) Find h_{max}, the maximum value of h for which the toboggan remains in contact with the ice at the top of the hill.
(b) If $h = \alpha h_{max}$, where h_{max} is the value found in part (a) and α is a constant obeying $0 < \alpha < 1$, find the angle θ at which the toboggan leaves the ice, in terms of any or all of α, g, h, and R.

10P26. Two blocks, with masses m_1 and m_2, are attached to opposite ends of a light inextendible cord, and the cord is draped over a frictionless peg, so that m_1 hangs vertically, and m_2 is free to slide without friction on a plane inclined at angle θ to the horizontal, as shown. Take the gravitational potential energy to be zero for both of the blocks when they are released from rest at time $t = 0$. As functions of time, find the energy E of (a) m_1 (b) m_2 (c) both together. (d) Is their total energy conserved?

10P27. A uniform, flexible rope of length ℓ and mass m can slip without appreciable friction on a peg, as shown. At time $t = 0$ the rope is at rest, the left end of the rope hangs a distance $x_0 - \Delta x_0$ below the center of the peg, and the right end hangs a distance x_0 below the center of the peg. The rope is released from rest. Find the speed of any point on the rope as a function of x, the distance the right-hand end hangs below the peg's center at any later time while the rope is still draped around the pulley.

Section 10.5 Spring potential energy

10P28. A block of mass m oscillates on the end of a spring of force-constant k. (a) If the distance between the two extreme positions of the block is D, what is the fastest speed v_{max} of the block during its motion? (b) If the block is brought to rest at its equilibrium point, what initial speed v_0 must then be given to the block so that it will reach a maximum distance $D/4$ from equilibrium?

10P29. A block of mass $m = 0.15$ kg oscillates on the end of a spring of force-constant $k =$ 0.75 N/m. (a) If the distance between the two extreme positions of the block is D= 25 cm, what is the fastest speed v_{max} of the block during its motion? (b) If the block is brought to rest at its equilibrium point, what initial speed v_0 must then be given to the block so that it will reach a maximum distance 8.0 cm from equilibrium?

10P30. Two springs, each of force-constant k, are attached to opposite sides of a block of mass m which slides on a frictionless surface as shown. (a) If the block is pulled to the right a distance A from equilibrium and released from rest, what is its speed v_0 when it returns to its original equilibrium position? (b) In terms of A, what is the distance d of the block from its equilibrium position when the kinetic and potential energies of the system are equal? What is its speed v at this point?

10P31. A block of mass m oscillates on the end of a spring of force constant k. Its position as a function of time is given by $x(t) = A\sin\omega t$, where A (the *amplitude*) and ω (the *angular frequency*) are constants. (a) Find the block's velocity $v(t)$. (b) Find its kinetic energy $K(t)$. (c) Find its potential energy $U(t)$. (d) Show that its total energy E is constant, if $\omega = \sqrt{k/m}$. Find E in terms of k and A.

10P32. A bullet of mass m is fired at a block of mass M which is initially at rest. The block is connected to a spring of force-constant k; the other end of the spring rests against a wall. The bullet is quickly brought to rest relative to the block, before the block has had time to move appreciably. We want to find the velocity v_0 of the bullet by measuring how much the spring is compressed (Δx) before the block and bullet are brought to rest. Find v_0 in terms of m, M, k, and Δx.

10P33. An astronaut of mass m is exploring an asteroid using a pogo stick to jump from place to place. The gravitational field near the asteroid's surface is g. The pogo stick contains a small but stiff spring of force-constant k, as shown. If at some point the spring is compressed by an amount Δx and the astronaut is instantaneously at rest, about how high can the astronaut rise above the surface?

10P34. A bead of mass m can move without friction on a straight wire. The bead is attached to two springs, each of force-constant k, as shown. The other ends of the springs are kept fixed. When the bead is at $x = 0$, each spring is at its equilibrium length ℓ_0. (a) Show that the net force on the bead as a function of x is

$$F_x = -2kx[\,1 - (1 + \frac{x^2}{\ell_0^2})^{-1/2}\,].$$

(b) Show that for small x ($x << \ell_0$), the force is proportional to x^3, and the potential energy of the block has the form $U(x) = Cx^4$. Find the constant C in terms of k and ℓ_0. *Hint*: Use the binomial expansion. (c) A second bead, also of mass m, is placed on the wire at the far left, and is given a velocity v_0 toward the right. It collides with the first bead, initially at rest at $x = 0$, and they stick together. For what value of x do the two beads subsequently instantaneously come to rest? (Assume $x << \ell_0$.)

Special Topic: Roller Coasters

10P35. A roller-coaster car starts with speed v_0 at point O on a frictionless track, as shown. In terms of g and h (the initial height of the track), what will be the speed of the car at points (a) A? (b) B? (c) C? Assume the car never leaves the track.

10P36. You are designing a roller coaster as pictured. Point B is an inflection point where two circular segments are tangent. A car of mass m is released from rest at a particular height h ($h > 2R$). The track is frictionless. (a) Find an expression for the speed v the cars at point A. (b) What is the upward normal force N exerted by the track on the car at point A? (c) Find h (as a multiple of R) if a car rider is weightless at point C. Use isolation diagrams in parts (b) and (c).

10P37. A roller-coaster car starts from rest at the top of a ramp of height h, and then does a loop as shown, where the car is upside down at the top. If the loop has radius $R = h/4$ and the car moves without friction, what is the magnitude and direction of the weight W of a rider of mass m when she is at the top of the loop? (Recall that the magnitude of the rider's weight is the normal force exerted upon her.)

10P38. A rider of mass m sits in a roller-coaster car which starts from rest at the top of a frictionless ramp of height h, and then travels down and makes a circular loop of radius R, as shown. Gravity is g.
(a) Find the speed of the car at points A, B, and C. (b) Find the magnitude and direction of the effective gravity g_{eff} acting on the rider at point A (The magnitude of g_{eff} is the *normal force /m*). Also find g_{eff} acting on the rider (c) at C, and (d) at B. (g_{eff} can be read by a spring-mass accelerometer carried by the rider.)

10P39. Designers want to build a roller coaster in which a single car begins at height h, with negligible initial speed, rolls down into a dip which is a circular arc of radius R, and then up to a crest which is also a circular arc of radius R. The crest is at altitude R above the bottom of the dip, as shown. Assuming energy conservation, (a) find the required R (as a multiple of h) so that the effective gravitational acceleration experienced by riders is 4.0 g's in the dip. (b) In that case, what is the effective gravitational acceleration experienced by riders at the crest?

10P40. Starting essentially from rest, a roller-coaster car descends a frictionless track a vertical distance h before entering a portion of track in the form of a horizontal ring of radius R. At each point in the ring the track is tilted at angle θ to the horizontal, as shown.

(a) What should be the angle θ (in terms of any or all of h, R, and g) so that passengers will find themselves pressed directly into their seats, with no tendency to slip sideways? (b) In that case, what is the effective gravity they feel while in this horizontal ring?

10P41. An L-shaped roller-coaster features a vertical section of height $h = 415$ ft, as shown. Riders enter a car on the bottom portion, and are then accelerated to a height $y = 370$ ft, $y = 370$ ft, reaching an upward velocity $v = 53$ ft/s. The power is then cut, and the car continues to rise, acted upon only by gravity. Neglect friction and air resistance. (a) What is the maximum height h_{max} achieved by the car? (b) How long a time t does it take to reach this height after the power is cut? (c) After reaching the top, the car falls straight down until it reaches a speed $v' = 100$ mi/hr (147 ft/s). How far (Δy) has the car fallen from the top? (d) What is the total length of time τ, from the beginning of the ride until they reach $v' = 100$ mi/hr, that the riders experience weightlessness (i.e., "zero gee")? (e) After the free-fall, the car moves along a quarter-circle of radius R, as shown. What is R? (f) If there were no braking or friction while the car moves along the quarter-circle, with what speed v_f would the car be moving by the time it reached the horizontal stretch? (g) What then would be the weight/mass of a rider at the bottom of the circular portion, just *before* entering the straight stretch? (Use an isolation diagram.) (h) What would be the weight/mass of the rider just *after* entering the straight stretch? (The sudden change shows that roller coasters need to be designed with smoother transitions.)

10P42. Design your own roller coaster, subject to the safety requirements that no positive effective gravity should exceed 5 gees, and that no negative gravity should exceed 2 gees.

Day Eleven

Work

"Work is the curse of the drinking classes." - William A. Spooner

The kinetic energy $K = \frac{1}{2}mv^2$ of a block changes when it moves back and forth on the end of a spring. What is the cause of this kinetic energy change? And why does the kinetic energy of a baseball change as the pitcher throws it, as the batter strikes it, as it rises and falls in response to gravity and air resistance, and as the outfielder's mitt brings it to rest? *What causes changes in kinetic energy*?

11.1 Work and the work-energy theorem

Start for simplicity with a block of mass m sitting on a frictionless plane, pulled to the right by a string, where the string tension F is kept constant. The acceleration $a = F/m$ of the block is therefore also constant, so we can use the constant-acceleration formula

$$v^2 = v_0^2 + 2as \tag{1}$$

to relate the block's speed v to its initial speed v_0, its acceleration a, and its displacement s. Multiplying Eq. 1 by $m/2$ and rearranging,

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = ma \cdot s = F \cdot s, \tag{2}$$

where the left-hand side is $K - K_0$, the change in kinetic energy of the block, positive if K has increased and negative if K has decreased. The right-hand side is the product of the applied force F and the displacement s.

So what has *caused* the block to change its kinetic energy? According to Eq. 2, it is the force F acting as the block moves through a distance s. Both F and s are important; a given force F can change the block's kinetic energy more and more as it moves farther and farther.

The same formula should work if we drop a ball, since the gravitational force $F_{grav} = mg$ is constant. The right-hand side of Eq. 2 in that case is $mg \cdot s$, which is responsible for the change in the ball's kinetic energy as it falls.

We define the *work*

$$W = F \cdot s \tag{3}$$

as the product of force and displacement, in the case where (*i*) F is constant, and (*ii*) the force is in the same direction as the displacement. Note from Eq. 2 that work has the dimensions of energy, so in SI units work can be expressed equivalently in units of either *Newton-meters* or *Joules*.

Sample Problem 1. A worker accidentally drops a bolt of mass $m = 12$ g off the top of the Sears Tower in Chicago, a height $h = 440$ m above the street. How much work is done by gravity as the bolt falls?

Solution: The gravitational force mg is constant, so the work done by gravity is $W_{\text{grav}} = mg\,s$, where s is the distance the bolt has fallen. Therefore

$$W_{\text{grav}} = F_{\text{grav}} \cdot s = mg\,s = 0.012 \text{ kg} \cdot 9.8 \text{ m/s}^2 \cdot 440 \text{ m} = 52 \text{ kg m}^2/\text{s}^2 = 52 \text{ Joules.}$$

If gravity is the only force acting (i.e., if we neglect air resistance), the work done by gravity must also be the change in kinetic energy of the bolt. We can check that directly. If the bolt started at rest at the top of the tower, at the bottom the square of its speed would be

$$v^2 = 2as = 2(9.8 \text{ m/s}^2)(440 \text{ m}) = 8.6 \cdot 10^3 \text{ m}^2/\text{s}^2,$$

using the constant-acceleration Eq. 9(d) from *Day Five*, so the change in kinetic energy would be

$$\Delta K = K_f - 0 = \frac{1}{2}mv^2 = \frac{1}{2}(0.012 \text{ kg})(8.6 \cdot 10^3 \text{ m}^2/\text{s}^2) = 52 \text{ Joules}$$

as expected.

When we drop an object in uniform gravity or pull a block with a horizontal string, the force in each case is in the same direction as the displacement. Suppose instead that we pull a block along a frictionless plane with a string oriented at angle θ to the horizontal, while the string tension F is kept constant, as shown in Fig. 1(a). An isolation diagram of

Fig. 1 (a) A block pulled by a string at angle θ. (b) Isolation diagram of the block

the block is shown in Fig. 1(b). The horizontal component of the string tension is $F\cos\theta$, so the block's acceleration to the right is $a = F\cos\theta / m$. In this case Eq. 2 becomes

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = ma \cdot s = F\cos\theta \cdot s. \tag{4}$$

The kinetic energy of the block is changed by the *component of force along the direction of motion*, multiplied by the distance the block moves. So in this case the work done by the constant force F on the block is

$$W = F\cos\theta \cdot s\,. \tag{5}$$

If $\theta = 0$, so the force **F** is lined up with the displacement **s**, the resulting change in kinetic energy is as large as it can be for given **F** and **s**. If $\theta = 90^\circ$, with **F** perpendicular to **s**, no work is done by **F** and there is no change in the block's kinetic energy.

In general *several* forces may act on the block; for each force F the work done by F can be defined by Eq. 5. The total work W_T done on the block is the sum of the work done by each individual force, $W_T = W_1 + W_2 + \ldots$ The work on the right-hand side of Eq. 4 must be the *total* work done on the block, since we used $F = ma$ in the final step. That is, Eq. 4 can be rewritten

$$\boxed{\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = W_T.} \tag{6}$$

We call Eq. 6 the *Work-Energy Theorem.* The total work W_T done on an object by all forces is equal to the change in the kinetic energy of the object. If we observe that the kinetic energy of a block, pool ball, automobile, ... has changed, we can be sure that a net amount of work has been done. So far we have proved the theorem only for the case where all the forces are constant forces, but will prove it in general later.

Sample Problem 2. A child pulls a sled of mass m at constant velocity along the snow by means of a rope at fixed angle θ relative to the horizontal. The coefficient of kinetic friction between the sled and snow is μ_k. (a) How much work is done by the child in pulling the sled a distance d? (b) How much work is done by friction?

Solution: The total work done on the sled is zero, since its kinetic energy remains unchanged. The child does work $W = Fd\cos\theta$ when pulling with force F. The frictional force of snow on the sled must do an equal but opposite (i. e., *negative*) work so thast the total work done on the sled is zero. We first solve for the particular value of F for which the sled moves without acceleration.

(a) An isolation diagram of the sled is shown.
Balancing the vertical forces gives

$$N + F\sin\theta = mg,$$

and balancing the horizontal forces gives

$$F\cos\theta = F_f = \mu_k N.$$

Eliminating N between these two equations gives $F\cos\theta = \mu_k(mg - F\sin\theta)$, which can be solved for F to give

$$F = \frac{\mu_k mg}{\cos\theta + \mu_k \sin\theta}$$

as the force due to the child. It follows that the normal force is

$$N = \frac{F\cos\theta}{\mu_k} = \frac{mg\cos\theta}{\cos\theta + \mu_k \sin\theta}.$$

The work done by the child is

$$W = Fd\cos\theta = \frac{\mu_k mgd\cos\theta}{\cos\theta + \mu_k \sin\theta} = \frac{\mu_k mgd}{1+\mu_k \tan\theta} .$$

(b) The work done by friction is

$$W_f = F_f \cdot d\cos 180^\circ = -F_f\, d$$

since the frictional force is opposite to the direction of the displacement. The frictional force has magnitude

$$F_f = \mu_k N = \frac{\mu_k mg\cos\theta}{\cos\theta + \mu_k \sin\theta} ,$$

so the work done by friction is

$$W_f = -F_f\, d = -\frac{\mu_k mgd\cos\theta}{\cos\theta + \mu_k \sin\theta} = -\frac{\mu_k mgd}{1+\mu_k \tan\theta} .$$

We knew this work had to be equal but opposite to the work done by the child, and it is.

11.2 Dot products

There is a simple way to express the work $W = F\cos\theta \cdot s$ done by a constant force, called the *dot product*. We define the dot product $\mathbf{F} \cdot \mathbf{s}$ of the two vectors $\mathbf{F}$ and $\mathbf{s}$ to be

$$\boxed{\mathbf{F} \cdot \mathbf{s} = F\, s\, \cos\theta ,} \qquad (7)$$

the product of the magnitudes of the vectors multiplied by the cosine of the angle θ between them. As we have seen, $F\cos\theta$ is the component of $\mathbf{F}$ parallel to $\mathbf{s}$; it is also true that $s\cos\theta$ is the component of $\mathbf{s}$ parallel to $\mathbf{F}$, as shown. Therefore as long as

the force $\mathbf{F}$ is constant, the work done by $\mathbf{F}$ as the object moves a distance $\mathbf{s}$ is

$$W = \mathbf{F} \cdot \mathbf{s} . \tag{8}$$

The work done when $\mathbf{F}$ and $\mathbf{s}$ are at various angles θ relative to one another is illustrated in Fig 2.

Fig. 2. The work done when $\mathbf{F}$ is at various angles to $\mathbf{s}$.

If a ball falls vertically downward a distance s , the work done by gravity is $mg\ s$, since $m\mathbf{g}$ is parallel to $\mathbf{s}$. But if a ball moves horizontally, the work done by gravity is *zero*, since $m\mathbf{g}$ is then perpendicular to $\mathbf{s}$. During *Day Ten,* when the block slid down the straight slope, the normal force $\mathbf{N}$ was constant and always perpendicular to the block's displacement. Therefore $\mathbf{N}$ did no work on the block, and so did not contribute to any change in the block's kinetic energy.

Sample Problem 3. Sisyphus slides a stone of mass m at constant velocity up the side of a mountain of slope-angle θ, as shown. He pushes parallel to the slope with force F_O, and there is a coefficient of kinetic friction μ_k between the stone and the surface.

(a) How much work does Sisyphus do on the stone, if he raises it to height h?

(b) After reaching this height, the stone rolls back down. How much work does gravity do on it? Is this greater or less than the work Sisyphus did to raise the stone?

Solution: (a) Four forces act upon the stone, as shown in the isolation diagram. The stone is not accelerating, so the *net* force is zero in any direction.

Perpendicular to the slope, $N - mg\cos\theta = 0$, where N is the normal force, so $N = mg\cos\theta$.

Parallel to the slope, $F_O - \mu_k N - mg\sin\theta = 0$. This gives

$$F_O = \mu_k mg\cos\theta + mg\sin\theta = mg(\sin\theta + \mu_k\cos\theta).$$

The force $\mathbf{F}_O$ is constant and in the direction of the displacement, so the work Sisyphus does on the stone is

$$W = \mathbf{F}_O \cdot \mathbf{s} = F_O s = F_O h/\sin\theta = mgh(1 + \mu_k\cot\theta) .$$

(b) On the way down, gravity does work

$$m\mathbf{g} \cdot \mathbf{s} = mgh ,$$

since h is the component of $\mathbf{s}$ parallel to $m\mathbf{g}$. This is less than the work Sisyphus did in raising the stone, unless $\mu_k = 0$. That is because in raising the stone, he had to overcome the work done by friction as well as the work done by gravity.

Using the dot product, Eq. 4 can be written

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \mathbf{F} \cdot \mathbf{s} = W. \tag{9}$$

Here $\mathbf{F}$ must be the total force acting on the block and this total force $\mathbf{F}$ must be constant throughout the motion. If $\mathbf{F}$ is parallel to $\mathbf{s}$, then the work done is simply the product of the total force and the displacement; if $\mathbf{F}$ is perpendicular to $\mathbf{s}$, then no work is done, and the block's kinetic energy does not change.

More generally, we define the dot product of any two vectors $\mathbf{A}$ and $\mathbf{B}$ to be

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta \tag{10}$$

where A is the magnitude of $\mathbf{A}$, B is the magnitude of $\mathbf{B}$, and θ is the angle between them. The dot product may be thought of in different ways: It is the magnitude of $\mathbf{A}$ times the component of $\mathbf{B}$ $(= B\cos\theta)$ along the direction of $\mathbf{A}$, as shown in Fig. 3(a). The dot product $\mathbf{A} \cdot \mathbf{B}$ is also equal to the magnitude of $\mathbf{B}$ times the component of $\mathbf{A}$ $(= A\cos\theta)$ along the direction of $\mathbf{B}$, as shown in Fig. 3(b).

Fig. 3. The dot product is either (a) the length of $\mathbf{A}$ times the component of $\mathbf{B}$ along the $\mathbf{A}$ direction, or (b) the length of $\mathbf{B}$ times the component of $\mathbf{A}$ along the $\mathbf{B}$ direction.

We can express the dot product in terms of the Cartesian components of the vectors. An arbitrary vector $\mathbf{A}$ can be written

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}\,, \tag{11}$$

using Cartesian components A_x, A_y, and A_z, and unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. For any two vectors $\mathbf{A}$ and $\mathbf{B}$, the dot product is $\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$, so in particular the dot products of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ with themselves are

$$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1 \tag{12}$$

because each vector is parallel to itself (with $\cos\theta = 1$) and has unit length, and the dot products with one another are

$$\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{x}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{y}} = 0 \tag{13}$$

because they are mutually perpendicular (with $\cos\theta = 0$). Therefore

$$\mathbf{A}\cdot\mathbf{B} = (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}})\cdot(B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})$$

$$= A_xB_x + A_yB_y + A_zB_z. \tag{14}$$

All other products of components (like A_xB_y) vanish, because of Eq. 13. If we dot a vector into itself, we get

$$\mathbf{A}\cdot\mathbf{A} = A^2 = A_x^2 + A_y^2 + A_z^2, \tag{15}$$

a three-dimensional Pythagorean theorem. The dot product of a vector with itself is just the square of its magnitude, the sum of the squares of its components.

Sample Problem 4. (a) Find the value of the constant a such that the vector $\mathbf{A} = -\hat{\mathbf{x}} + 2\hat{\mathbf{y}}$ is perpendicular to the vector $\mathbf{B} = 4\hat{\mathbf{x}} + a\hat{\mathbf{y}}$. (b) Find the magnitudes of the vectors $\mathbf{A}$ and $\mathbf{B}$, using the value of a found in part (a).

Solution: (a) The dot product of the vectors $\mathbf{A}$ and $\mathbf{B}$ is $\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$, where θ is the angle between them. Now $\theta = 90^{\circ}$ if $\mathbf{A}$ and $\mathbf{B}$ are perpendicular to one another, so $\mathbf{A} \cdot \mathbf{B} = AB\cos\theta = 0$ in this case. Dotting the vectors as given,

$$\mathbf{A} \cdot \mathbf{B} = (-\hat{\mathbf{x}} + 2\hat{\mathbf{y}}) \cdot (4\hat{\mathbf{x}} + a\hat{\mathbf{y}})$$

$$= -4(\hat{\mathbf{x}} \cdot \hat{\mathbf{x}}) + 2a(\hat{\mathbf{y}} \cdot \hat{\mathbf{y}}) \quad (\text{since } \hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = 0)$$

$$= -4 + 2a \quad (\text{since } \hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = 1 \text{ and } \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = 1)$$

$$= 0 \quad \text{if and only if } a = 2.$$

That is, $\mathbf{A}$ is perpendicular to $\mathbf{B}$ if and only if $a = 2$.

(b) The magnitude of $\mathbf{A}$ is $\sqrt{\mathbf{A} \cdot \mathbf{A}} = \sqrt{A_x^2 + A_y^2} = \sqrt{1+4} = \sqrt{5}$, and the magnitude of $\mathbf{B}$ is similarly $\sqrt{16+4} = \sqrt{20}$.

11.3 Work done by variable forces

What if the force on an object changes as the object moves? Then it doesn't make sense to define work as the dot product $W = \mathbf{F} \cdot \mathbf{s}$, because $\mathbf{F}$ has no fixed value throughout the motion, so it isn't clear which $\mathbf{F}$ to use. Suppose a block moves on the end of a spring, for example. The force $F = -kx$ of the spring depends on the position x of the block, where $x = 0$ is the equilibrium position. How do we define work in this case, when F depends upon position?

Certainly the force on an object stays nearly constant as the object moves through a very small distance. So a natural definition of the infinitesimal work dW done by a force $\mathbf{F}$ on the object as it moves an infinitesimal distance $\mathbf{ds}$ is $dW = \mathbf{F} \cdot \mathbf{ds}$. The work done by $\mathbf{F}$ over the entire path (as the object travels from point a to point b) is the sum of these,

$$W = \int_a^b dW = \int_a^b \mathbf{F} \cdot \mathbf{ds}. \qquad (16)$$

If the force and displacement are both in the x direction, for example, then $\mathbf{F} = F\,\hat{\mathbf{x}}$ and $\mathbf{ds} = dx\,\hat{\mathbf{x}}$, so $dW = \mathbf{F} \cdot \mathbf{ds} = F(x)\,dx$, where F is the force acting on the object at x. The work done along the entire one-dimensional path is then

$$W[a \to b] = \int_a^b F\,dx\,. \qquad (17)$$

If the force actually turns out to be constant after all, F can be taken outside the integral, so

$$W[a \to b] = F \int_a^b dx = F\,x\Big|_a^b = F(x_b - x_a)\,, \qquad (18)$$

the product of force times the distance traveled, our original definition of work for constant force. Whether the force is constant or not, the work done by $F(x)$ is the area under the curve $F(x)$ between x_a and x_b, as illustrated in Fig. 4. Sometimes $F(x)$ is simple enough so the integral can be carried out explicitly. Even if it isn't, the area under the graph can be estimated in various ways, by numerical methods with the help of a computer, or simply by plotting $F(x)$ on a piece of graph paper with squares and counting squares under the function. See Special Topic II for a description of the numerical methods.

Fig. 4. The work $W[a \rightarrow b]$ done on an object by a force F is the area under the graph of $F(x)$ as the object moves from a to b. It is the sum of the $\Delta W = F\Delta x$ rectangles, in the limit $\Delta x \rightarrow 0$.

Sample Problem 5. A block of mass m is placed on the end of an unstretched spring of force constant k. How much work is done on the block by the spring if we pull the block a distance x from its equilibrium position?

Solution: The work done by the spring is

$$W_{\text{spring}}[0 \rightarrow x] = \int_0^x F\,dx = \int_0^x -kx\,dx = -\frac{1}{2}kx^2.$$

In this case the work is *not* simply the force times the distance $Fx = -kx^2$, because the spring force is not constant, so we have to do the integral. The work is negative because the spring force is to the left while the displacement is to the right.

In Section 11.1 we derived the work-energy theorem

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = W_T, \tag{6}$$

relating the change in kinetic energy of a particle to the total work W_T done on it by all the forces acting. We actually derived the result only for the special case where every force $\mathbf{F}$ is constant. Is the theorem still valid even when the forces are *not* constant?

It is easy to see that it *is* still valid. If the particle moves a sufficiently small distance $\Delta s = s_1 - s_0$, then a force $\mathbf{F}$ stays nearly constant over that distance, so the small amount of work done by $\mathbf{F}$ as the particle moves from s_0 to s_1 is $\Delta W = \mathbf{F} \cdot \Delta s$. The change in kinetic energy over this small distance is

$$\Delta K = K_1 - K_0 = \Delta W_T = \mathbf{F}_T \cdot (s_1 - s_0) \tag{19a}$$

according to the Eq. 6, where ΔW_T is the work done by the sum $\mathbf{F}_T$ of all the forces. When the particle moves a further short distance $\Delta s = s_2 - s_1$ from s_1 to s_2, the change in kinetic energy during that interval is similarly

$$\Delta K = K_2 - K_1 = \Delta W_T = \mathbf{F}_T \cdot (s_2 - s_1) \tag{19b}$$

where $\mathbf{F}_T$ is the total force acting during that interval, and so on.

As the particle travels farther and farther, over N steps (say), we just sum all these equations, so that summing the total change in kinetic energy on the left-hand side gives

$$\Delta K = (K_N - K_{N-1}) + \ldots + (K_3 - K_2) + (K_2 - K_1) + (K_1 - K_0) = K_N - K_0$$

and the sum of the total work elements on the right-hand side is $\sum \mathbf{F}_T \cdot \Delta\mathbf{s}$. Taking the limit of an infinite number of infinitesimal steps $\mathbf{ds}$, the sum becomes the integral $\int \mathbf{F}_T \cdot \mathbf{ds}$, so the work-energy theorem is still $\Delta K = K_{\text{final}} - K_{\text{initial}} = W_T$, where now the total work done is the integral over the total force,

$$\boxed{W_T = \int \mathbf{F}_T \cdot \mathbf{ds}\ .} \tag{20}$$

Sample Problem 6. A block of mass m is attached to one end of a spring of force-constant k, and the other end of the spring is held fixed. The block is then pulled a distance A from equilibrium, stretching the spring as shown. What is the speed of the block when it returns to its equilibrium position?

Solution: As the block returns to its original position, the work done by the spring is $+\frac{1}{2}kx^2$, evaluating the integral in *Sample Problem* 5 with the spring force in the *same* direction as the displacement this time. The spring force is the *only* horizontal force acting on the block, so the *total* work on the block is $+\frac{1}{2}kx^2$.

The block's initial kinetic energy is zero, so it follows from the work-energy theorem that the kinetic energy of the block is $\frac{1}{2}mv^2 = \frac{1}{2}kx^2$ when it returns to its original position. The speed of the block at this point is therefore $v = \sqrt{\frac{k}{m}}\,x$.

Sample Problem 7. A roller-coaster car accelerates down the long slope shown, and then travels around a circular loop of radius R.

(a) Evaluate the total work $W_T = \int \mathbf{F} \cdot d\mathbf{s}$ done on the car, as it moves from the bottom of the loop up to a position at angle θ_0. Neglect air resistance and frictional forces.

(b) Evaluate the integral all the way around the loop, starting at the bottom and going counter-clockwise back to the bottom again.

Solution: (a) The only forces acting on the car are gravity $m\mathbf{g}$ and the normal force $\mathbf{N}$. The normal force does no work on the car, because it is everywhere perpendicular to the path $d\mathbf{s}$, which is tangent to the circular loop. Therefore $\mathbf{N} \cdot d\mathbf{s} = 0$ everywhere, and the total work done is

$$\int \mathbf{F} \cdot d\mathbf{s} = \int m\mathbf{g} \cdot d\mathbf{s} .$$

From the diagram, $\mathbf{g} \cdot d\mathbf{s} = g\, ds \cos(\theta + \pi/2)$. But $\cos(\theta + \pi/2) = -\sin\theta$, using the trig identity $\cos(A + B) = \cos A \cos B - \sin A \sin B$. Therefore $\mathbf{g} \cdot d\mathbf{s} = -g\, ds \sin\theta$, which is the length of $d\mathbf{s}$ multiplied by the component of $\mathbf{g}$ parallel to $d\mathbf{s}$.

The differential path length is $ds = Rd\theta$, because angles in radians are *defined* to be the subtended arc-length divided by the radius, as shown. So the integral is

$$W = -mgR\int_0^{\theta_0} d\theta \sin\theta = mgR\cos\theta \Big|_0^{\theta_0} = -mgR(1 - \cos\theta_0).$$

(b) Going all the way around the loop, the integration limits are $\theta = 0$ and $\theta = 2\pi$. The integral is then $-mgR(1 - \cos 2\pi) = -mgR(1 - 1) = 0$: The integral around the closed loop is *zero*. The total work done on the car is therefore zero, so from the work-energy theorem we conclude that the car must have the same speed each time it reaches the bottom. (Remember we assumed there is no friction!)

11.4 Power

The kinetic energy of a rocket increases as it rises from the launchpad. Therefore there must be a net amount of work done on it by all the forces involved, including the pressure of expanding vapors as the fuel burns, gravity, and air resistance. Depending on the balance of forces, the kinetic energy K may increase very slowly if the forces are nearly balanced, very quickly if the exhaust pressure forces are much larger than gravity, or any rate in between. So an interesting question is: How fast does the kinetic energy change? In other words, what is the rate of change $\frac{dK}{dt}$?

The change in a particle's kinetic energy $K - K_O$ is due to the total work done W_T, according to the work-energy theorem. Here K_O is the initial kinetic energy. If a small amount of work ΔW_T is done during some short time interval, the particle's kinetic energy will change by the small amount

$$\Delta K = \Delta W_T. \tag{21}$$

The total work done is $\Delta W_T = \mathbf{F}_T \cdot \Delta \mathbf{s}$ where the total force is $\mathbf{F}_T = \mathbf{F}_1 + \mathbf{F}_2 + \ldots$ Now divide Eq. 21 by the small time interval Δt and take the limit $\Delta t \rightarrow 0$. The result is

$$\frac{dK}{dt} = \mathbf{F}_T \cdot \frac{d\mathbf{s}}{dt} = \mathbf{F}_T \cdot \mathbf{v} \equiv P_T \tag{22}$$

where we will call $P_T \equiv \mathbf{F}_T \cdot \mathbf{v}$ the *total power input* to the particle. The total power input is the dot product of the net force $\mathbf{F}_T = \mathbf{F}_1 + \mathbf{F}_2 + \ldots$ with the velocity $\mathbf{v}$ of the particle. The total power input can be thought of as the sum of the power inputs $P_i \equiv \mathbf{F}_i \cdot \mathbf{v}$ from each individual force $\mathbf{F}_i$.

It is the total power input that is responsible for the change in kinetic energy of the particle. For example, the kinetic energy of a ball falling only under the influence of gravity increases with time, because the power input $P_T \equiv \mathbf{F}_T \cdot \mathbf{v} = mgv$ is positive. If the net force is zero, there is no net power input. If the particle is not moving, there is no power input. If the net force is perpendicular to the velocity, there is no net power input.

The dimensions of power are those of energy/time and the units of power are Joules/second in the SI system. We call this unit *Watts:* That is, 1 Watt = 1 W = 1 Joule/second.

Sample Problem 8. A locomotive pulls a chain of train cars of mass M at steady speed v up a straight grade of angle θ. A frictional force F_f acts directly backwards on the cars. (a) Find the total power input P_T to the cars. (b) Find the individual power inputs to the cars due to the normal force, the frictional force, gravity, and the force of the locomotive.

Solution: (a) The kinetic energy of the cars is not changing, so the total power input P_T must be zero. (b) An isolation diagram of the cars is shown. The normal force is perpendicular to the velocity, so $P_N = \mathbf{N} \cdot \mathbf{v} = 0$. The frictional force is opposite to the velocity, so

$$P_f = \mathbf{F}_f \cdot \mathbf{v} = F_f v \cos 180^\circ = -F_f v .$$

The power input from gravity is

$$P_g = m\mathbf{g} \cdot \mathbf{v} = mgv \cos(90^\circ + \theta) = -mgv \sin\theta.$$

The sum of these power inputs is $-(F_f + mg \sin\theta)v$, so the power input from the locomotive must be

$$P_{loco} = +(F_f + mg \sin\theta)v$$

to make the total power input equal to zero. We can verify that from the isolation diagram. The acceleration of the cars is zero, so the net force must be zero as well. Therefore the force due to the locomotive up the slope must be $F_f + mg \sin\theta$, parallel to the velocity of the cars. Therefore the power input from the locomotive is $(F_f + mg \sin\theta)v$, as expected.

The total power input to a particle is the cause of its time rate of change of kinetic energy. The term "power" is also often used to describe the time rate of change of energy in other contexts as well. A 100 Watt bulb uses 100 Joules of electrical energy each second, for example. Electric power plants, whether they are coal-burning, oil-burning, gas-burning, or nuclear, are rated by the number of Watts (usually Megawatts (1 MW =

10^6 W) or Gigawatts (1 GW = 10^9 W)) of power they produce. This power can be used to add energy to systems, including the heating of water, the running of motors, etc.

The dimensions of power are those of Energy/Time, so the dimensions of Energy are those of Power · Time. The electrical energy used by a household over some period of time is often measured in *kilowatt hours*. If one kilowatt hour of energy is used, for example, this is equivalent to an energy expenditure of

$$10^3\,\mathrm{W} \cdot 1\ \mathrm{hr} = 10^3\,\frac{\mathrm{J}}{\mathrm{s}} \cdot 1\ \mathrm{hr} \cdot 60\frac{\mathrm{min}}{\mathrm{hr}} \cdot 60\frac{\mathrm{s}}{\mathrm{min}} = 3{,}600{,}000\ \text{Joules}.$$

Sample Problem 9. Water flows downward a vertical distance h inside a dam, and then turns giant turbines which generate electrical power.

(a) Find an expression for the mass/second of falling water needed to generate electrical power P_e, if the conversion efficiency of gravitational potential energy into electrical energy is η.

(b) Find the volume/second of water required to generate 1000 Megawatts of electricity inside Hoover Dam, where $h = 200$ m. Assume $g = 10$ m/s^2 and $\eta = 0.5$.

Solution: (a) Suppose that in a period of time t a mass M of water falls through the dam. Its gravitational potential energy decreases by Mgh as it falls and the efficiency of electrical power generation from this change is η ($\eta < 1$ because of frictional and other heat losses from the water flow and from the turbines), so the electrical power generation is

$$P_e = \eta\,\frac{Mgh}{t}.$$

The mass/sec of falling water required is therefore $\dfrac{M}{t} = \dfrac{P_e}{\eta\, gh}$.

(b) Using the numbers given,

$$\frac{M}{t} = \frac{1000 \cdot 10^6\ \mathrm{W}}{0.5\ (10\ \mathrm{m/s^2})\ (200\ \mathrm{m})} = 10^6\ \mathrm{kg/s} = 1000\ \text{tonnes/s}.$$

One cubic meter of water has a mass of one metric tonne, so the required volume/sec of water is 1000 m^3/s.

State of the Search

1. The kinetic energy of an object changes as a result of the total work W_T done on it by all the forces acting. If a force $\mathbf{F}$ is constant, the work done by $\mathbf{F}$ on the object is

$$W = \mathbf{F} \cdot \mathbf{s},$$

the dot product of the force and the displacement of the object. The dot product $\mathbf{F} \cdot \mathbf{s} = Fs \cos\theta$, where θ is the angle between the two vectors. The dot product can be thought of as the displacement s multiplied by the component of $\mathbf{F}$ in the direction of $\mathbf{s}$. (The dot product is also the force F multiplied by the component of $\mathbf{s}$ in the direction of $\mathbf{F}$.) If $\mathbf{F}$ and $\mathbf{s}$ are pointed in the same direction, the work is just Fs, a positive quantity. If $\mathbf{F}$ and $\mathbf{s}$ are pointed in opposite directions, the work done is $-Fs$.

2. The dot product of *any* two vectors $\mathbf{A}$ and $\mathbf{B}$ is $\mathbf{A} \cdot \mathbf{B} = AB \cos\theta$, the product of the magnitudes of each vectors and the cosine of the angle between them.

3. If a force $\mathbf{F}$ depends on the position of the particle (such as the spring force whose magnitude is $F = -kx$) then the work done by $\mathbf{F}$ when the particle moves from point a to point b is the integral

$$W[a \rightarrow b] = \int_a^b \mathbf{F} \cdot d\mathbf{s} .$$

4. The total work done on the particle due to all the forces is the sum of the work done by each individual force. The total work is also the integral over the net force $\mathbf{F}_T$. The *work-energy theorem*

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0{}^2 = W_T$$

shows that the change in kinetic energy of a particle is equal to the total work done on it by all the forces acting. If the speed of a particle does not change, no net work has been done.

5. The time rate of change of a particle's kinetic energy is $\frac{dK}{dt} = \mathbf{F}_T \cdot \frac{d\mathbf{s}}{dt} = \mathbf{F}_T \cdot \mathbf{v} \equiv P_T$, where $P_T \equiv \mathbf{F}_T \cdot \mathbf{v}$ is the total power input to the particle, the dot product of the total force with the velocity of the particle. Each individual force $\mathbf{F}_i$ contributes a power input $P_i \equiv \mathbf{F}_i \cdot \mathbf{v}$, and the total power input is the sum of these individual power inputs.

Compare two integrals of $\mathbf{F} = m\mathbf{a}$: the time integral and the distance integral. We found the *impulse equation* (see Day Five) by integrating Newton's second law $F = ma$ in one dimension with respect to *time*. During Day Eleven we have integrated the same equation $F = ma$ with respect to the distance x instead, and obtained the *work-energy theorem*. How do the results and the physical meaning of the two integrals compare? Note the close analogy between them.

Time Integral	**Distance Integral**
The impulse equation	The work-energy theorem
is $I_T = \int F_T\, dt = \Delta p$.	is $W_T = \int F_T\, dx = \Delta K$.
Momentum $p = mv$ is conserved if $I_T = 0$.	Kinetic energy $K = \frac{1}{2}mv^2$ is conserved if $W_T = 0$.
The time derivative of I_T is $\frac{dI_T}{dt} = F_T$, the net force acting on the particle.	The time derivative of W_T is $\frac{dW_T}{dt} = P_T$, the net power input to the particle. The power may also be written $P = F\,v$

We have added the subscript "T" to the impulse as well as to the work, since the change in momentum of a particle is due to the total impulse acting on it, just as the change in the kinetic energy of the particle is due to the total work done on it. The units of impulse and momentum are kg m/s ; the units of work or kinetic energy are Joules, with 1 J = 1 N m = 1 kg m^2/s^2 ; and the units of power are Watts, with 1 W = 1 J/s.

How do the two sets of integrals compare for *two-* or *three*-dimensional motion? Some of the physical quantities become vectors, and their products have to be expressed as dot products, as shown below.

Time Integral	**Distance Integral**
The impulse equation	The work-energy theorem
is $\mathbf{I}_T = \int \mathbf{F}_T\, dt = \Delta\mathbf{p}$.	is $W_T = \int \mathbf{F}_T \cdot d\mathbf{s} = \Delta K$.
Momentum $\mathbf{p} = m\mathbf{v}$ is conserved if $\mathbf{I}_T = 0$.	Kinetic energy $K = (1/2)mv^2$ is conserved if $W_T = 0$.
The time derivative of $\mathbf{I}_T$ is $\frac{d\mathbf{I}_T}{dt} = \mathbf{F}_T$, the net force acting on the particle.	The time derivative of W_T is $\frac{dW_T}{dt} = P_T$, the net power input to the particle. The power may also be written $P = \mathbf{F} \cdot \mathbf{v}$.

Appendix A: Numerical Integration

In physics we often need to integrate something. We may need to find the work

$$W[a \to b] = \int_{x_a}^{x_b} F\,dx$$

done by some force F on a particle. We may need to find the velocity v of a particle of mass m, given the net force $F(t)$ acting on it, using

$$v(t) = v_0 + \int_0^t a(t)\,dt = \int_0^t (F/m)\,dt.$$

Sometimes such integrals can be carried out by well-known techniques such as substitution, by parts, by partial fractions (or by looking them up in tables or by using a package like *Maple* or *Mathematica.*)

However, it can happen that none of these techniques is of much use. The integrand can be too complicated, for example, or it may only be a set of experimental numbers that do not match any common function. What then?

Suppose we have an arbitrary smooth function $y(x)$ such as the one shown below. We want to know the integral of the function between given limits, which is the *area* under the graph.

There are many ways to find the area. One is to draw the function on a sheet of paper covered by squares, and count the number of squares. Another is to draw the function on a sheet of uniform cardboard, cut out the area and weigh it, and then divide by the weight per unit area of the cardboard.

We will demonstrate two ways of calculating the area under smooth functions by *numerical* means, i. e., using a computer. The results are almost never exact, but the approximation can be as accurate as needed.

The idea is to find the value of the function $y(x)$ at $N + 1$ discrete points $x_0, x_1, x_2, \ldots x_N$ separated by the same distance Δx. That is, we need $y(x)$ at points x_0, $x_1 = x_0 + \Delta x$, $x_2 = x_0 + 2\Delta x$, etc,, as illustrated below.

THE TRAPEZOIDAL RULE

In the *trapezoidal* rule, we approximate the function $y(x)$ between two neighboring points by a straight line, as shown at the right. The geometric shape of the area between two neighboring points is a trapezoid, whose area is its width (Δx) times its average height, $\frac{y_1 + y_2}{2}$. Thus the total area under the curve is

$$A = \Delta x\left[\frac{y_0 + y_1}{2} + \frac{y_1 + y_2}{2} + \frac{y_2 + y_3}{2} + \ldots \frac{y_{N-1} + y_N}{2}\right]$$

or more simply

$$\boxed{A = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + \ldots + 2y_{N-1} + y_N\right).}$$

TRAPEZOIDAL RULE

For example, suppose we want to find the integral $\int_0^3 x^2 dx$ using the trapezoidal rule. This is not a very clever thing to do, because in this case we can carry out the integration exactly, That is,

$$\int_0^3 x^2 dx = \left.\frac{x^3}{3}\right|_0^3 = \frac{27}{3} - 0 = 9.$$

Nevertheless, let us see how well we do using the trapezoidal rule. We begin by splitting up the range of x from 0 to 3 into N intervals. If we choose $N = 3$, for example, as shown, then $\Delta x = 1$. We need to find the function $y(x) = x^2$ at $x = 0$, $x = 1$, $x = 2$, and at $x = 3$. These values are $y_0 = 0$, $y_1 = 1$, $y_2 = 4$, and $y_3 = 9$. The trapezoidal rule then gives

$$A = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + \ldots + 2y_{N-1} + y_N\right)$$

$$= \frac{1}{2}(0 + 2\cdot 1 + 2\cdot 4 + 9) = \frac{19}{2} = 9.5,$$

about 5% too large. (We *expect* the trapezoidal rule to be too large in this case, because the areas of the trapezoids are always somewhat greater than the actual area under the function $y = x^2$ as shown in the picture above.)

Now reduce the step-size Δx by half, from 1.0 to 0.5. There are then *six* rather than three intervals, so the area under the curve, using the trapezoid approximation, is

$$A = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + \ldots + 2y_{N-1} + y_N\right)$$

$$= \frac{1/2}{2}\left(0 + 2\left(\frac{1}{2}\right)^2 + 2(1)^2 + 2\left(\frac{3}{2}\right)^2 + 2(2)^2 + 2\left(\frac{5}{2}\right)^2 + 3^2\right)$$

$$= \frac{1}{4}\left(0 + \frac{1}{2} + 2 + \frac{9}{2} + 8 + \frac{25}{2} + 9\right) = \frac{1}{4}(19 + \frac{35}{2}) = 9.125,$$

less than 1.5% too large. Decreasing the step size typically gives better accuracy. We could make the step size still smaller and get still better accuracy at the sacrifice of more computation time (which is obviously not a problem with current calculators or computers).

SIMPSON'S RULE

Simpson's rule gives greater accuracy than the trapezoidal rule with the same step-size Δx. Whereas the trapezoidal rule imposes a *linear* function $y = a + bx$ between any two successive heights $y(x)$ and $y(x+\Delta x)$, Simpson's rule fits a *quadratic* function

$$y = a + bx + cx^2 ,$$

where a, b, and c are constants, between *three* successive values of the function, $y(x)$, $y(x+\Delta x)$, and $y(x+2\Delta x)$. Suppose for example the first point is $x = 0$, the second is $x = \Delta x$, and the third is $x = 2\Delta x$, as illustrated, with corresponding heights y_0, y_1, and y_2. It follows that

$$y_0 = a + 0 + 0$$

$$y_1 = a + b\Delta x + c\Delta x^2$$

$$y_2 = a + b(2\Delta x) + c(2\Delta x)^2 .$$

From the first equation we find that the constant $a = y_0$. To solve for b, multiply the y_1 equation by 4 and subtract the y_2 equation, giving

$$4y_1 - y_2 = 4y_0 - y_0 + 2\,b\Delta x,$$

so

$$b\Delta x = (1/2)\,[4y_1 - y_2 - 3y_0].$$

Substituting this result into the y_1 equation gives

$$y_1 = y_0 + (1/2)\,[4y_1 - y_2 - 3y_0] + c\Delta x^2,$$

so

$$c\Delta x^2 = -y_1 + (1/2)y_0 + (1/2)y_2.$$

Now the area under the function $y = a + bx + cx^2$ between $x = 0$ and $x = 2\Delta x$ is the integral

$$\int_0^{2\Delta x} (a + bx + cx^2)\, dx = (ax + \frac{b}{2}x^2 + \frac{c}{3}x^3)\Big|_0^{2\Delta x}$$

$$= a(2\Delta x) + \frac{b}{2}(2\Delta x)^2 + \frac{c}{3}(2\Delta x)^3) = 2\Delta x\left[a + b\Delta x + \frac{4}{3}c(\Delta x)^2\right]$$

$$= 2\Delta x\left[y_o + \frac{1}{2}\left(4y_1 - y_2 - 3y_o\right) + \frac{4}{3}(-y_1 + \frac{1}{2}y_o + \frac{1}{2}y_2)\right],$$

substituting in the values of a, $b\Delta x$, and $c\Delta x^2$. Collecting like terms together, the area under the curve between $x = 0$ and $x = 2\Delta x$ is

$$A = \frac{\Delta x}{3}\left(y_o + 4y_1 + y_2\right)$$

where we have curve-fit $y(x)$ using a quadratic function.

If there are additional areas to consider, from $x = 2\Delta x$ to $4\Delta x$, from $4\Delta x$ to $6\Delta x$, etc., the total area is the sum

$$A = \frac{\Delta x}{3}\left(y_o + 4y_1 + y_2\right) + \frac{\Delta x}{3}\left(y_2 + 4y_3 + y_4\right) + \frac{\Delta x}{3}\left(y_4 + 4y_5 + y_6\right) + \ldots$$

$$= \frac{\Delta x}{3}\left(y_o + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + \ldots\right).$$

We begin by splitting the range of integration into N segments, where for Simpson's rule N *must be an even number*. Then the integral using this technique is

$$\int_{x_o}^{x_f} y(x)\, dx = \frac{\Delta x}{3}\left(y_o + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 2y_{N-2} + 4y_{N-1} + y_N\right).$$

SIMPSON'S RULE

with a (1/3) in front and the pattern of coefficients 1, 4, 2, 4, 2, 4, ...2, 4, 1.

We can apply Simpson's rule to the integral $\int_0^3 x^2 dx$, again pretending that we don't know how to do it exactly. In using the trapezoidal rule we used the points $x = 0$, 1, 2, 3, so $N = 3$ in this case, which is an *odd* number, so we can't use Simpson's rule for these points. We repeated the trapezoidal rule calculations for the points $x = 0$, 1/2, 1, 3/2, 2, 5/2, 3, which corresponds to $N = 6$, an *even* number. So we will apply Simpson's rule to this problem using $N = 6$. The area under the curve according to Simpson's rule is

$$\int_0^3 x^2\, dx = \frac{1/2}{3}\left(0 + 4\left(\frac{1}{2}\right)^2 + 2(1)^2 + 4\left(\frac{3}{2}\right)^2 + 2(2)^2 + 4\left(\frac{5}{2}\right)^2 + (3)^2\right)$$

$$= \frac{1}{6}(0 + 1 + 2 + 9 + 8 + 25 + 9) = \frac{54}{6} = 9$$

which is the *exact* value of the integral! This of course is no coincidence in this case, since the Simpson's rule approximation of fitting a quadratic form between three points is no approximation at all here, since our function $y = x^2$ is already quadratic. Simpson's rule, like the trapezoidal rule, is generally only an approximation in nearly all cases, an approximation that gets better as the spacing Δx is decreased.

Computer Question and Problems

11AQ1. What is the most general function for which both the trapezoidal rule and Simpson's rule give the *exact* integral, for *any* spacing?

11AP1. (a) Write a computer program to implement the trapezoidal rule, and use it to find the work done on a particle by the force $F = 3\sqrt{x}$, where F is in Newtons and x is in meters, if the particle moves from $x = 0$ to $x = 4$ m. Use spacings Δx = 1.0 m and 0.5 m. (b) What is the largest spacing needed so the result agrees with the *exact* answer within 0.1%?

11AP2. Write a computer program to implement Simpson's rule, and use it to find the velocity of a particle at time $t = 2.0$ s, given that the particle starts at rest at time $t = 0$, and that it has acceleration $a(t) = (1 + t^2 e^{-t})^{1/2}$, where a is in m/s if t is in seconds. Try various spacings until you are confident that you know $v(2\text{ s})$ within 0.1%.

11AP3. Write computer programs to implement both the trapezoidal rule and Simpson's rule in evaluating the integral $\int_0^1 \sqrt{1 + x^2}\, dx$. Using various spacings, compare results for each rule with the exact answer, to test how much better Simpson's rule is than the trapezoidal rule.

PICTURE QUIZ 11

1. A weightlifter raises a barbell of mass m from the floor to height h, when it again comes instantaneously to rest. The work done on the barbell by the weightlifter is

(A) $-mgh$ (B) 0 (c) $+mgh$

2. The weightlifter in question 1 then drops the barbell. Using the work-energy theorem, how fast is it moving just before it hits the floor?

(A) $\sqrt{gh}$ (B) $\sqrt{2gh}$ (C) $2gh$ (D) gh (E) $\sqrt{gh/2}$

3. A custodian pushes with force F_0 directly along the handle of a mop of mass m. The handle is oriented at angle θ relative to the horizontal, as shown, and there is a coefficient of kinetic friction μ_k between the mop and floor. The custodian finds that the mop moves at constant velocity v along the floor. The work done by friction as the mop moves a distance s is

(A) $\dfrac{\mu_k mgs}{1+\mu_k \tan\theta}$ (B) $\dfrac{\mu_k mgs}{1-\mu_k \cot\theta}$ (C) $-\dfrac{\mu_k mgs}{1-\mu_k \tan\theta}$ (D) $-\dfrac{\mu_k mgs}{1+\mu_k \tan\theta}$ (E) $\dfrac{\mu_k mgs}{1+\mu_k \cot\theta}$

4. In the situation described in question 3, the total power input to the mop is

(A) $\dfrac{\mu_k mg}{1+\mu_k \tan\theta} v$ (B) $\dfrac{\mu_k mg}{1-\mu_k \cot\theta} v \cos\theta$ (C) $-\dfrac{\mu_k mg}{1-\mu_k \tan\theta} v$

(D) $\dfrac{\mu_k mgs}{1+\mu_k \cot\theta} v \cos\theta$ (E) 0

PROBLEM QUIZ 11

1. Ten Alaskan huskies are tethered to a sled and rider of total mass $m = 220$ kg.

(a) What total work is done by the dogs on the slide and rider in bringing them from rest up to the speed $v = 7.5$ m/s?
(b) If this speed is reached after pulling for a distance $s = 17$ m, what average force was exerted by each dog? Assume negligible friction between snow and sled.

2. A hailstone falls from rest a distance $d = 4.0$ km. Its mass stays constant at $m = 2.2$ g. If it strikes the ground with speed $v = 25$ m/s, what is the total work W done by air resistance as the hailstone falls?

3. The motor molecule *dynein* moves, one step at a time, along microtubules in a living organism. A dynein molecule has a step size of approximately 8 nm, and pulls with a force estimated to be in the range 4 -10 pN. Estimate the maximum work that can be done by the molecule in a single step, in Joules.

4. A person of mass m rides in a seat of a Ferris wheel. She is moving in a vertical circle of radius R at constant angular speed ω, and at time $t = 0$ she is at the lowest point of the circle. As functions of time, find the net work done on her by (a) gravity (b) the seat (c) both together. Neglect air resistance.

5. (a) Find the value of the constant a such that the vector $\mathbf{A} = 2\hat{\mathbf{x}} - 3\hat{\mathbf{y}} - a\hat{\mathbf{z}}$ is perpendicular to the vector $\mathbf{B} = 3\hat{\mathbf{x}} - a\hat{\mathbf{y}} + 4\hat{\mathbf{z}}$. (b) For this value of a, what is the magnitude of each of the vectors $\mathbf{A}$ and $\mathbf{B}$?

quiz continued on reverse

6. If **A**, **B**, **C** are vectors representing the sides of a triangle, derive the law of cosines by dotting the vector $\mathbf{C} = \mathbf{A} - \mathbf{B}$ into itself.

7. When stretched by amount x, a particular nonlinear spring exerts a restoring force $F = -k\sqrt{x}$, where k is a constant. A block of mass m is attached to one end of the spring and the other end of the spring is kept fixed. (a) If the block is pulled from the spring's equilibrium position $x = 0$ to $x =$ A, what is the work done by the spring on the block? (b) If the block is then released from rest at $x =$ A, what is the speed of the block just before it reaches $x = 0$?

8. A member of the ΔH *Club* with mass m rappells down a vertical rope, touching nothing but the rope. Find the power inputs P to her, as functions of time, from (i) gravity; (ii) the rope; and (iii) both together, if she is descending (a) with constant speed v; (b) with constant acceleration $a = g/10$, assuming she starts at rest at time $t = 0$.

9. A sled slides down a straight slope at constant speed v under the influence of gravity, the normal force, and a frictional drag force. The power input from gravity is P. What is the power input from friction?

Special Topic I: Energy from the Sun

Life on Earth depends upon energy radiated from the Sun in the form of light, including visible, infrared, and ultraviolet radiation. What is the source of this energy?

The Sun and solar system were formed from a large cloud of gas and dust, which gradually contracted due to its self-gravitation. As the cloud shrank, gravitational potential energy was converted into heat energy of atoms in the cloud. When the temperature reached a few thousand degrees Kelvin the protosun began to glow; it then consisted of an ionized gas, mostly protons and electrons, but a substantial number of helium nuclei and small percentages of the other stable atomic nuclei as well. When the Sun's core reached a few million degrees, some of the protons moved fast enough to overcome their mutual electrical repulsion and so get close enough to allow nuclear reactions to take place. Some of these reactions are responsible for the Sun's energy; they release a net amount of kinetic energy. Such nuclear fusion reactions began about four and one-half billion years ago, and will keep going for about the same length of time before the Sun begins to change radically.

The first reaction in the "proton-proton chain", the most important set of energy-producing reactions in the Sun, is

$$p + p \rightarrow d + e^{+} + \nu$$

in which two protons come together to form a deuteron d (the hydrogen isotope with one proton and one neutron), a positron e^{+} (an antielectron), and a neutrino ν. The positron subsequently annihilates with an electron in the Sun to form two or more photons (γ). The total energy released in this first step is 1.19 MeV: That is, the total kinetic energies of the final particles (d, ν, and γ's) exceeds the total kinetic energy of the original protons by 1.19 MeV.

The second reaction in the chain is

$$p + d \rightarrow {}_2He^3 + \gamma$$

in which another proton interacts with the deuteron formed in the first reaction, producing a helium-three isotope and a photon. The energy released in this second reaction is 5.49 MeV.

When enough ${}_2He^3$ isotopes have been formed, the third reaction

$$ {}_2He^3 + {}_2He^3 \rightarrow {}_2He^4 + p + p $$

begins to kick in, in which two ${}_2He^3$ isotopes interact to form the helium isotope ${}_2He^4$ (which is the same as an α-particle) and two protons. The energy produced in this third reaction is an impressive 12.85 MeV.

Note that to make one ${}_2He^4$ nucleus we need two of the first reactions, two of the second, and one of the third. The total energy released in turning four protons into one ${}_2He^4$ nucleus, through the net reaction $4p + 2e^- \rightarrow {}_2He^4 + \nu$'s $+ \gamma$'s is therefore

$$ \Delta E = 2(1.19\text{ MeV}) + 2(5.49\text{ MeV}) + 12.85\text{ MeV} = 26.2\text{ MeV}. $$

A small fraction of this energy is removed from the Sun by neutrinos produced in the first reaction (nearly all of which leave the Sun without losing appreciable energy.) The rest is energy which slowly diffuses out of the core of the Sun in the form of light, i.e., photons.

The Sun emits the total power $P_{Sun} = 3.8 \cdot 10^{26}$ Watts, radiated out as light essentially equally in all directions. How much of this power is captured by Earth? The fraction of the Sun's power that strikes Earth's atmosphere is the ratio of the cross-sectional area of Earth,

$$ A_{Earth} = \pi r^2, $$

where r is Earth's radius, to the area of the huge spherical surface whose radius R is the distance of Earth from the Sun,

$$ A_{Sphere} = 4\pi R^2 . $$

This tiny fraction is

$$f = \frac{\pi r r^2}{4\pi R^2} = \left(\frac{r}{2R}\right)^2 = \left(\frac{6400\text{ km}}{2\left(1.5\cdot 10^8\text{ km}\right)}\right)^2 = 4.55\cdot 10^{-10}.$$

The total power reaching Earth's upper atmosphere is therefore

$$f\cdot P_{\text{Sun}} = 4.55\cdot 10^{-10}\cdot 3.8\cdot 10^{26}\text{ Watts} = 1.7\cdot 10^{17}\text{ W}.$$

Dividing by πr^2, the cross-sectional area of Earth, the power per unit area striking the top of the atmosphere amounts to

$$1.3\text{ kW/m}^2,$$

which is called the "solar constant". A one meter-squared piece of cardboard held perpendicular to the Sun's rays at the top of the atmosphere would receive 1.3 kilowatts of power, enough to run thirteen 100-Watt electric light bulbs. Only about 47% of this radiation reaches Earth's surface; the rest is reflected back into space by the atmosphere and by clouds. Nevertheless, about 80,000 Terawatts of power reach Earth's surface (1 Terawatt = 1 TW = 10^{12} W.)

We can compare this received power of 80,000 TW with the total power used by humans around the world, which is roughly 15 TW. Therefore if we could capture even a very small fraction of the Sun's energy reaching the surface and convert it into electrical power, we could in principle power human needs without burning fossil fuels (coal, oil, natural gas) or using nuclear reactors.

Various techniques have been explored for capturing sunlight directly, including passive solar heating (heating water during the daytime and using it for hot water and heat at night) and solar cells (solid-state devices which absorb photons and produce electric currents).

At the present time the direct conversion of solar energy into electricity is about twice as expensive as burning fossil fuels, although solar cells are used to power a few homes in sunny areas and also to provide small amounts of power at remote locations, such as emergency telephones along freeways. However, in a few years the use of solar power may become competitive for the production of large-scale electric power, if the cost of solar cells keeps dropping and the cost of fossil fuels rises. This could reduce air pollution and the other environmental consequences of fuel burning.

Questions and Problems from Day Eleven

11Q1. Can *static* friction ever do *positive* work on an object? If so, give an example. If not, why not?

11Q2. Can a normal force $\mathbf{N}$ ever do work on an object? If not, state why not; if so, give an example.

11Q3. Roughly how much work is done by your arm muscles when you chin yourself?

11Q4. Roughly how much work is done by a fly's legs when it does a pushup?

11Q5. The Sun exerts a gravitational force directed toward itself. If a planet travels in a perfectly circular orbit around the Sun, does the Sun do work on the planet?

11Q6. A water-skier is pulled by a rope attached to a powerboat. The boat and skier are both moving at constant velocity. Is the rope doing positive or negative work on the powerboat? Since the speed of the skier is constant, does that mean the rope does no work on the skier?

11Q7. A magnetic field $\mathbf{B}$ exerts a force on a moving particle possessing an electric charge. The force is always perpendicular to both $\mathbf{B}$ and the instantaneous velocity of the particle. If the particle enters a magnetic field with speed v_0, what is its speed when it leaves the magnetic field?

11Q8. A building has two elevators, each carrying a passenger. One elevator is at rest and the other is moving upward at constant velocity. In the frame of reference of the building, is work done on either passenger by the corresponding elevator floor? If so, how can this be reconciled with the work-energy theorem?

11Q9. Work is done on an object by a force F. Does the amount of work done by F depend upon the reference frame from which the process is viewed? Use a gravitational force and a spring force as examples.

11Q10. Is $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$ for arbitrary vectors $\mathbf{A}$ and $\mathbf{B}$?

11Q11. Can $\mathbf{A} \cdot \mathbf{B}$ be negative? If so, under what conditions?

11Q12. If a is a number, is it true for any vectors $\mathbf{A}$ and $\mathbf{B}$ that $\mathbf{A} \cdot a\mathbf{B} = a(\mathbf{A} \cdot \mathbf{B})$?

11Q13. What can be concluded if the dot product of two vectors is $\mathbf{A} \cdot \mathbf{B} = 0$?

11Q14. A pendulum swings back and forth as shown. The bob (of mass m) is subject to the forces of gravity and string tension. Separate the motion into four parts: (i) vertical to far right (ii) far right back to vertical (iii) vertical to far left, (iv) far left back to vertical. (a) What is the power input from string tension for each of these parts? (b) What is the sign (plus or minus) of the power input from gravity for each of these parts? (Not counting the extreme endpoints of the motion, where the power input is zero.)

11Q15. If a constant net force acts on an object initially at rest, how does the power input depend on time?

11Q16. The NOVA laser at Lawrence Livermore National Laboratories could put out a power of 30 TW for a time interval 3 ns. How much energy (in Joules) was in such a pulse? (1 TW = terawatt = 10^{12} W, 1 ns = 1 nanosecond = 10^{-9} s)

Section 11.1 Work and the Work-Energy Theorem

11P1. A bolt of mass $m = 83$ g is dropped into a vertical mineshaft of depth $d = 1.2$ km. (a) How much work (in Joules) does gravity do on the bolt as it falls to the bottom? (b) If air resistance were negligible, what would be the speed of the bolt just before it strikes the bottom of the shaft?

11P2. A large block of marble can be dragged slowly along either of two horizontal paths by a team of horses, from the quarry to a building site a distance $d = 210$ m away. The block has mass $m = 820$ kg, and its coefficient of kinetic friction with the ground is $\mu_k = 0.75$. How much extra work ΔW must be done by the horses on the block if they pull it along a semi-circular arc, than if they pull it straight to its destination? (The figure is a view from above.)

11P3. Two people push a stalled car of mass M along a level road, each exerting a constant horizontal force F_0. If the car started from rest, (a) how long a time does it take the car to achieve speed v? (b) What total work was done by each person? Assume negligible friction between the car and road.

11P4. A rock of mass m is thrown with speed v_0 vertically upward from the surface of a large asteroid. The gravitational field near the surface is g_A. (a) What has been the work done on the rock by gravity by the time the rock comes to rest? (b) What is the total work done on the rock by gravity by the time it returns to the surface?

11P5. A United Federation Starship pulls a captured alien vessel of mass $M = 1.5 \cdot 10^8$ kg by means of a cable of fixed length. How much work was done by the cable (in the initial rest-frame of the Starship) if the vessel has moved a distance $d = 3.0 \cdot 10^3$ km in exactly one hour? Assume the cable is kept at constant tension.

11P6. Starting from rest relative to the ground, a sled of mass m is pulled along the level ground by a horizontal rope with tension T for a time interval t. Other horizontal forces are negligible. How much work has been done on the sled by the rope (a) in the rest-frame of the ground? (b) in the reference frame of a car moving at speed v_0 in the forward direction relative to the ground?

11P7. We push a box of mass m straight along a horizontal floor with constant force F_0 for a distance d. The velocity of the box is constant. How much work has been done by (a) us? (b) gravity? (c) the normal force? (d) friction with the floor?

11P8. An owner pulls a stubborn dog of mass $m = 16$ kg, originally sitting at rest, by a leash oriented at angle $\theta = 30^0$ to the horizontal, icy surface. The owner does work $W = 77$ J in pulling the dog a distance $d = 3.6$ m. (a) With what (constant) force did the owner pull the dog? (b) How fast was the dog moving at the end? (c) How long did it take the dog to reach this speed?

11P9. A golfball is thrown directly upward, away from the surface of the Earth, with initial velocity v_0. Find its speed v as a function of its altitude and in terms of g), using three different approaches, using (a) the uniform-acceleration Eq. 9(d) from *Day Five;* (b) conservation of energy; (c) the work done by gravity to find the change in the ball's kinetic energy.

11P10. A boat is pulled up a river by means of two cables attached to two railroad locomotives on the opposite shores, where the rails are parallel to the river as shown.

The distance between the locomotives is 200 m and the boat is 150 m behind the locomotives. The tension in each cable is kept constant at 5000 N. (a) If the boat moves at the steady speed 2.0 m/s upstream for 30 minutes, what total work is done on the boat by each locomotive? (b) Neglecting air resistance, how much work is done on the boat by the water during this time?

11P11. Starting at rest, a crate of mass m is slowly dragged up a steep path at angle θ relative to the horizontal, by means of a cable oriented parallel to the path, as shown. The cable tension T is kept constant. If the crate is dragged for a distance D and then brought to rest, (a) how much work has been done on it by gravity? (b) How much work has been done on it by the rope? (c) How much work has been done on it by friction with the ground? (d) What is the total work done on it by all forces?

Section 11.2 Dot products

11P12. Find the angle between the vectors $\mathbf{A} = \hat{\mathbf{x}} + 2\hat{\mathbf{y}}$ and $\mathbf{B} = 2\hat{\mathbf{x}} - \hat{\mathbf{y}}$.

11P13. Find the angles the vector $\mathbf{A} = \hat{\mathbf{x}} + 2\hat{\mathbf{y}} - 2\hat{\mathbf{z}}$ makes with respect to (a) the x-axis; (b) the y-axis; (c) the z-axis.

11P14. (a) Find the value of a such that $\mathbf{A} = 2\hat{\mathbf{x}} + \hat{\mathbf{y}} - a\hat{\mathbf{z}}$ is perpendicular to $\mathbf{B} = -a\hat{\mathbf{x}} - 3\hat{\mathbf{y}} + \hat{\mathbf{z}}$.

(b) For this value of a, what is the magnitude of each of the vectors?

11P15. Find (a) the magnitude of each of the two vectors $\mathbf{A} = \sqrt{3}\hat{\mathbf{x}} - \hat{\mathbf{y}}$ and $\mathbf{B} = \sqrt{3}\hat{\mathbf{x}} + \hat{\mathbf{y}}$ (b) the angle θ between them.

11P16. Find the dot product of vectors **A** and **B**, where
(a) $\mathbf{A} = 3\hat{\mathbf{x}} - 5\hat{\mathbf{y}} + \hat{\mathbf{z}}$ and $\mathbf{B} = -2\hat{\mathbf{x}} + 10\hat{\mathbf{y}} - \hat{\mathbf{z}}$.
(b) **A** has magnitude $A = 10$ and points at angle $\theta_1 = 30^\circ$ to the x-axis, while **B** has magnitude $B = 5$ and points at angle $\theta_2 = -95^0$ with respect to the x-axis, as shown.

11P17. Show that the vectors $\mathbf{A} = -\hat{\mathbf{x}} + 3\hat{\mathbf{y}} - 5\hat{\mathbf{z}}$, $\mathbf{B} = -2\hat{\mathbf{x}} - \hat{\mathbf{y}} + 4\hat{\mathbf{z}}$, and $\mathbf{C} = -3\hat{\mathbf{x}} + 2\hat{\mathbf{y}} - \hat{\mathbf{z}}$ form a right triangle.

11P18. Do the vectors $\mathbf{A} = -\hat{\mathbf{x}} + 3\hat{\mathbf{y}} - 5\hat{\mathbf{z}}$, $\mathbf{B} = -2\hat{\mathbf{x}} - 3\hat{\mathbf{y}} + 2\hat{\mathbf{z}}$, and $\mathbf{C} = \hat{\mathbf{x}} - 2\hat{\mathbf{y}} + 3\hat{\mathbf{z}}$ form a right triangle? Explain why or why not.

11P19. Prove that the diagonals of a rhombus are mutually perpendicular. (A rhombus is a parallelogram whose sides all have the same length.)

11P20. (a) Given vectors $\mathbf{A} = 2\hat{\mathbf{x}} + 2\hat{\mathbf{y}}$ and $\mathbf{B} = \hat{\mathbf{x}} - \sqrt{3}\,\hat{\mathbf{y}}$, find the angle between **A** and **B**. (b) Sketch the two vectors on a two-dimensional Cartesian system of coordinates. (c) Express the same vectors **A** and **B** in terms of unit vectors $\hat{\mathbf{x}}'$ and $\hat{\mathbf{y}}'$ in a coordinate system rotated by angle $\theta = 30^\circ$ in the counterclockwise sense, as shown. (d) Find the dot product $\mathbf{A} \cdot \mathbf{B}$, using the expressions found in part (c). Compare with $\mathbf{A} \cdot \mathbf{B}$ found using the original expressions for **A** and **B**.

11P21. Find the work done by a force $\mathbf{F} = (-2\hat{\mathbf{x}} + \hat{\mathbf{y}} - 3\hat{\mathbf{z}})$ N on a particle as it moves from the origin directly to the point $\mathbf{d} = (5\hat{\mathbf{x}} - \hat{\mathbf{y}})$ m

11P22. Prove that the dot product obeys the distributive rule

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

for arbitrary vectors **A**, **B**, **C**.

11P23. Starting from rest, a sled of mass m slides a distance d down a frictionless slope inclined at angle θ to the horizontal, as shown. The gravitational field g is directed vertically downward. (a) Let x and y be the horizontal and vertical directions. In terms of unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, find expressions for (i) the normal force **N**, (ii) the gravitational force $\mathbf{F}_{grav}$ acting on the sled, and (iii) the vector displacement **d**. (b) Evaluate the work $W_N = \mathbf{N} \cdot \mathbf{d}$ done on the sled by the normal force and the work $W_{grav} = \mathbf{F}_{grav} \cdot \mathbf{d}$ done by gravity. (c) Find the kinetic energy of the sled after it has moved the distance d. (d) What is the sled's final speed?

11P24. A force $F = Cx^2$ acts upon a particle at position x, where C is a constant. How much work is done by F on the particle as it moves from (a) $x = 0$ to $x = A$? (b) $x = A$ to $x = 0$?

11P25. A force $F = C\sin kx$ acts upon an object at position x, where C and k are constants. What is the work done by F as the object moves (a) from $x = 0$ to $x = \pi k^{-1}$? (b) from $x = \pi k^{-1}$ to $x = 2\pi k^{-1}$? (c) from $x = 2\pi k^{-1}$ to 0 ?

11P26. A force $F = Ce^{\alpha x}$ acts upon an object at position x, where C and α are constants. What is the work done by F as the object moves (a) from $x = 0$ to $x = 2\alpha^{-1}$? (b) from $x = 2\alpha^{-1}$ to $x = 4\alpha^{-1}$? (c) from $x = 4\alpha^{-1}$ to 0 ?

11P27. An unusual spring exerts the force $F(x) = -kx^5$ on a car of mass m to which it is attached, where x is the extension of the spring and k is a constant. What is the work done by the spring on the car if the car moves (a) from its equilibrium position $x = 0$ to $x = A$ (b) from $x = A$ to $x = 2A$? (c) directly from $x = 0$ to $x = 2A$?

11P28. The figure shows the force $F(x)$ acting on a particle. Estimate the work done by the force on the particle as the particle moves from $x = 0$ to $x = 3.0$ m.

11P29. A block of mass m is held at rest on a horizontal surface with which it has coefficient of kinetic friction μ_k. The block is in contact with a spring of force-constant k which is is compressed by Δx, as shown. The block is released and allowed to slide to the left. (a) What is the speed v of the block at the moment when the spring is no longer compressed? (b) The block then loses contact with the spring. How much farther does it slide before coming to rest?

11P30. A worker lifts a crate of mass m using the pulley arrangement shown. The pulleys have equal radius, and they and the rope have negligible mass. Both the upper end of the rope and a strut supporting the upper pulley are attached to the ceiling. (a) What force F must the worker exert to begin lifting the crate? (b) If the crate rises a distance h and then comes to rest, how much energy E has it gained? (c) In terms of h, how far downward has the worker pulled the rope? (d) How much work W has been done by the worker?

11P31. A damped oscillator consists of a mass m attached to a spring k, with frictional damping forces. (a) If A (the *amplitude*) is the initial displacement of the mass from equilibrium, and $A/2$ is the amplitude after 100 oscillations, what is the total work W done by friction during the 100 oscillations? (b) Suppose only one-quarter as much work is done by friction during the *next* 100 oscillations. What is A_f, the amplitude of oscillation at the end of this time?

11P32. A skier of mass m slides at constant speed straight down a slope of length D, tilted at angle θ to the horizontal, as shown. Air resistance is negligible. Find (a) the coefficient of kinetic friction μ_k between the skis and snow; (b) the work W_{grav} done by gravity on the skier; (c) the work W_{snow} done by the snow on the skier; (d) the total work W_T done on the skier.

11P33. A ski run on frictionless ice goes from the top of the lift at altitude $h_o = 3250$ m across a valley at altitude $h_v = 3070$ m up to a crest at altitude $h_c = 3180$ m, as shown. If a skier of mass $m = 75.0$ kg starts from rest, what would be her speed v (neglecting air resistance and assuming she uses no poles) (a) in the valley? (b) at the

crest? (c) On the next run, after a fresh snowfall, she only barely reaches the crest. What has been the total work done on her by the snow?

11P34. A skier descends a slope angled at $\theta = 30^o$ to the horizontal. Starting from rest, she slides with uniform acceleration for a distance $d = 52$ m in time $t = 11$ s. Gravity is $g = 9.8$ m/s^2. Neglect air resistance, and assume she does not use her poles. (a) Find the coefficient of kinetic friction μ_k between the snow and skis. (b) Find her change in energy ΔE (kinetic plus potential) from the beginning to the end of the trip. Compare ΔE with the work W done by friction during her descent. Take her mass to be $m = 65$ kg.

11P35. A block of mass m slides along a frictionless table at initial speed v_o, as shown. It then starts sliding along a surface with which it has coefficient of kinetic friction μ_k. After moving a distance d_o, it runs into a spring of force-constant k; friction with the surface continues. Find the total distance d the block moves while in contact with the frictional surface, by the time it has compressed the spring and instantaneously come to rest. Check the (rather complicated) answer in as many ways as you can.

11P36. A three-man bobsled team at the Winter Olympics pushes a bobsled to speed v_0 before jumping on at the beginning of a run. Descending a vertical distance h, they have speed v_f when they reach the bottom. (a) What fraction of their energy at the start of the run was lost to the snow and in heating the bobsled, in terms of v_0, v_f, g, and h? (Measure their potential energy from the bottom.) b) What is this fraction if $v_0 = 5$ m/s, $v_f = 31$ m/s, and $h = 67$ m?

11P37. A toboggan of mass m slides a distance D down a straight frictionless slope tilted at angle θ to the horizontal, starting at A. After reaching the bottom, the toboggan then starts up a slope tilted at angle ϕ to the horizontal. (a) How much work was done by gravity on the toboggan's way down? (b) What distance d up the second slope must the toboggan move so that the total work done by gravity, including the work done along each slope, is zero?

11P38. A roller-coaster car of mass $m = 1000$ kg moves without friction around a closed-loop track with the shape shown (a) If the work done by gravity on the car as it moves from a to b is $W_{ab} = -25{,}000$ J, and the work done by gravity as it moves from b to c is $W_{bc} = -9{,}500$ J, how much work W_{ca} is done on the car by gravity as it moves from c to a? (b) If at c the car has speed $v_c = 4.0$ m/s, what is its speed v_a at a?

11P39. A block of mass m is attached to a spring of force constant k. The block is pulled so the spring is stretched by the distance A, and then released from rest. (a) Find the work W_1 done by the spring on the block, as the spring returns to its equilibrium length. Is W_1 positive or negative? (b) What is the kinetic energy K of the block at this instant? (c) The block keeps moving, compressing the spring. How much work W_2 is done by the spring in bringing the block instantaneously to rest? Is W_2 positive or negative? (d) What is the distance x the spring has been compressed in bringing the block to rest?

11P40. In the absence of gravity, a weight is placed on one end of a spring; the other end of the spring is held in place. Gravity is then slowly turned on, gradually stretching the spring, and keeping the weight in equilibrium. When g has reached its final value, the spring is fully stretched and the weight is at rest. If W_1 is the work done by gravity, what is the work done by the spring? Is W_1 positive or negative?

11P41. A chain of mass M and length L, with half its length hanging over the edge of a table, is held in place as shown. The chain is surrounded by a frictionless plastic tube whose left end is glued to the tabletop, so the chain is free to slide inside. (a) What is the work W required to pull all of the chain up on the table? (b) If instead the chain is released, with what speed v will the last link be moving just as it leaves the tabletop?

Section 11.4 Power

11P42. The total power input P to a particle as a function of time is shown in the graph. If the particle is at rest at time $t = 0$, estimate its kinetic energy K at time $t =$ (a) 1 s (b) 3 s

11P43. An object of mass $m = 0.5$ kg moves in one dimension with the velocity shown in the graph. Find the total power input P to the object (in Watts) at time $t =$ (a) 3 s (b) 6 s (c) 9 s

11P44. A child pulls a sled of mass M along a horizontal snowy street, using the rope as as shown. There is a coefficient of sliding fricction μ between the sled and snow. (a) If the acceleration of the sled is a constant a_0, what is the constant horizontal force F_0 with which the child pulls the sled? If the sled starts at rest at time $t = 0$, as functions of time, (b) What is the net power input to the sled? (c) What is the power input to the sled due to the rope? (d) What is the power input to the sled due to friction?

11P45. A member of the "Δh" Club with mass m is climbing out of a crevass, touching nothing but the rope. Find the power inputs P to her, as functions of time, from (1) gravity (2) the rope (3) both together, if she is ascending (a) with constant speed v; (b) with constant upward acceleration $a = g/20$, assuming she is at rest at $t = 0$.

11P46. A motor pulls a Buick of mass M vertically upward against gravity by means of an attached cable of negligible mass. Find the power input $P(t)$ from the motor to the car, as a function of time, if (a) the car has constant upward speed v; (b) the car has constant upward acceleration a, with velocity $v = 0$ at time $t = 0$; (c) the car has constant *downward* acceleration a, with upward velocity $v = v_0$ at time $t = 0$.

11P47. A ball of mass m is hung on a spring of force-constant k in uniform gravity g. The ball is dropped from rest when the spring is unstretched. In terms of m, g, and k, (a) How far has the ball fallen when it reaches its maximum speed? At this instant, (b) what is the power input to the ball from gravity? (c) what is the power input to the ball from the spring? (d) What is the total power input?

11P48. On a calm day, a fit cyclist is riding at speed $v = 23$ miles/hr (10.3 m/s) on the flat Lake Ontario Parkway near Hamlin Beach State Park, New York. She is burning 660 Calories per hour, and has just eaten a 105-Calorie banana to keep herself going. At any time, nearly all of her power output is used to fight wind resistance. The magnitude of this force is $F = kAv^2$, where A is her frontal area and k is a constant. (a) What is her power consumption P in Watts? (b) How many miles will the banana last before she will need to pull more food from her jersey pocket? (c) Find the value of k in units kg/m^3, if her frontal area is 1.0 m^2. (d) If another bicyclist passes her moving at speed $v' = 25$ mph (11.2 m/s), and she catches up and partially rides in the slipstream by keeping her front wheel a couple of feet behind the leader, her effective frontal area drops to 0.60 square meters. Under this circumstance, how many miles does the banana last her? (1 Calorie = 1000 calories; 1 cal = 4.19 J)

11P49. Hoover Dam is 221 m high. If water in effect falls the entire 221 m to the bottom, and its kinetic energy is converted there into electrical energy with 100% efficiency, how many metric tonnes of water must fall per second to provide Hoover Dam's full electrical capacity of 1345 MW? (1 MW = 10^6 W, 1 tonne = 1000 kg).

11P50. A hailstone of mass m falls vertically with constant speed v in gravity g. Find the power input P to the hailstone from (a) gravity (b) air resistance. (c) What is the net power input? Express all in terms of m, g, and v.

11P51. A bobsled of total mass M starting from rest accelerates uniformly with $a = g/4$ down a straight slope tilted at angle $\theta = 30^\circ$ to the horizontal. Find the power input P_{grav} to the bobsled due to gravity as a function of time, and the power output P_{fric} due to friction and air resistance as a function of time.

11P52. A car has an engine which puts out $P = 95$ HP when the car is moving at its top speed, $v_{max} = 85$ mi/hr. Assuming that nearly all of this power is going into fighting air resistance, what force F is exerted by the air on the car at this speed?

11P53. West of Raton Pass in New Mexico, the Santa Fe tracks rise 175 feet per mile of track. Two 4400 horsepower engines pull an 1880 ton train up the slope at constant speed (1 ton = 2000 lbs.) (a) At full throttle, pretending there is no friction or air resistance, what is the speed v of the train in miles/hour? (b) What percent of the engine's power is used to overcome friction and air resistance, if the maximum speed of the train on the slope is $v_{max} = 21$ miles/hour?

11P54. Starting from rest, a roller-coaster car of mass m slides down a straight, frictionless slope inclined at angle θ to the horizontal. As functions of time, find (a) the speed v of the car; (b) its kinetic energy K, (c) the power input $P = Fv$ to the car due to gravity. (note that the gravitational force F used must be the component parallel to the slope.) (d) Verify directly that the time rate of change of the car's kinetic energy is equal to the power input.

11P55. Two constant forces $\mathbf{F}_1 = 30\ \hat{\mathbf{x}}$ N and $\mathbf{F}_2 = 50\ \hat{\mathbf{y}}$ N act upon a pig of mass $m = 20$ kg. At time $t = 0$, the pig is at rest. At time $t = 5$ s, find the power input to the pig from (a) $\mathbf{F}_1$ (b) $\mathbf{F}_2$ (c) What is the time rate of change of its kinetic energy at time $t = 5$ s?

11P56. A car starts from rest at point A, and then rolls without friction down into a depression and up over a rise, as shown. At each of the following points is the power input to the car positive, negative, or zero due to the force stated? (a) Point A, due to gravity (b) Point B, due to the normal force (c) Point B, due to gravity (d) Point C, due to gravity (e) Point D, due to the normal force.

11P57. Starting from rest, a bicyclist coasts down a slope tilted at angle θ to the horizontal, as shown. The total mass of the bicycle and rider is M. (a) Find the speed v of the bicycle as a function of the distance s it has moved along the slope, using Eq. 9d from *Day Five*. (b) Find the kinetic energy K as a function of s. (c) Differentiate this equation with respect to time, and verify that the time rate of change of K is equal to the power input due to the component of gravity parallel to the slope. Neglect friction and air resistance.

11P58. Every October, a time-trial bicycle race is held in which bicyclists race from the corner of Euclid Avenue and Foothill Blvd in Upland, California, to the parking lot at the bottom of the ski lifts on Mt. Baldy. This 16-mile route has an elevation gain of 5000 feet (1524 meters) and has been raced in a record time of 57 minutes. Assume the masses of the record-holding racer and his bicycle were 65 kg and 9 kg, respectively. (a) How much gravitational potential energy did the racer and his bicycle gain? (b) Assuming only 25% of the racer's efforts actually brought about an increase in potential energy, what was his average power output in horsepower? (1 hp = 780 W)

11P59. A batter hits a baseball of mass m so that it has the velocity v_0 directed at initial angle θ_0 to the horizontal. Pretending there is no air resistance, find the net power input P to the ball due to gravity as a function of the time t after the ball is struck (a) directly from the definition $P = \mathbf{F} \cdot \mathbf{v}$; (b) from the relationship $P = \frac{d}{dt}(\frac{1}{2}mv^2)$. Are the results consistent?

11P60. A skateboarder of mass m coasts back and forth within a large hemispherical bowl of radius R. His speed is zero at the top of the bowl. (a) Using energy conservation, find his speed as a function of the angle θ, as shown. (b) Find the power input to him from gravity, $P_{grav} = m\mathbf{g} \cdot \mathbf{v}$, as a function of θ. (c) Find the power input P_N to him from the normal force $\mathbf{N}$. (d) Using his speed from part (a), show directly that $\frac{d}{dt}(\frac{1}{2}mv^2) = \mathbf{F}_T \cdot \mathbf{v}$, where $\mathbf{F}_T$ is the total force acting on him.

Day Twelve

Conservative and Nonconservative Forces

"When you come to a fork in the road - take it!" – attributed to Yogi Berra

12.1 Energy conservation and the work-energy theorem

The energy of a block is conserved when it slides down frictionless slopes or when it moves on the end of a spring. In each case the sum of the kinetic and potential energies stays constant, so the total energy is conserved. Any change in the kinetic energy of the block is always accompanied by an opposite change in its potential energy to make up for it.

We have also derived the work-energy theorem

$$W_T = \frac{1}{2}m\,v_b^2 - \frac{1}{2}m\,v_a^2, \tag{1}$$

showing that any change in kinetic energy of a block is caused by the total work W_T done on it as it moves from some point a to another point b .

So which is right? Is the change in kinetic energy due to a change in potential energy or to the total work done? What is the relation between energy conservation and the work-energy theorem?

First, note that *if* the total work W_T done on the block is equal to the (negative) change in its potential energy $-(U_b - U_a)$, then we can rewrite Eq. 1 as a conservation of energy equation of the form $K_a + U_a = K_b + U_b$ = constant, and so *both explanations would be valid.* Drop a ball of mass m, for example. Its kinetic energy grows because of the work done by gravity. Neglecting air resistance, the total work done on the ball is

$$W_T = \mathbf{F}_{grav} \cdot \mathbf{s} = mgs = mg\,(y_a - y_b) \tag{2}$$

since the distance s the ball falls is its initial altitude y_a minus its later altitude y_b, as shown. So the work done by gravity really is the difference between the gravitational potential energies $U = mgy$ before and after.

Substituting Eq. 2 into the left-hand side of Eq. 1 and rearranging, we get

$$\frac{1}{2}m\,v_a^2 + mg\,y_a = \frac{1}{2}m\,v_b^2 + mgy_b \tag{3}$$

which is the conservation of energy equation $K + U_{grav} = E =$ constant we found during Day Ten! The work-energy theorem is the same as energy conservation in this case.

The energy diagram for uniform gravity is reproduced in Fig. 1, showing the constant-slope potential energy $U = mgy$ and the horizontal line E representing the constant total energy. In the case of uniform gravity with no other forces acting, the work-energy theorem is the same as energy conservation. Can we write a similar energy conservation equation for other forces as well?

Fig. 1 Energy diagram for uniform gravity

Try next the spring force $F_{spring} = -kx$ acting on a block of mass m, where $x = 0$ is the block's equilibrium position. If the block is pulled, stretching the spring from some x_a to x_b, the work done on the block by the spring is

$$W_{spring}[x_a \rightarrow x_b] = \int_{x_a}^{x_b} F\,dx = \int_{x_a}^{x_b} -kx\,dx = -\frac{1}{2}kx^2\Big|_{x_a}^{x_b} = -\frac{1}{2}kx_b^2 + \frac{1}{2}kx_a^2. \tag{4}$$

If there is no friction or any other force acting on the block, this is the total work done it, so the work-energy theorem (Eq. 1) can be written

$$\frac{1}{2}m\,v_a^2 + \frac{1}{2}kx_a^2 = \frac{1}{2}mv_b^2 + \frac{1}{2}kx_b^2 \tag{5}$$

which again represents the conservation of energy for a spring-mass system, with spring potential energy $U(x) = \frac{1}{2}kx^2$. The work-energy theorem is equivalent to the conservation of energy for springs as well as for uniform gravity. The corresponding energy diagram for the spring mass system is reproduced in Fig. 2.

Fig. 2 Energy diagram for the spring-mass system

Sample Problem 1. (a) Show that the work-energy theorem is equivalent to energy conservation for a ball of mass m hanging in uniform gravity g on a spring of force-constant k, that is, for a ball subject to both gravity and the spring force, if the total potential energy is $U = U_{grav} + U_{spring}$, the sum of the gravitational and spring potential energies.

(b) Construct the energy diagram for the system.

(c) If the ball is released from rest with an unstretched spring, how far does it fall before it starts to rise again?

Solution: (a) Let x be the stretch in the spring, positive downward, and let the altitude of the ball be $y = 0$ when $x = 0$, corresponding to the unstretched spring as shown. Then as the spring is stretched a distance x, the altitude of the ball decreases to $y = -x$. The work done by gravity as the ball moves from an arbitrary point x_a to another arbitrary point x_b is $W_{grav} = mg(x_b - x_a) = mg(-y_b + y_a)$. According to equation 4, the work done by the spring between the same two points is $W_{spring} = -\frac{1}{2}kx_b^2 + \frac{1}{2}kx_a^2$, so the work-energy theorem becomes

$$W_T = W_{grav} + W_{spring} = mg(y_a - y_b) - \frac{1}{2}kx_b^2 + \frac{1}{2}kx_a^2 = \frac{1}{2}m\,v_b^2 - \frac{1}{2}m\,v_a^2.$$

Rearranging terms gives $(K + U_{grav} + U_{spring})_a = (K + U_{grav} + U_{spring})_b$, where $U_{grav} = mgy$ and $U_{spring} = \frac{1}{2}kx^2$. In other words, energy is conserved if the potential energy is taken to be the sum of the gravitational and spring potential energies. The sum of all the energies can be set equal to E, the total energy of the system.

(b) The total potential energy can be written in the form

$$U_{total} = mgy + \frac{1}{2}kx^2 = -mgx + \frac{1}{2}kx^2$$
$$= \frac{1}{2}k\left(x - x_o\right)^2 + U_o$$

where $x_o = mg/k$ and $U_o = -(m^2g^2/2k)$ are constants. Therefore U_{total} is a parabola (just as U_{spring} is a parabola all by itself), but the parabola has now been shifted to the right by the distance x_o and its minimum has become U_o. The ball acts as though it were in a spring potential alone, except its equilibrium position has been shifted by the distance $x_o = mg/k$.

(c) If the ball is released from rest at a location corresponding to the left turning point in the diagram, it falls to lower elevations (i.e., to the right in the diagram above) until it reaches the right turning point, a distance $2x_o = 2mg/k$ from where it started.

12.2 Conservative forces

We have found that the work-energy theorem is equivalent to conservation of energy for both uniform gravity and the spring force. We will call these forces *conservative forces*, because a potential energy function U exists for each and the energy $E = K + U$ is conserved for each. Are *all* forces conservative? Can we use the theorem to find a potential energy function for *any* force?

As for gravity and springs, we have to try to write the work $W[a \to b] = \int_a^b \mathbf{F}\cdot d\mathbf{s}$

as the difference between potential energies at the endpoints of the motion a and b. We can do that if the work integral depends *only* on the endpoint locations of a and b.

What *else* might the integral depend upon? It might depend upon

(i) the velocity v of the particle

(ii) the time t when the particle moves

(iii) the path the particle takes between the endpoints.

It is easy to eliminate possibilities (i) and (ii). We will simply restrict ourselves to forces that depend only on position, $\mathbf{F} = \mathbf{F}(x, y, z)$, and not on velocity or time! Then the work integral *can't* depend on v or t, because there is nothing in the integrand that depends on either of them.

Even if $\mathbf{F}$ depends only on position, the work $\int_a^b \mathbf{F} \cdot \mathbf{ds}$ might still depend on the particular path taken. Generally there are an infinite number of ways to travel from a to b. Three of these are illustrated in Fig. 3.

Fig. 3. Three sample paths between a and b

When is the work done independent of the path taken? It is easy to see that for any one-dimensional motion, with the net force in the same direction as the motion, the work integral $W[a \to b] = \int_a^b F \cdot ds$ depends only on the endpoints and not on the path. Even in one dimension there can be many possible paths: The particle might overshoot the final endpoint b all the way to c, for example, and then return to b. The integral from b to c is equal but opposite to the integral from c to b, so the work done by the extra path cancels out by the time the particle finally arrives back at b. For two or three-dimensional motion, the integral is not necessarily path-independent, but it can be. The very important path-independent example of spherically symmetric gravity is worked out in Section 12.4.

If the work done by some force $\mathbf{F}$ is path-independent, then it is also true that the round-trip work done by $\mathbf{F}$ is *zero*. This interesting result can be shown as follows. Given that the work done by $\mathbf{F}$ along path ① is the same as the work done along any other path ②,

$$W_{①}[a \to b] = W_{②}[a \to b].$$

If the particle travels *backward* along path ② from b to a, the work done is

$$W_{②}[b \to a] = -W_{②}[a \to b]$$

since the work along any path is given by the integral $W_{②}[a \to b] = \int_b^a \mathbf{F}\cdot d\mathbf{s}$, and exchanging the limits of integration changes the sign of the integral. Therefore

$$W_{①}[a \to b] + W_{②}[b \to a] = 0. \qquad (7)$$

If the work integral is path-independent, the total work done around a closed path is zero.

If the only forces acting on a particle are conservative, then the total work done depends only on the endpoints a and b,

$$W[a \to b] = -(U_b - U_a). \qquad (8)$$

Note that $W = 0$ if a and b are the same point, i.e. if the particle takes a round trip back to where it started. The work-energy theorem then becomes

$$W[a \to b] = -(U_b - U_a) = \frac{1}{2}m\,v_b^2 - \frac{1}{2}m\,v_a^2, \qquad (9)$$

equivalent to energy conservation $K_a + U_a = K_b + U_b$.

The potential energy can be calculated directly from the force by evaluating the indefinite integral

$$\boxed{U(x) = -\int^{x} F(x)\,dx,} \tag{10}$$

which has the opposite sign of the work integral because of the first minus sign in Eq. 8. Any constant added to the potential energy will cancel out in Eq. 9, so we can add any constant of integration we want. Once we choose a constant, however, we have to keep the same constant throughout a calculation.

Conversely, the force is the negative derivative of the potential energy,

$$\boxed{F(x) = -\frac{dU(x)}{dx},} \tag{11}$$

just as was true for gravity and for the spring force. *The force $F(x)$ is the negative of the slope of the potential energy function $U(x)$.*

An energy diagram for the one-dimensional motion of a particle might look as shown in Fig. 4a. The curve is the potential energy $U(x)$, and a straight, horizontal line

Fig. 4. (a) An energy diagram for a particle with the potential energy $U(x)$ shown. Note that the vertical axis is energy, not altitude; the particle is moving in one dimension (x) only. (b) The force $F(x)$ is the slope of $U(x)$.

represents a particular constant energy E. The turning points are located at the intersections of E with $U(x)$. If the particle has energy E_1 (the larger energy shown) the only turning point is located at x_1. So if the particle moves initially toward the left it comes to rest at x_1 and then starts moving back toward the right. There is then no further turning point, so the particle moves off toward the right indefinitely. The particle moves fast where the kinetic energy is large (i.e., where the potential energy is small, as in the valleys of the potential), and the particle moves slow where the potential energy is large. The force $F(x)$ is positive (pushing the particle toward the right) where the slope is negative, and $F(x)$ is negative (pushing the particle toward the left) where the slope is positive, as illustrated in Fig. 4b.

If the particle has energy E_2 (the smaller energy shown in Fig. 4(a), there are *three* turning points. If the particle starts far to the right and moves toward the left, it approaches the right-hand turning point at x_4, stops and turns around, moving off toward the right indefinitely. If the particle starts somewhere between the turning points at x_2 and x_3, it moves back and forth between these points forever, trapped in a potential energy "well". If it has energy E_2 the particle is never found between points x_3 and x_4, because in this region $U(x) > E$. That region is forbidden, because it implies that the kinetic energy K would have to be negative, which is not possible for $K = \frac{1}{2}mv^2$, since the mass is positive and the velocity is real.

Sample Problem 2. An atom in a diatomic molecule like O_2 feels a force something like that shown. If the atom is far from the other atom it feels a weak attractive force called the *Van der Waals* force. We assign such an attractive force a *minus* sign, the same as the attractive force $F = -kx$ of a stretched spring. As the atom approaches the other atom the force first becomes more and more attractive, but if the atoms get too close together the force becomes repulsive. How do we find the potential energy of the atom, given the force $F(x)$?

We can model the force acting on the atom by the function $F(x) = \frac{a}{x^3} - \frac{b}{x^2}$, where a and b are positive constants. (a) Sketch $F(x)$. (b) Find the corresponding potential energy function $U(x)$, and sketch it. (c) For what value(s) of x is the atom in equilibrium?

Solution: (a) For small x the first term in $F(x)$ dominates the second term (inverse cubed beats inverse square for small x). Then $F(x)$ is repulsive (i.e., positive), going to ∞ as $x \to 0$. For sufficiently large finite x, the second term in $F(x)$ is larger than the first (inverse square beats inverse cubed for large x). Then $F(x)$ is attractive (i.e., negative). When $x = a/b$, the force $F(x) = 0$; when $x = 3a/2b$, the slope $F'(x) = \dfrac{-3a}{x^4} + \dfrac{2b}{x^3} = 0$. It follows that $F(x)$ looks as shown.

(b) The corresponding potential energy is

$$U(x) = -\int^x F(x)\,dx = -\int^x \left(\frac{a}{x^3} - \frac{b}{x^2}\right) dx = -a\int^x \frac{dx}{x^3} + b\int^x \frac{dx}{x^2}$$

$$= \frac{a}{2x^2} - \frac{b}{x}.$$

The potential energy vanishes at infinity, is negative for large finite x, and positive for very small x. The potential energy $U(x)$ is sketched at the right.

(c) The atom is in equilibrium as $x \to \infty$, where the molecule has been dissociated, and also at the potential energy minimum. In both cases the force on the atom is $F(x) = -\dfrac{dU(x)}{dx} = 0$. In the diatomic molecule the atom sits at the potential energy minimum, i.e. at $x = a/b$, in its lowest energy state.*

* This is according to Newtonian mechanics. In reality it is necessary to use *quantum mechanics* to understand fully the motion of an atom in a molecule. According to quantum mechanics, in its lowest energy state the atom will almost always be found near (but not necessarily at) the position of the potential energy minimum.

12.3 Friction: conservative or nonconservative?

What about *friction*? Is friction a conservative force?

In Section 12.2 we showed that for one-dimensional motion, *any* force that depends only on position, and not on velocity or time, is a conservative force. That is because for one-dimensional motion the work done depends only on the endpoints of the path. Examples include gravity and spring forces.

We can test frictional forces in the same way. The force of sliding friction on a block moving along a horizontal table is $F_f = \mu_k N = \mu_k mg$, with the force always opposite to the direction of motion.

Assuming that the coefficient of kinetic friction μ_k is constant along the table, the quantity $\mu_k mg$ is also constant, so the frictional force is constant, just like gravity. So sliding friction would seem to be conservative, just like gravity! There must be a corresponding potential energy for friction, and the energy of the block can go back and forth between kinetic energy and frictional potential energy.

That doesn't make sense, however! If we shove a block on the table it starts with a lot of kinetic energy, and then slows down and stops. It doesn't start moving again - it never regains its kinetic energy. It is different when we throw a ball upward. The ball comes to rest at some maximum altitude, and then starts falling back down and regains its kinetic energy.

What's wrong with the argument that friction should have a potential energy, since $F_f = \mu_k mg$ seems to be a constant force?

What's wrong is that friction really *isn't a* constant force. It depends on velocity in a subtle way: It depends on the velocity's *direction*. If the block moves to the right, the friction is $F_f = -\mu_k mg$, *minus* meaning to the left. And if the block moves to the left, the friction is $F_f = +\mu_k mg$, *plus* meaning to the right. A force that changes sign isn't constant.

Energy is conserved for one-dimensional motion if the force depends only on the position x of a particle, and not (say) on the particle's velocity or the time. Then we can find a potential energy $U(x)$ that is a unique function of x, so that every time the particle returns to some particular point it will have the same potential energy it had before.

For a nonconservative force like surface friction, the work done along the return path is not equal but opposite to the original, outgoing path, so the round-trip work is not zero. If a block is moved along a level surface the work done by surface friction is negative

on the initial leg, since the frictional force is opposite to the displacement. As the block is moved back to its initial position, the work done by friction is still negative, because again the frictional force is again opposite to the displacement. So the round-trip work is not zero, confirming that surface friction is not a conservative force.

Sample Problem 3. A block of mass m is pulled along the floor from point A to point B a distance d away, and then back to A along the two sides of a square as shown. Find the round-trip work done by friction if the coefficient of kinetic friction is μ_k.

Solution: The frictional force is $F_f = \mu_k N = \mu_k mg$, always opposite to the direction of motion. The frictional work done along the diagonal is therefore $-F_f d = -\mu_k mgd$. On the return trip the total distance moved is $\sqrt{2}\,d$, so the work done is $-\sqrt{2}\,\mu_k mgd$. The round-trip frictional work is the sum of these, $-\left(1+\sqrt{2}\right)\mu_k mgd$. Friction is clearly not a conservative force, both because the work depends on the path and because the round-trip work is nonzero.

12.4 Spherically symmetric gravity

The gravitational force on a planet, asteroid, or comet of mass m a distance r from the center of the Sun is given by the inverse square law $\mathbf{F} = -\dfrac{GMm}{r^2}\hat{\mathbf{r}}$, where M is the mass of the Sun, G is Newton's gravitational constant, and $\hat{\mathbf{r}}$ is a unit vector pointing away from the center of the Sun. The minus sign means that $\mathbf{F}$ is attractive, acting toward the center. Is the inverse-square gravitational force conservative? And if so, what is the corresponding potential energy function $U(r)$?

Suppose a comet moves from a to b in its orbit, as shown. The distances of these points from the center of the Sun are r_a and r_b, respectively. If the comet moves along path ①, work is done by gravity only on the first (radial) leg of the trip, since gravity is perpendicular to the second (tangential) leg. The total work done by gravity along this path is

$$W = \int \mathbf{F}\cdot d\mathbf{s} = \int_{r_a}^{r_b} -\frac{GMm}{r^2}\,\hat{\mathbf{r}}\cdot d\mathbf{r} = -GMm\int_{r_a}^{r_b}\frac{dr}{r^2} = \left.\frac{GMm}{r}\right|_{r_a}^{r_b}.$$

$$= -\frac{GMm}{r_a} + \frac{GMm}{r_b}. \qquad (12)$$

If the comet moves instead along path ②, we can approximate that path arbitrarily closely by using small radial and tangential steps, as shown. Work is done by gravity only on the radial portions of the path, since gravity is perpendicular to the tangential portions. The sum of the radial portions is the same as it is by path ①. The total work done is therefore the same for *all* paths between a and b, so the inverse-square gravitational force is conservative.

What is the corresponding potential energy $U(r)$? According to the work-energy theorem $\Delta K = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 = W_T[a \to b]$, the total work done on a particle is equal to its change in kinetic energy. Therefore if spherically symmetric gravity is the only force acting on the comet, it follows that

$$\frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 = W_T[a \to b] = -\frac{GMm}{r_a} + \frac{GMm}{r_b} \qquad (13)$$

using the results of Eq. 12. This shows that the comet's energy is conserved, i.e. $E = K + U$ = constant, if we take the potential energy of the comet to be

$$\boxed{U(r) = -\frac{GMm}{r}} \qquad (14)$$

as shown in the sketch. Here m is the mass of the comet, M is the mass of the Sun, and r is the distance between their centers. The potential energy function is everywhere negative, approaching zero as $r \to \infty$; it is valid only *outside* the Sun, because the inverse-square force law is only valid outside.

How is inverse-square gravity and its corresponding potential energy related to uniform gravity $\mathbf{F} = m\mathbf{g}$ (with $\mathbf{g}$ = constant) and its potential energy $U = mgh$? Near the surface of a planet or star we can let $r = R + h$, where R is the radius of the body and h is small ($h << R$). The gravitational force is then

$$\mathbf{F} = -\frac{GMm}{(R+h)^2}\hat{\mathbf{r}} \cong -\frac{GMm}{R^2}\hat{\mathbf{r}} = m\mathbf{g},$$

where the gravitational field at the surface is $\mathbf{g} = -\frac{GM}{r^2}\hat{\mathbf{r}}$. If we use the Earth's mass and radius, the magnitude of $\mathbf{g}$ is $g = GM/R^2 = 9.8$ m/s^2. Near the surface, $U(r)$ is

$$U = -\frac{GMm}{R+h} = -\frac{GMm}{R}\left(1+\frac{h}{R}\right)^{-1} \cong -\frac{GMm}{R}\left(1-\frac{h}{R}\right) = -\frac{GMm}{R} + mgh \qquad (15)$$

where in the next-to-last step we used the binomial approximation $(1+x)^n \cong 1 + nx$ for $|x| << 1$, and in the final step we used $g = \frac{GM}{R^2}$. The potential energy U is always

arbitrary within an additive constant, so we can just throw out the constant term $-\frac{GMm}{R}$. This is because such an additive constant always cancels out when we write conservation of energy, $K + U = K_0 + U_0$, and also because the force $F(r) = -\frac{dU(r)}{dr}$ is unchanged when a constant is added to the potential energy. Throwing out the constant term in this case means that the zero of potential energy is at the surface $r = R$ instead of at $r = \infty$.

Sample Problem 4: *Escape Velocity*

(a) Find a minimum velocity, called the *escape velocity* v_{esc}, required to shoot a particle from the surface of the spherically symmetric body so it never falls back. Find v_{esc} in terms of the mass M and radius R of the body.

(b) Find v_{esc} from Earth and the Sun.

Solution: (a) A particle at rest at the surface $r = R$ has energy

$$E_0 = K_0 + U_0 = 0 - \frac{GMm}{R} = -\frac{GMm}{R}.$$

The particle has to raise its energy to $E = 0$ to escape to arbitrarily large r; for any smaller energy, the particle eventually runs into the potential energy barrier for some finite value of r, and then falls back to the surface. If $E \geq 0$, the particle never hits the potential energy barrier, so can keep on increasing its distance from the surface forever. Conserving energy for a particle with the escape energy,

$$E = 0 = \frac{1}{2}mv_{esc}^2 - \frac{GMm}{R}.$$

The escape velocity from a spherical body of mass M and radius R is therefore

$$v_{esc} = \sqrt{\frac{2GM}{R}}.$$

(b) The escape velocity from Earth, with mass $M = 6.0 \cdot 10^{24}$ kg and radius $R = 6400$ km, is

$$v_{esc,\ Earth} = \sqrt{\frac{2GM_{Earth}}{R_{Earth}}} = \sqrt{\frac{2 \times 6.67 \cdot 10^{-11}\ \mathrm{m^3/kg \cdot s^2} \times 6.0 \cdot 10^{24}\,\mathrm{kg}}{6.4 \cdot 10^6\,\mathrm{m}}} = 11.2\ \mathrm{km/s}.$$

The Sun has mass $M = 2.0 \cdot 10^{30}$ kg and radius $R = 7.0 \cdot 10^8$ m, so

$$v_{esc,\ Sun} = \sqrt{\frac{2GM_{Sun}}{R_{Sun}}} = 620\ \mathrm{km/s}.$$

Sample Problem 5. *Gravitational Assists*

Sending a spacecraft to another planet requires large amounts of fuel. In traveling out to Neptune, for example, a spacecraft needs to overcome the attractive gravitational force of the Sun. A craft of mass m has to raise its gravitational potential energy due to the Sun from $-\frac{GM_{Sun}m}{d_{Earth}}$ to $-\frac{GM_{Sun}m}{d_{Neptune}}$ as shown, where d_{Earth} and $d_{Neptune}$ are the distances of Earth and Neptune from the Sun.*

There is a way to reach the outer planets which requires much less fuel than a direct flight. The method was exploited in NASA's Voyager Program and in other spaceflights. Planets between

*The craft actually gets some *help* from the fact that in the Sun's frame its kinetic energy is higher near the Earth than it is near Neptune, because Neptune's orbital velocity around the sun is smaller than Earth's. So the decrease in the craft's kinetic energy can be used to overcome gravitational potential energy.

Earth and Neptune, assuming they are positioned just right, can be used to give the spacecraft a boost as it passes by. A spacecraft passing Jupiter, for example, can get a *gravitational assist* from Jupiter as it leaves for Saturn; Saturn can give it another gravitational assist as it leaves for Uranus, and so on.

How does this work? Suppose the spacecraft swings toward Jupiter, increasing its kinetic energy as it falls. It reaches a point of closest approach to the planet, and then starts to rise again. As it leaves Jupiter, the spacecraft loses all the kinetic energy it gained as it climbs back out of Jupiter's potential energy well. When it is far from Jupiter, the spacecraft's kinetic energy, and therefore its speed as well, is the same when it departs from Jupiter as it was when it approached, as seen in Jupiter's reference frame. This follows from the conservation of energy.

The Sun's reference frame is also important, however, because it is the craft's potential energy due to the *Sun* that must be increased to reach Saturn, the next destination. Suppose in particular that Jupiter is moving to the left in the Sun's frame (Picture the Sun as stationary way below the bottom of the figure on the right.) Then if the craft comes up behind Jupiter and swings around it, even though its speed before and after is the same relative to *Jupiter,* its speed afterward can be quite a bit greater in the *Sun's* frame than it was before.

To see how much velocity is gained by the Jupiter flyby, we can use vector addition of velocities, as described during *Day Two* in the *Bubble*. Because Jupiter moves with velocity $\mathbf{v}_J$ to the left relative to the Sun, all we have to do is add the vector $\mathbf{v}_J$ to the before and after velocities of the spacecraft in Jupiter's frame, to find the before and after velocities of the spacecraft in the Sun's frame.

Problem: (a) A spacecraft moves with speed v_O before and after in Jupiter's frame, and turns by 90° as it passes by, as shown. Jupiter's speed relative to the Sun is also $v_J = v_O$. Find the initial and final spacecraft velocities in the Sun's frame, and the angle through which the spacecraft orbit turns in the Sun's frame. (b) By what factor has the spacecraft's kinetic energy increased in the Sun's frame?

Solution: (a) By vector addition, the speed of the spacecraft in the Sun's frame before reaching the vicinity of Jupiter is $\sqrt{2}\,v_O$, and the speed afterward is $2v_O$, as shown. The spacecraft has turned through an angle of only 45° in the Sun's reference frame.

(b) The kinetic energy has increased from

$$\frac{1}{2}m(\sqrt{2}v_O)^2 = mv_O^2 \qquad \text{to} \qquad \frac{1}{2}m(2v_O)^2 = 2mv_O^2 .$$

in the Sun's frame; it has *doubled*. This extra energy can go a long way toward overcoming the Sun's gravitational potential energy as the spacecraft heads out toward Saturn.

Where does the added energy of the spacecraft come from? Note from the Sun's-frame picture above that on average the spacecraft's gravity pulls on Jupiter in a direction that is "up" and to the "right". The "upward" component can deflect Jupiter's velocity vector slightly, without changing its magnitude. The "rightward" component can slow down Jupiter, stealing some of its kinetic energy. That is where the spacecraft's energy *would* come from, if it were not for the fact that the Sun is an important player in all of this. What actually happens is that the spacecraft *does* remove an infinitesimal part of Jupiter's energy, which makes Jupiter "fall"slightly toward the Sun, decreasing its orbital radius. But this actually makes Jupiter move *faster*! The net effect is that Jupiter gains kinetic energy but loses even more gravitational potential energy, and the energy left over is transferred to the spacecraft.

State of the Search

1. The total work done on an object by all forces changes the kinetic energy of the object according to the work-energy theorem

$$\Delta K = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 = W_T[a \to b]$$

where the work done by any particular force $\mathbf{F}$ is $W[a \to b] = \int_a^b \mathbf{F} \cdot d\mathbf{s}$. The work integral may depend upon the velocity of the object and the time (if the force depends on those quantities.) Even if the force depends only upon the position of the object, the work done by $\mathbf{F}$ may depend upon the *path* taken between a and b.

2. If the work done by a particular force $\mathbf{F}$ is independent of the path between given endpoints, then the force F is *conservative*, and a corresponding potential energy

$$U(\mathbf{r}) = -\int^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{s}$$

exists, so that the energy $E = K + U(\mathbf{r})$ is conserved if $\mathbf{F}$ is the only force acting. The integral is an indefinite integral. For one-dimensional motion the potential energy is $U(x) = \int^x F\,dx$ where any convenient constant of integration can be added. Uniform gravity and the spring force are examples of conservative forces; friction is a *non*conservative force, since the work done by friction depends upon the path.

3. Spherically symmetric gravity is an example of a three-dimensional conservative force. The gravitational force on a spherically-symmetric body of mass m due to another spherically-symmetric body of mass M is $\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$, where r is the distance between their centers, G is Newton's gravitational constant, and $\hat{\mathbf{r}}$ is an outward unit vector, pointing away from M. The corresponding gravitational potential energy is $U(r) = -\frac{GMm}{r}$.

4. The escape velocity is the minimum velocity that a particle must have at the surface of a spherically-symmetric planet or star, so that in coasting outward the particle will escape to infinity. In terms of the planet's (or star's) mass M and radius R, the escape velocity is $v_{esc} = \sqrt{\frac{2GM}{R}}$.

PICTURE QUIZ 12

1. A particle moves with the potential energy $U(x)$ and energy E shown below. Answer each of the following questions, giving the appropriate letters A to J. At what point(s)

(a) does the particle experience a force in the $+x$ direction?

(b) is the particle's speed greatest?

(c) is the force on the particle zero?

(d) does the particle accelerate *toward* $x = 0$?

(e) is the particle's speed zero?

2. An atom experiences the potential energy $U(r)$ shown. At time $t = 0$ the atom is at some large value of r. (a) What is the minimum energy the atom must have to reach $r = 0$? (b) What is the minimum energy the atom must have to reach arbitrarily large values of r? (c) At which point(s) is there no force on the atom?

(d) For given energy, at which point is the kinetic energy of the atom a maximum?

3. The approximate trajectory of a spacecraft of mass m around a particular asteroid is shown, drawn in the rest-frame of the asteroid. The trajectory is nearly straight as it approaches with initial speed v_0, and nearly straight as it leaves as well. Which of the following pictures most closely approximates the trajectory and velocities in the Sun's frame, if the asteroid is moving with speed $v_0/2$ to the right relative to the Sun?

(A)

(B)

(C)

(D)

PROBLEM QUIZ 12

1. A spring is hung from the ceiling and a weight is attached as shown, held in place by hand so the spring is unstretched. If the weight is then slowly lowered, the spring stretches a distance d when the weight is in equilibrium. If instead the weight is suddenly released from rest, how far will it fall, in terms of d, when it (a) achieves its greatest speed? (b) is instantaneously brought to rest by the stretched spring?

2. Which of the following forces are conservative? The quantities x, y, z are Cartesian coordinates, r, θ are plane polar coordinates. Unit vectors for each of these coordinates are $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, etc. Also v_x, v_y, v_z are velocity components and C is a constant.

(A) $\mathbf{F} = Cx\,\hat{\mathbf{x}}$ (B) $\mathbf{F} = -Cr\,\hat{\mathbf{r}}$ (C) $\mathbf{F} = -C\,\hat{\theta}$

(D) $\mathbf{F} = -Cv_x\,\hat{\mathbf{y}}$ (E) $\mathbf{F} = -Cy\,\hat{\mathbf{x}}$

3. A skier of mass m slides at constant speed v down a straight slope of length L oriented at angle $\theta = 30^\circ$ to the horizontal. (a) What was the total work done on the skier? What was the work done by (b) gravity (c) the normal force (d) friction?

4. A spherical planet has mass M and radius R. A spaceship hovering at rest a distance $R/2$ above the planet's surface finds that its power suddenly fails. With what speed will it crash into the planet?

5. A typical white dwarf star has the mass of our Sun ($2 \cdot 10^{30}$ kg), and a radius 10^4 km, comparable to the radius of Earth. (a) Calculate the average density of the star, in kg/m^3. (b) How many metric tonnes (1 tonne = 1000 kg) are in one cubic centimeter of the star pretending that the density is uniform? (c) Find the gravitational field g at the surface of the star. (d) Find the escape velocity from the star in m/s, and as a multiple of the escape velocity from Earth.

Special Topic: White Dwarfs, Neutron Stars, and Black Holes

If it didn't stay blazingly hot, the Sun would collapse. Without the enormous heat energy it needs to resist gravity, it would quickly fall in upon itself. The big ball of hot gas radiates energy continuously, however, so the heat energy must replenish itself to keep the Sun at a stable size.

The needed heat energy is generated in the core of the Sun by nuclear fusion reactions. Most stars, including the Sun, convert hydrogen nuclei into helium nuclei, releasing kinetic energy in the process. They remain in equilibrium for billions of years, gradually "burning" their hydrogen into helium. As the hydrogen begins to get used up, the core of such a star starts to contract and gets even hotter with time, while the outer layers expand and cool off, visible from outside as a so-called *red giant* star. When enough energy has been radiated away, it settles down to form a very compact star called a *white dwarf.* A typical white dwarf has about the mass of the Sun, but a radius that is only about that of Earth. The mass of the Sun is $3.3 \cdot 10^5$ times as large as the mass of Earth, so the "surface gravity" $g = \frac{GM}{R^2}$ of a typical white dwarf is about $3.3 \cdot 10^6$ m/s^2, over 300,000 times larger than Earth's gravity. Our Sun will become a white dwarf in four or five billiion years.

White dwarfs gradually cool off, getting dimmer and dimmer; they are supported against gravitational collapse not by nuclear burning, but by what is called electron degeneracy pressure, in which electrons resist being squeezed too close together, an effect related to the Pauli exclusion principle in atoms.

Stars that are much heavier than the Sun have even hotter cores, and burn their hydrogen more quickly. As hydrogen becomes depleted the core of such a star is compressed a bit, getting still hotter, able now to burn its helium into carbon, and then to burn successively heavier and heavier nuclei. This process ultimately comes to an end; energy generation in the core ceases, so the giant star is no longer supported. The enormous gravity causes a sudden, catastrophic, free-fall collapse. The entire star, including previously unburned material, collapses in less than one minute. It reheats during this gravitational collapse to such high temperatures that it undergoes a thermonuclear explosion, becoming a "Type II Supernova", a star so bright that for several days it can outshine a galaxy of ordinary stars. Most of the star is driven outwards, expanding at high speed. The core continues to collapse, however; it can form either a *neutron star* or *black hole*, depending upon its mass.

A neutron star is composed primarily of neutrons, although it contains protons, electrons, and other particles as well. A typical neutron star has one or two solar masses (One solar mass $M = 2 \cdot 10^{30}$ kg) but a radius of only about 10 km. A typical surface gravity is therefore

$$g = \frac{GM}{R^2} = 3.3 \cdot 10^6 \text{ m/s}^2 \left(\frac{R_{\text{Earth}}}{10 \text{ km}}\right)^2 = 1.4 \cdot 10^{12} \text{ m/s}^2$$

(using $R_{\text{Earth}} = 6400$ km), nearly a million times the gravity on a white dwarf.

Rotating neutron stars are sometimes observed to pulse on a regular basis; they behave like giant lighthouses, in which a rotating searchlight beam originating on the star may strike the earth once every revolution. These stars are called pulsars. The Crab Nebula is a reddish cloud of ionized gas left over from a supernova explosion observed on earth in 1054 A.D. Within the nebula is a pulsar which rotates 30 times per second, so we receive one of its pulses every .033 s.

If the supernova core is too heavy, larger than two or three solar masses, it can't form a stable neutron star, and so it collapses still further to form a black hole. The core collapses to a very small size, and is surrounded by a so-called "event horizon" from which nothing can escape, not even light (hence the name "black hole".) The radius of the event horizon of a nonspinning black hole is $R = \frac{2GM}{c^2}$, where M is the black-hole mass and c is the speed of light. Gravity at the event horizon is therefore *

$$g = \frac{GM}{R^2} = \frac{GM}{\left(2GM/c^2\right)^2} = \frac{c^4}{4GM} .$$

For a black hole of mass $2\,M_{\text{sun}} = 4 \cdot 10^{30}$ kg, for example, the event-horizon radius is about 6 km, and the event-horizon gravity is

$$g = \frac{\left(3 \cdot 10^8 \text{ m/s}\right)^4}{4\left(6.67 \cdot 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2\right)\left(4 \cdot 10^{30} \text{ kg}\right)} = 7.6 \cdot 10^{12} \text{ m/s}^2 .$$

This is the typical acceleration a particle would achieve as it falls into the black hole. Probable black holes have been observed in double-star systems (Cygnus X-1 is a well-known example) and in the nuclei of galaxies. Our own Milky Way galaxy most likely harbors a block hole at its center of over one million solar masses.

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*Einstein's *general theory of relativity* is needed to accurately describe gravity in the vicinity of black holes. The Newtonian calculations given here just give a rough idea how gravity behaves in this case.

Questions and Problems from Day Twelve

12Q1. Is air resistance a conservative or nonconservative force?

12Q2. A particle moves with potential energy shown in the figure. Describe qualitatively the motion of a particle with total energy (a) $E = 2U_0$ (b) $E = U_0$ (b) $E = U_0/2$. Where in each case does the particle come to rest, where is its kinetic energy greatest, where is its speed greatest, and in which direction is the force acting during each part of its motion?

12Q3. What is an explicit function $U(x)$ that has the shape shown in 12Q2?

12Q4. Astronomers discover a new comet falling into the inner solar system, which passes around the Sun and heads back out again. How can we tell whether this comet will later revisit, or instead depart the solar system, never to return?

12Q5. Why does the Moon have virtually no atmosphere? Would it likely have more or less atmosphere if it were orbiting Mars rather than Earth?

12Q6. Is it possible for a rocket to escape from Earth if it never achieves Earth's escape velocity $v_{esc} = 11.2$ km/s?

Section 12.1 Energy conservation and the work-energy theorem

12P1. A ball of mass m is attached to the lower end of an initially unstretched hanging spring of force-constant k. The ball is then slowly lowered so it reaches its equilibrium position under the influence of the spring and gravity g. (a) How much potential energy resides in the spring? (b) How much has the ball's gravitational potential energy been reduced? (c) Which of these quantities is larger, and by what factor? (d) How much work was done by the spring? (e) How much work was done by gravity?

12P2. A spring of force-constant k and equilibrium length ℓ is hung from the ceiling, which is a height h ($h > \ell$) above the floor. A ball of mass m is pressed upward against the spring, compressing it a distance d. The ball is released from rest, and the spring (and gravity g) propel it downward. What is the ball's speed v when it hits the floor? Solve the problem (a) using energy conservation (b) Using the work-energy theorem.

12P3. Two springs, each of force-constant k, are attached to a ball of mass m and to two walls as shown. The ball is a distance L from each wall; assume that the unstretched length of each spring is *zero*. If the ball is dropped from rest at the point midway between the springs, how far does it fall before it is brought instantaneously to rest? Solve the problem (a) using energy conservation (b) using the work-energy theorem.

12P4. A spring gun consists of a hollow cylinder capped at one end, with a spring of force-constant k inside. A ball of mass m is pressed against the spring, compressing it by Δx. The gun is held horizontally, a distance h above the floor. The ball is released, flies out of the gun, and follows a parabolic trajectory to the floor under the influence of gravity g. Using energy conservation, find the speed v of the ball (a) when it leaves the gun; (b) when it reaches the floor.

12P5. Prove that the sum of a straight line and a parabola is a parabola. (See *Sample Problem* 1 for an example.)

12P6. In *Sample Problem 1* the energy diagram was constructed for a block hanging on the end of a spring. Suppose instead that a block of mass M is attached to the top end of a spring of force-constant k, whose other end is kept fixed on a table top. Let (+) x be the distance of the block below the top end of the spring, if the spring were uncompressed. (a) Show that energy is conserved for this system, if both gravitational and spring potential energies are included. (b) Draw a diagram of the total potential energy. (c) If the ball is released from rest when the spring is unstretched, how far will it fall before it begins to rise again?

12P7. A block of mass m is placed on a table, and a spring of force-constant k and negligible mass is attached to the block as shown. (a) A second block of mass m is attached to the top end of the spring. How much is the spring compressed by this second block? (b) Now someone pushes down on the top block, compressing the spring an additional distance d, and then suddenly releases the block. The top block springs upward. What must be the distance d so that when the top block reaches its highest point, the bottom block just barely lifts off the table?

Section 12.2 Conservative forces

12P8. A long straight rod is given a uniform positive electric charge, and a negative point charge is allowed to move close to the center of the rod, a distance x away, as shown. Then it can be shown that the force on the point charge is $F = -C/x$, where C is a positive constant and the negative sign means that the force is attractive, toward the rod. (a) Find an expression for the potential energy of the point charge. (b) Find the work done on the point charge if it is moved from x_1 to x_2.

12P9. The force between a positive electric charge Q and a negative charge $-q$ is $F = -kQq/r^2$, where r is the distance between them and k is a constant. (a) Find the potential energy between them as a function of r. (b) Find the work done in separating them from a distance r_0 to a larger distance r.

12P10. A particle moves in a straight line in response to a conservative force. The corresponding potential energy diagram for the particle's motion is shown.

The energy of the particle is $E = 0.8U_0$. For the following questions, answer by giving the appropriate letters A - H. At what point(s) (a) does the particle experience a force in the $+x$ direction? (b) is the speed of the particle greatest? (c) is the force on the particle equal to zero? (d) does the particle accelerate *toward* $x = 0$? (e) is the particle's speed zero? (f) can the particle never be found?

12P11. Solve 12P10, except assume the energy E of the particle is equal to U_0 (instead of $0.8U_0$).

12P12. The potential energy of an object is represented by the curve shown; the constant energy E of the object is represented by the horizontal straight line. At which point(s) is the kinetic energy a maximum? At which point(s) is the kinetic energy zero?

12P13. The potential energy of an object is represented by the curve shown; the constant energy E of the object is represented by the horizontal straight line. At which point(s) is the kinetic energy a maximum? At which point(s) is the kinetic energy zero?

12P14. A particle moving in a straight line has the potential energy given by the solid line shown. The total energy of the particle is given by the horizontal dotted line shown. Answer the following questions by listing *all* appropriate letters *A, B, C, ...*

(a) Where is the particle not moving? (b) Where is the particle accelerating toward the right? (c) Where is the particle moving with constant velocity? (d) Where does the particle experience a force toward the left?

12P15. A particle of mass $m = 2.0$ kg moves with the potential energy $U(x) = 3|x|$ shown, where x is measured in meters and $U(x)$ in Joules. When the particle is at $x = 0$, its speed is v_O = + 3.0 m/s. (a) Find the total energy E of the particle. (b) What is the position x_L of the left turning point of the particle? (c) What is the position x_R of the right turning point? (d) What is the speed v_2 of the particle when it reaches $x = 2.0$ m? (e) What is its speed v_{-1} when it reaches $x = -1.0$ m? (f) How long a time t does it take the particle to move from x_L to x_R?

12P16. A particle of mass $m = 0.20$ kg moves with the potential energy $U(x)$ graphed in the figure. When the particle is at $x = 2.5$ m, its speed $v = 5.0$ m/s. (a) Find the energy E of the particle. (b) What is the location x_L of the left-hand turning point of this particle? (c) What is the location x_R of the right-hand turning point? (d) What is the speed v_{-2} of the particle when it reaches $x = -2.0$ m? (e) How long a time t does it take the particle to move from x_L to x_R?

12P17. Show that spring forces are conservative for two-dimensional motion. *Hint:* Given a mass m attached to one end of a spring of force-constant k and unstretched length r_O. The other end of the spring is attached to a fixed point by a gimble, so the spring can orient itself in any direction. Then consider various paths by which the mass can move between two given points, and explain why the work done by the spring doesn't depend upon the path taken.

12P18. A particle of mass m moves with potential energy $U(x) = ax^4 - bx^2 + b^2/4a$ where a and b are constants. (a) Sketch $U(x)$ for the choice $a = 1$ and $b = 2$ in appropriate units. *Then for arbitrary values of a and b,* (b) Find the energy E_0 below which m always lives at positive x or always lives at negative x.
(c) The particle has energy $E_0/2$, and is at positive x. Find the values of x where it has zero velocity, and the location x_0 where the particle has its largest kinetic energy.

12P19. An atom of mass m is bound with potential energy $U(x) = a/x^5 - b/x^4$ where a and b are positive constants. (a) Sketch $U(x)$. Then find (b) the force $F(x)$ acting on the atom. (c) the equilibrium position x_0 of the atom. (d) the dissociation energy D, the minimum energy required to raise the atom from its lowest possible energy so that it is no longer bound.

12P20. The "6-12 potential" $U(r) = -2a/r^6 + b/r^{12}$, where a and b are positive constants, is sometimes used to approximate the potential energy between two atoms in a diatomic molecule, where the atoms are separated by the distance r. (a) Sketch $U(r)$, and find the location r_0 and potential energy $U(r_0)$ at the equilibrium point of the atoms. (b) What is the dissociation energy of the molecule, the energy required to separate the atoms infinitely far apart, starting at their equilibrium position?

12P21. The "Morse potential" $U(r) = D_e(1 - e^{-a(r - r_0)})^2$ is a function which fits molecular potential energies quite well. Here r is the separation between two atoms in a diatomic molecule, and D_e, a, and r_0 are constants. In terms of the three constants,
(a) Find the radius r and potential energy U at the equilbrium point.
(b) What is the dissociation energy of the molecule, the energy required to separate the atoms infinitely far apart, starting at their equilibrium position?

Section 12.3 Nonconservative forces

12P22. A downhill racer of mass $m = 77.2$ kg starts at altitude $h_0 = 3220$ m with speed $v_0 = 1.5$ m/s, and thereafter skis without poles down to altitude $h_f = 2770$ m, arriving with speed $v_f = 31.5$ m/s. How much energy did the skier lose to frictional heating of the snow, skis, and air?

12P23. A toboggan and riders of total mass $M = 262$ kg starts at altitude $h_0 = 2330$ m with speed $v_0 = 4.5$ m/s, and then slides down to altitude $h_f = 1560$ m, arriving with speed $v_f = 42.3$ m/s. How much energy was lost to frictional heating of the snow, toboggan, and air?

12P24. A cannonball of mass $m = 8.0$ kg is shot vertically upward from Earth's surface with initial speed $v_0 = 150$ m/s. However, because of air resistance the cannonball comes to rest at an altitude half as great as it would without air resistance. (a) How much work W_1 (in Joules) did gravity do on the cannonball during its ascent?
(b) How much work did air resistance do on the cannonball? (c) How much work W_2 does gravity do on the cannonball as it falls back to the ground?

12P25. A block of mass m is hung vertically from an ideal massless spring. The spring stretches by an amount d. The spring is then attached at the base of a plane inclined at angle θ, as shown. The block is released from rest a distance ℓ from the spring. The coefficient of kinetic friction between the block and the incline is μ_k. (a) Draw an isolation diagram of the block. (b) Using $F = ma$, find the speed of the block when it first touches the spring, in terms of given parameters. (c) We note that the maximum compression of the spring is x. Find an expression for μ_k in terms of ℓ, x, θ, and d.

12P26. A skydiver of mass $m = 73$ kg exits an airplane moving at $v_0 = 65$ m/s, and then falls a vertical distance $d = 1.5$ km. If he strikes the ground with speed $v = 3.4$ m/s, what has been the total work W done on him by air resistance?

12P27. On Earth, with $g = 9.8$ m/s^2, a cannonball of mass $m = 21$ kg is shot vertically upward at $v_0 = 130$ m/s. (a) If the ball rises to the height $h = 790$ m before it starts to fall, find the work W_a done by air resistance on the ball during its ascent. (b) If air resistance does the same work W_a on the ball during its descent, find the speed v_f of the ball when it hits the ground. (Neglect the length of the cannon.) [Note: the effect of air resistance is such that the acceleration of the cannonball varies.]

12P28. A cow of mass m wandering through the mountains stumbles and slides down a frictionless icy cliff shaped like a quarter-circle of radius R. The cow then has a coefficient of sliding friction $\mu_k = 1/2$ in the level snow at the base of the cliff. After sliding a distance R on the snow, it reaches a vertical cliff of height R, and falls into a soft snowbank at the bottom. (a) What was its speed v_1 at the bottom of the circular cliff? (b) What was its speed v_2 just before falling off the final cliff? (c) If all of its kinetic energy is converted into heat when it stops in the snowbank, what heat energy H is produced there? (Steuard Jensen)

.Section 12.4 Spherically-symmetric gravity

12P29. The Sun orbits the center of our Milky Way galaxy with a period of about 250 million years. Estimate the number of solar masses in the galaxy which are inside the orbit of the Sun. (The mass of the Sun is $M_{sun} = 2.0 \cdot 10^{30}$ kg and its orbital radius is $R \approx 8.5$ kpc. Note that 1 kpc is 10^3 parsecs (pc); 1 pc = 3.26 light years; 1 light-year is the distance light travels at speed $c = 3 \cdot 10^8$ m/s for 1 year; and 1 yr = $3.16 \cdot 10^7$ s.)

12P30. A satellite orbits Earth in a circular path of radius R. (a) Using $\mathbf{F} = m\mathbf{a}$, find the speed v of the satellite in terms of Newton's gravitation constant G, M_{Earth}, and R. (b) Show that the kinetic energy of the satellite is half the magnitude of its gravitational potential energy. (c) Find its total energy E in terms of the same three quantities. (d) If the satellite is in low-earth orbit (altitude $\ll R_{Earth}$), find the approximate period T of the orbit in minutes.

12P31. A hypothetical Earth-intersecting asteroid has its perihelion at $r_p = 0.800$ AU, its closest distance to the Sun; there it has speed $v_p = 38.5$ km/s. What is its speed v_a when it is at aphelion $r_a = 1.60$ AU ? (Note: $1 \text{ AU} = 150 \cdot 10^6$ km is the average distance of Earth from the Sun.)

12P32. An alien spaceship drifts into the solar system. Long before it gets close, its speed is v_0. If its perihelion is a distance R from the Sun's center, how fast is it moving there?

12P33. Communications satellites orbit in the equatorial plane of Earth with a period of 24 hours so they can remain above the same point on the equator. (a) Using $\mathbf{F} = m\mathbf{a}$, find how many Earth's radii the satellite is from the center of Earth. (b) If we eventually place 10,000 satellites in this "geosynchronous" orbit, how far apart will nearest neighbors be?

12P34. Stymied by its inability to find politically viable radioactive waste storage sites, the government consults with you about how to send waste permanently away from Earth on spacecraft. Two possibilities arise: (1) "dropping" the loaded spacecraft into the Sun, and (2) sending it on a trajectory to escape the solar system, never to return. Which requires the least rocket fuel? The change in velocity Δv of the spacecraft during a burn is a measure of the fuel used; suppose in each case the rocket burns only at the beginning of its journey and coasts thereafter. In each case the rocket must start by escaping from Earth, so assume the rocket has already escaped and is in orbit around the Sun with Earth's orbital radius and velocity. In terms of G, M_{sun}, and R, the radius of Earth's orbit, find (a) the minimum Δv required to send the spacecraft into the Sun (b) the minimum Δv required to eject the spacecraft from the solar system. For each case, in which direction should the rocket be fired? Which is the method requiring the least fuel?

12P35. The Galileo orbiter spacecraft is exploring the outer three Galilean moons of Jupiter (Europa, Ganymede, and Callisto), by making close flybys of each in turn. Gravitational boosts provide most of the extra speed needed to go from one moon to the next one farther out. Suppose the spacecraft passes by Ganymede along the path shown, on its way out to Callisto. The spacecraft both approaches and departs Ganymede with speed v_0, but turns its trajectory by 90°, as shown in Ganymede's rest frame. Now suppose that Ganymede is moving to the right at speed $v_0/2$, from the point of view of observers in Jupiter's frame. (a) Make a qualitative sketch of the spacecraft's trajectory as seen in Jupiter's frame. Then in Jupiter's frame, find the spacecraft's (b) initial speed v_i, (c) final speed v_f, in terms of v_0.

12P36. A spacecraft passes the planet Jupiter as shown in Jupiter's rest-frame, with speed v_S both long before and long after the flyby. Both the approach and departure are at an angle $\theta = 45°$ to the y-axis. Jupiter is moving to the left (i.e., in the $-x$ direction) at speed v_J in the rest-frame of the sun; suppose for simplicity that $v_J = v_S / \sqrt{2}$. Find the spacecraft's velocity v in the sun's frame, including both its magnitude in terms of v_S, and direction (a) long before the flyby (b) long after the flyby.

Special Topic: White dwarfs, neutron stars, and black holes

12P37. A particular neutron star has a mass 1.5 times that of our Sun and a radius of 10.0 km. (a) Calculate the density ρ of the star, in kg/m^3. (b) How many metric tonnes (1 tonne = 1000 kg) are in one cubic centimeter of the star? (c) Find the gravitational field g at the surface of the star. (d) Find the escape velocity v_{esc} from the star, in m/s and as a fraction of the speed of light. (See mass and radius data in a table at the end of this book.)

12P38. (a) Find the minimum radius R a spherical body of mass M can have so that light of speed c can escape from the surface, pretending that light is a classical particle which slows down as it gains altitude. (The result was first obtained by John Michell in 1783 and then independently by Pierre Simon Laplace in 1797.) Then evaluate R for a sphere with mass that of (b) Earth (c) the Sun (See mass data in a table at the end of this book.)

Dimensional analysis problems

12P39. (a) Find an expression for the Planck time t_p, the unique combination of the three fundamental constants G (Newton's gravitational constant), c (the speed of light) and $\hbar$ (Planck's constant divided by 2π, with the dimensions of energy x time and value $\hbar = 1.055 \cdot 10^{-34}$ J s) and then evaluate t_p in seconds. It is thought that prior to the Planck time in the history of the universe, space and time may be essentially quantum mechanical, lacking the smooth structure we normally ascribe to it. (b) Find an expression for the Planck energy E_p in terms of the same fundamental constants, and then evaluate it numerically in Joules. It is thought that typical particles existing at the Planck time have the Planck energy.

12P40. An exploding nuclear bomb creates a rapidly-expanding shock wave in the air surrounding the blast. Within the shock wave the air glows brightly because it has been strongly heated, giving the appearance of a "fireball". The radius R of the expanding ball of hot air depends upon time t, the energy E of the blast, and the density ρ of the air. (It might depend also upon the ambient air pressure, but to a good approximation it does not, because the ambient pressure is so much smaller than the pressures created by the blast.)
(a) Using dimensional reasoning, find out how R depends upon t, E, and ρ.
(b) If the shock wave has radius R_o at time 0.1 s, what is its radius at 1.0 s?
(c) The picture shows the fireball of the Trinity test, the first nuclear explosion, at 05.30 hours, 16 July 1945, Alamogordo, New Mexico, at time 0.006 s after the detonation. The diameter is about 150 m, as shown. Estimate the energy of the blast in Joules and in equivalent tonnes of TNT, assuming the dimensionless coefficient in the expression for $R(t, E, \rho)$ is unity. (The coefficient has been calculated to be 1.003). The explosive energy of one tonne (1000 kg) of TNT is $4.2 \cdot 10^9$ J. [Pictures of the first explosion were published in *Life Magazine*. From the pictures and using dimensional reasoning, a number of physicists around the world deduced the yield of the first nuclear explosion.]

Day Thirteen

Elastic and Inelastic Collisions

"The universe is made up of stories, not of atoms." - Muriel Rukeyser

Three large hard-rubber balls occupy most of the floor in a storage closet. One is stamped "1-kg" and two are stamped "2-kg"; they all have thumbholes like bowling balls. When we drop one on the concrete floor it bounces back almost to the same height. It must compress when it hits the floor and then re-expand to push itself back up. As it falls, its energy first changes from gravitational potential into kinetic, and then to a spring-like potential energy as it compresses, back to kinetic, and so forth. The total energy seems to stay virtually unchanged.

As the ball bounces up and down its motion is also nearly time reversible: If we made a video film and later screened it, we would have to watch awhile to tell whether it was being played forward or backward. For two or three bounces the motion would look the same backward as forward in time.

There is no air resistance outside our room and the ice is nearly frictionless. We can see what happens when we launch the balls along the ice with virtually no energy loss. So we pull on our gear, stuff the balls into a backpack, and venture outside.

Exploring in a new direction we discover a long horizontal slot in the ice, barely large enough to accept the lower half of a ball. The slot rises at the far end in a circular arc. When we bowl along it, the ball whips up the arc, comes to rest, and then slides back down, returning a few seconds later.

13.1 Elastic collisions

We set one of the 2-kg balls at rest in the slot about halfway along, and bowl the other 2-kg ball at it. As expected from the collisions last week in the Bubble, the bowled ball almost completely stops after it collides with the target ball, and the target ball captures all its velocity. The target ball then shoots up the circular arc, slides back down, hits the originally bowled ball giving it back almost all the velocity (the "target" ball now nearly stops), and the originally bowled ball returns to us.

In Day Three the incident ball had only a small velocity after it hit an equal-mass target, and most of its original velocity went to the target. Here the velocity transfer is nearly perfect. We call idealized collisions with no kinetic energy loss at all "elastic collisions".

We should be able to predict collision results based on conservation laws. Call the bowled ball a and the target ball b, as shown. Then since b is initially at rest, kinetic energy

conservation predicts

$$\frac{1}{2}m_a(v_a)_i^2 = \frac{1}{2}m_a(v_a)_f^2 + \frac{1}{2}m_b(v_b)_f^2. \tag{1a}$$

The kinetic energy of a just before the collision equals the sum of the kinetic energies of a and b after. Momentum conservation likewise predicts

$$m_a(v_a)_i = m_a(v_a)_f + m_b(v_b)_f. \tag{2a}$$

Cancelling the 1/2 's in Eq. 1a, bringing all ball "a" quantities to the left, and dividing by

m_a, the two equations (representing conservation of kinetic energy and conservation of momentum) become

$$(v_a)_i^2 - (v_a)_f^2 = \left(\frac{m_b}{m_a}\right)(v_b)_f^2 \tag{1b}$$

and

$$(v_a)_i - (v_a)_f = \left(\frac{m_b}{m_a}\right)(v_b)_f . \tag{2b}$$

Dividing Eq. 1b by Eq. 2b, using the algebraic identity $A^2 - B^2 = (A + B)(A - B)$, and rearranging terms we get

$$(v_a)_i = (v_b)_f - (v_a)_f. \tag{3}$$

The right-hand side of Eq. 3 is the relative velocity of the two balls *after* the collision, and (since ball b is initially at rest) the left-hand side is the relative velocity of the balls *before* the collision. Note however that whereas before the collision $v_a > v_b$, afterwards $v_b > v_a$. Therefore

> *in one-dimensional elastic collisions, the relative velocity of the two objects has the same magnitude before and after.*

Eliminating $(v_a)_f$ between Eqs. 2b and 3 and then solving for $(v_b)_f$ we find that

$$(v_b)_f = \frac{2m_a}{m_a + m_b}(v_a)_i . \tag{4}$$

Combining again with Eq. 3, we find also that

$$(v_a)_f = \left(\frac{m_a - m_b}{m_a + m_b}\right)(v_a)_i . \tag{5}$$

Equations 4 and 5 give the final velocities for one-dimensional elastic (i.e., kinetic-energy conserving) collisions, if the target ball b is initially at rest. When a and b have equal mass, these equations agree with our observation that ball a stops and ball b gets all its velocity.

Sample Problem 1. Bowl a 2-kg ball with initial speed v_o at a 1-kg ball at rest. If the collision is elastic, what are the final velocities of the bowled ball and the target ball?

Solution: Equation 5 predicts the final velocity of the bowled ball:

$$(v_a)_f = \left(\frac{2-1}{2+1}\right)v_o = \frac{1}{3}v_o$$

Equation 4 gives the recoil velocity of the target ball:

$$(v_b)_f = \left(\frac{2 \cdot 2}{2+1}\right)v_o = \frac{4}{3}v_o \,.$$

Momentum is conserved: $2 \cdot v_o = 2 \cdot (\frac{1}{3}v_o) + 1 \cdot (\frac{4}{3}v_o)$. Kinetic energy is also conserved: $\frac{1}{2} \cdot 2v_o^2 = \frac{1}{2} \cdot 2\,(\frac{1}{3}v_o)^2 = \frac{1}{2} \cdot 1\,(\frac{4}{3}v_o)^2$. After the collision both balls move to the right, the bowled ball at speed $v_o/3$ and the target ball at speed $4v_o/3$.

13.2 Exploiting time-reversibility: a pictorial approach

A pictorial approach helps us understand what's goes on in elastic collisions. In the center-of-mass (CM) frame, the total momentum of the balls is zero, by definition (See *Day Four* in *the Bubble*). Therefore prior to the collision, two equal-mass balls must approach one another with equal but opposite velocities in the CM frame. Momentum is conserved if the total momentum remains zero, which means the equal-mass balls must have equal but opposite velocities *away* from each other after the collision. *Any* final velocity will do, to conserve momentum. As long as the final velocities are equal but opposite, momentum is conserved.

In elastic collisions, however, the final *speeds* must equal the *initial* speeds v' in the (primed) CM frame, as shown in Fig. 1(a). That is because elastic collisions are time reversible; if we watched a movie taken of the collision, we shouldn't be able to tell whether it is being shown forward or backward.

Fig. 1. The collision of equal-mass balls (a) in their CM frame; (b) in the "lab" frame in which b is initially at rest. The lab frame moves at speed $V = v'$ to the left relative to the CM frame.

If we look at this exact same experiment from the point of view of a "lab" frame moving at speed $V = v'$ to the *left* relative to the CM frame, by addition of velocities it is easy to see that in the lab frame the three pictures shown in Fig. 1(a) look instead as shown in Fig. 1(b) .

Ball a with speed $2v'$ hits ball b at rest; afterwards ball a has stopped dead and ball b has speed $2v'$. This is what we got before using algebra! That is, in the lab frame the initial speed is $v_0 = 2v'$ (ball a moves twice as fast in the laboratory as in the CM frame.) The speed is exactly the same after the collision. So this technique of starting with the CM frame, in which we can "see" what will happen, and then transforming to the lab frame, is a way of doing the problem with little or no math, and makes the result more understandable.

Sample Problem 2. An elastic collision using the pictorial approach

A 2-kg ball with velocity v_o collides with a 1-kg ball, initially at rest in the laboratory. Find the final velocities of each ball in the laboratory, using the pictorial approach.

Solution: In the (primed) CM frame, the relative velocities of the two balls must still be v_o, and their momenta must also sum to zero. Their initial velocities must therefore be $v_o/3$ and $-2v_o/3$, as shown. That is, their velocites are in opposite directions, with a ratio of 1:2, since the ratio of their masses is 2:1 ; dividing by 3 makes their relative velocity v_o, just as it is in the (unprimed) lab frame. Note that the CM frame moves to the right with speed $V = 2v_o/3$ relative to the lab, because the 1-kg ball is initially at rest in the lab, and has velocity $-2v_o/3$ in the CM frame. (Alternatively, we can find V more formally, using

$$V = V_{CM} = \frac{P_{total}}{M_{total}} = \frac{m_a v_a + m_b v_b}{m_a + m_b} = \frac{m_a v_a}{m_a + m_b} = \frac{2\,v_a}{2+1} = \frac{2}{3}v_o$$

given as the definition of V_{CM} in *Day Four.*)

After the collision the velocity of each ball has just turned around, as shown; the sum of their momenta is still zero, and the picture is the time reverse of the preceding picture.

The final velocities of the balls in the *lab* frame are easily found by adding $2v_o/3$ to the final velocity of each in the CM frame, as shown below. (This is just the velocity transformation $v = v' + V$ between the two frames.) The final velocity of the 2-kg ball is $v_o/3$ to the right, and the final velocity of the 1-kg ball is $4v_o/3$ to the right.

Three checks: (i) The relative velocity of the two balls (v_o) is the same before and after.
(ii) Momentum is conserved in the lab: $2\text{ kg}\,(v_o) = 2\text{ kg}\,(v_o/3) + 1\text{ kg}\,(4v_o/3)$.
(iii) Energy is conserved in the lab: $\frac{1}{2}(2\text{ kg})v_o^2 = \frac{1}{2}(2\text{ kg})(v_o/3)^2 + \frac{1}{2}(1\text{ kg})\,(4v_o/3)^2$.

13.3 Elastic collisions in three dimensions

One-dimensional collisions only happen if the incident ball makes a head-on collision with the target ball. More generally, in the lab frame the incident ball a makes a glancing collision with the initially stationary target ball b, scattering off it at some angle while the target ball recoils at some other angle, as illustrated in Fig. 2. The entire motion takes place in a plane containing both final velocity vectors, $\mathbf{v}_a$ and $\mathbf{v}_b$, since (by momentum conservation) the initial velocity vector $\mathbf{v}_o$ of ball a must lie in this plane as well.

Fig. 2. A general collision in the lab frame between incident ball a and target ball b.

Let the incident ball a scatter off with speed v_a at angle θ to the forward direction, while ball b recoils with speed v_b at angle ϕ. Momentum is a vector, so momentum conservation leads to two equations, one corresponding to the forward (x) direction and the other corresponding to the perpendicular (y) direction.

$$Px: \qquad m_a v_o = m_a v_a \cos\theta + m_b v_b \cos\phi \tag{6a}$$

$$Py: \qquad 0 = m_a v_a \sin\theta - m_b v_b \sin\phi \tag{6b}$$

Kinetic energy is also conserved in elastic collisions. Energy is a scalar quantity, so there is only a single equation corresponding to conservation of kinetic energy. (It makes no sense to conserve horizontal and vertical "components" of kinetic energy, since kinetic energy is not a vector.) The energy equation is

$$K.E.: \qquad \frac{1}{2} m_a v_o^2 = \frac{1}{2} m_a v_a^2 + \frac{1}{2} m_b v_b^2 \tag{7}$$

Given the ball masses and the initial velocity v_o, the equations 6a, 6b, and 7 contain four unknowns, the two final speeds v_a and v_b and the two angles θ and ϕ. There are

only three equations, however, so we cannot determine the four unknowns uniquely. This makes physical sense, since when we throw one ball at another, the thrown ball can bounce off at all sorts of angles depending on just how it hits the target*. If we measure one of the four unknowns (say the scattering angle θ of the incident ball) then we can determine all three of the other unknowns v_a, v_b, and ϕ by using the three conservation equations.

*That is, the initial offset of the ball centers is the fourth piece of information we would need to determine the scattering uniquely.

Sample Problem 3. Ball a, of mass m_a and speed v_o, strikes ball b, of mass $m_b = 2m_a$, initially at rest. After the collision ball a moves at 90° to its initial direction of motion, as shown. Find the angle ϕ at which ball b recoils from the collision.

Solution: Using $\cos 90^o = 0$ and $\sin 90^o = 1$, Eqs. 6(a), 6(b), and 7 become, respectively,

$$v_o = 2v_b\cos\phi\ , \qquad v_a = 2v_b\sin\phi, \qquad \text{and} \qquad v_o^2 = v_a^2 + 2v_b^2\ .$$

Substitution of the second equation into the third and taking the square root yields

$$v_o = v_b\sqrt{2\left(1+2\sin^2\phi\right)}.$$

Then dividing this result by the first equation and squaring gives

$$2(1 + 2\sin^2\phi) = 4\cos^2\phi \equiv 4(1 - \sin^2\phi)$$

using the trig identity $\cos^2\phi = 1 - \sin^2\phi$. Therefore $\sin^2\phi = 1/4$, so $\sin\phi = 1/2$, giving the result

$$\phi = 30^o.$$

We could also solve for v_a and v_b in terms of v_o by substituting $\phi = 30^o$ back into the equations.

It is interesting to view elastic collisions in the center-of-mass frame. Before the collision the balls approach one another with equal but opposite momenta. If they do not collide head-on, they emerge at angles different from their approach angles. Their total momentum is zero after the collision as well, so if one of the balls comes off in a particular direction, the other ball has to come off in the exact opposite direction, as shown in Fig. 3. Only in this way can their momenta cancel one another. Since elastic collisions are time-reversible, the speed of each ball has to be the same as it was before; otherwise we could tell the difference between a movie of the collision shown forward and one shown backward.

Fig. 3. An elastic collision (a) before (b) after, drawn in the CM frame

Sample Problem 4. Two balls of equal mass collide elastically. Suppose that in the CM frame the balls emerge from the collision oriented at angles θ' and $\pi - \theta'$ to the x-axis, as shown. How do they emerge from the collision as seen in the original lab frame?

Solution: In the (unprimed) lab frame the initial velocity of ball a is $\mathbf{v}_o$ while ball b is at rest, as shown in the left-hand picture below. Therefore in the (primed) CM frame the initial velocities must be $\mathbf{v}_o/2$ and $-\mathbf{v}_o/2$, to make their total momentum equal to zero. The CM frame moves to the right relative to the lab frame with velocity $\mathbf{V} = \mathbf{v}_o/2$.

After the collision the balls emerge in opposite directions, still with the same speeds in order to conserve kinetic energy. Their velocities are now $\mathbf{v}'_a$ and $\mathbf{v}'_b = -\mathbf{v}'_a$ as shown in the right-hand picture, each of magnitude $v_o/2$. We can find the final velocities of the balls in the lab frame by using vector addition of velocities $\mathbf{v} = \mathbf{v}' + \mathbf{V}$.

The diagram below shows the vector additions $\mathbf{v}_a = \mathbf{v}'_a + \mathbf{V}$ and $\mathbf{v}_b = \mathbf{v}'_b + \mathbf{V}$.

It is interesting that these two vectors are perpendicular to one another! To prove this, take the dot product

$$\mathbf{v}_a \cdot \mathbf{v}_b = (\mathbf{v}'_a + \mathbf{V}) \cdot (-\mathbf{v}'_a + \mathbf{V}) = -\mathbf{v}'_a \cdot \mathbf{v}'_a + \mathbf{v}'_a \cdot \mathbf{V} - \mathbf{v}'_a \cdot \mathbf{V} + \mathbf{V} \cdot \mathbf{V}.$$

The right-hand side of this expression is *zero*. The terms in the middle cancel one another and the first and last terms cancel as well, since $\mathbf{v}'_a$ and $\mathbf{V}$ both have magnitude $v_0/2$. The dot product of $\mathbf{v}_a$ and $\mathbf{v}_b$ is therefore zero, so the velocities must be perpendicular to one another (unless one of the two velocities is zero). In an elastic collision of two equal-mass balls (pool balls, for example) we may not be able to predict which angle the incident ball will have after the collision (unless we are very good at pool), but we *do* know that the two balls will emerge at right angles to one another (or *nearly* so, since real collisions are not perfectly elastic.)

A liquid hydrogen bubble-chamber photograph is reproduced on the next page, showing several elastic collisions of protons. A proton enters the chamber from the top center and moves downward, colliding with an equal-mass proton (a hydrogen nucleus), initially at rest. The two protons separate, and each subsequently collides with one or more additional protons, as can be seen, before coming to rest. (There are other particles entering the chamber, and the tight spirals are electrons ejected from hydrogen atoms by passing electrically charged particles. Curvature in the tracks is caused by an applied magnetic field, used to measure the momentum of moving particles and the sign of the electric charge carried by each.) Note that the tracks made by the two protons emerging from a collision all seem to make about a right angle with one another.*

* The photograph is of course a projection of the actual three-dimensional scene onto a plane. Therefore even if the scattering angle is exactly 90° in the plane of the collision, it will generally be somewhat different in the plane of the photograph.

13.4 Inelastic collisions

Looking back at our data from the one-dimensional collisions of metal balls during our third day in the Bubble, even when we crashed a ball into an *equal-mass* target ball at rest, the incident ball did not quite stop, but moved forward a little bit after the collision. Why?

Momentum was exactly conserved in the collisions, but now we know that energy was not conserved. Was energy gained or lost?

In the CM frame collision of equal-mass balls that *lose* energy, the before-and-after pictures would be as shown in Fig. 4(a), where $v_f' < v_o'$. Then the kinetic energy after would be less than it was before.

Fig. 4. An energy-losing (inelastic) collision of equal-mass balls (a) in their CM frame (b) in the lab frame, in which b is initially at rest. The final CM speed v_f' is less than the initial speed v_o'.

The lab frame (in which the target ball is at rest) moves at speed v_o' to the left relative to the CM frame. The before-and-after pictures in the lab frame must therefore be as shown in Fig. 4(b). If kinetic energy is lost in the collision, ball a will move forward in the lab frame afterwards, with the relatively small speed $v_o' - v_f'$; the target ball b will move forward with speed $v_o' + v_f'$, a slightly smaller speed than it would have had if the collision were elastic. So kinetic energy was *lost* in the collisions back in the Bubble. What happened to it? Did it simply vanish?

Energy conservation has been such a useful concept that we should try hard to hold onto it! Is there an alternative to the loss of kinetic plus potential energy? Are there additional forms of energy beyond those we have found so far?

If kinetic plus potential energy is lost in a collision, some new form of energy may increase to make up for the loss. The energy might go into the balls themselves, for example, as some kind of "internal" energy: A single ball might oscillate after being struck, so that it possesses an internal *kinetic* energy. It might be that when the balls hit each other they are temporarily deformed, giving each a *potential* energy like a compressed spring. If the collision is hard enough the balls might even become permanently deformed, so the initial energy has gone into some kind of energy of *deformation*.

Some of the macroscopic kinetic energy (the large-scale kinetic energy of the colliding macroscopic objects) is lost in inelastic collisions. In *totally* inelastic collisions, as much of the kinetic energy is lost as possible, consistent with momentum conservation. In the center-of-mass frame, the total momentum is zero before and after; therefore, in totally inelastic collisions the colliding objects stick together and are at rest in the CM frame after the collision. Every bit of the initial kinetic energy in the CM frame is lost is such collisions.

In the lab frame the final particles (now stuck together) move as a unit. They cannot lose all their initial kinetic energy in the lab frame, because momentum must be conserved. It can be shown that the same *amount* of kinetic energy is lost in the lab frame as in the CM frame.

Conservation of energy is more general than is obvious in watching inelastic collisions. For an *isolated* system (isolated so that energy doesn't somehow flow in or out, or so that nothing from the outside is doing work on the system, and the system is not doing work on the outside world), it is possible that energy is *always conserved.* Sometimes the energy is just the sum of the kinetic and potential energies, $K + U$. But sometimes $K + U$ may decrease and the energy may appear as internal energy to compensate. So within the macroscopic world, we can say that "mechanical energy", defined to be this sum of the macroscopic kinetic and potential energies, may change. But when internal energy is accounted for, the overall energy is conserved. *

*Internal energy can take many forms. An object struck might begin to oscillate, for example, ringing like a bell while the kinetic energy of oscillation is gradually turned into random molecular motions, which we call heat energy. An object struck might also suffer a rearrangement of its structure, so that it is dented or bent. It takes energy to do this, so the molecules within the object may have a greater potential energy between themselves than they had before. Energy in a collision may also be carried away in the sound waves generated if the collision takes place in air.

Sample Problem 4. A bullet of mass m is fired with velocity v_o into a wooden block of mass M sitting on a frictionless horizontal surface, in which the bullet becomes embedded in the block. The collision of bullet and block is therefore totally inelastic.

(a) Find the velocity of the combined bullet + block system after the collision.

(b) What fraction of the original kinetic energy is lost in the collision?

(c) The combined bullet + block system then slides up a frictionless ramp. To what height h does it slide?

Solution: (a) Conservation of momentum $mv_o = (m+M)v_f$ during the collision gives

$$v_f = \left[\frac{m}{m+M}\right]v_o .$$

The sum of the kinetic energies of the bullet and block is not conserved in this totally inelastic collision, but internal energy is produced to make up for it: The bullet and block are deformed and get hot as the block stops the bullet inside. Part of the collision energy goes into deformation; it takes energy to compress the bullet and make a hole in the block. Another part of the collision energy goes into the random motions of molecules in the bullet and block, showing up as an increase in their temperatures.

(b) Conserving total energy (including internal energy),

$$\frac{1}{2}mv_o^2 = \frac{1}{2}(m+M)\,v_f^2 + \text{internal energy}.$$

Dividing by the initial kinetic energy K_o,

$$\frac{\text{Internal energy}}{K_o} = \frac{(1/2)m\,v_o^2 - (1/2)(m+M)\,v_f^2}{(1/2)m\,v_o^2} = 1 - \frac{(m+M)}{m\,v_o^2}\left[\frac{m}{m+M}\right]^2 v_o^2$$

$$= 1 - \frac{m}{m+M} = \frac{M}{m+M} .$$

If the bullet and block have the same mass, for example, *half* of the bullet's kinetic energy is converted into internal energy!

During the collision, momentum is conserved because there is no external horizontal force on the system. "Mechanical" energy (the sum of the observed kinetic energies of the bullet and block) is *not* conserved, because the collision is inelastic; some of the energy is converted into internal energy, so the bullet and block heat up. Momentum is conserved during the collision because there is no external horizontal force on the system.

(c) After the collision the system of block + bullet has speed $v_f = mv_o/(M+m)$, and kinetic energy $K = \frac{1}{2}(M+m)v_f^2 = \frac{m^2}{2(M+m)}v_o^2$. As the system rises, momentum is *not* conserved, because there is an external force: Both gravity and the normal force act on the system. But now mechanical energy *is* conserved; no more internal energy is produced. Kinetic energy is gradually converted into potential energy as the block rises. Conserving mechanical energy, we have

$$K_{\text{after collision}} = \frac{m^2}{2(M+m)}v_o^2 = U_{\text{final}} = (M+m)gh, \text{ so}$$

$$h = \frac{1}{2g}\left[\frac{m}{M+m}\right]^2 v_o^2 .$$

The problem therefore has two parts: During the collision, momentum is conserved but not mechanical energy; after the collision, mechanical energy is conserved but not momentum.

13.5 The center-of-mass frame and internal energy

There is a nice way to repackage the total kinetic energy of two balls. Suppose the balls, with masses m_a and m_b, move with velocities $\mathbf{v}_a$ and $\mathbf{v}_b$. Their total kinetic energy

$$K = \frac{1}{2}m_a v_a^2 + \frac{1}{2}m_b v_b^2$$

is the sum of their individual kinetic energies. It is interesting that K can also be written in a different way, as the sum of the kinetic energy of a *single* particle with the total mass $M = m_a + m_b$ moving at the velocity of the center of mass, and the kinetic energy K' of the two balls in their mutual center-of-mass frame. Here is how it works.

The velocity of the CM frame is

$$\mathbf{V}_{CM} = \frac{m_a \mathbf{v}_a + m_b \mathbf{v}_b}{M}, \tag{8}$$

as given during *Day Four*. By vector subtraction, the velocities of the two particles in the CM (primed) frame are therefore

$$\mathbf{v}_a' = \mathbf{v}_a - \mathbf{V}_{CM} \qquad \text{and} \qquad \mathbf{v}_b' = \mathbf{v}_b - \mathbf{V}_{CM}. \tag{9}$$

(These velocities have to be in opposite directions, since the net momentum in the CM frame is always zero.) The total kinetic energy of the two particles in the original frame is

$$\begin{aligned} K &= \tfrac{1}{2}m_a v_a^2 + \tfrac{1}{2}m_b v_b^2 \equiv \tfrac{1}{2}m_a(\mathbf{v}_a \cdot \mathbf{v}_a) + \tfrac{1}{2}m_b(\mathbf{v}_b \cdot \mathbf{v}_b) \\ &= \tfrac{1}{2}m_a(\mathbf{v}_a' + \mathbf{V}_{CM}) \cdot (\mathbf{v}_a' + \mathbf{V}_{CM}) + \tfrac{1}{2}m_b(\mathbf{v}_b' + \mathbf{V}_{CM}) \cdot (\mathbf{v}_b' + \mathbf{V}_{CM}) \\ &= \tfrac{1}{2}m_a(\mathbf{v}_a' \cdot \mathbf{v}_a') + \tfrac{1}{2}m_b(\mathbf{v}_b' \cdot \mathbf{v}_b') + (m_a\mathbf{v}_a' + m_b\mathbf{v}_b') \cdot \mathbf{V}_{CM} + \tfrac{1}{2}(m_a+m_b)\mathbf{V}_{CM}^2 \\ &= \tfrac{1}{2}m_a v_a'^2 + \tfrac{1}{2}m_b v_b'^2 + \tfrac{1}{2}(m_a+m_b)\mathbf{V}_{CM}^2 \end{aligned} \tag{10}$$

where we have used the fact that $m_a\mathbf{v}_a' + m_b\mathbf{v}_b' = 0$; the total momentum is *always* zero in the CM frame.

The last term $\frac{1}{2}(m_a+m_b)\mathbf{V}_{CM}^2$ in Eq. 10 is the same as the kinetic energy of a single ball of mass $M = m_a+m_b$ moving at speed V_{CM}. The other two terms are the "internal" kinetic energy of the system, the total kinetic energy of the two balls in the CM frame. Using Eq. 10, this internal kinetic energy K' is

$$K' = \frac{1}{2}m_a v_a'^2 + \frac{1}{2}m_b v_b'^2 = K - \frac{1}{2}MV_{CM}^2$$

$$= \frac{1}{2}m_a v_a^2 + \frac{1}{2}m_b v_b^2 - \frac{1}{2}M\left(\frac{m_a \mathbf{v}_a + m_b \mathbf{v}_b}{M}\right)^2$$

$$= \frac{1}{2}\left(\frac{m_a m_b}{m_a + m_b}\right)(\mathbf{v}_a - \mathbf{v}_b)^2 \equiv \frac{1}{2}\mu\,(\mathbf{v}_a - \mathbf{v}_b)^2 \qquad (11)$$

using Eq. 8 and a little algebra. Here we call $\mu \equiv \dfrac{m_a m_b}{m_a + m_b}$ the *reduced mass* of the system of two balls. The total kinetic energy of the balls in the original reference frame is

$$\boxed{K = \frac{1}{2}MV_{CM}^2 + \frac{1}{2}\mu v_{rel}^2,} \qquad (12)$$

where $M = m_a + m_b$ is the total mass, and $v_{rel} = \left|\mathbf{v}_a - \mathbf{v}_b\right|$ is the magnitude of the relative velocity of the two balls. The first term in Eq. 12 is the kinetic energies of the balls if both moved as a unit at velocity V_{CM}, and the second term is their "internal" kinetic energy, which accounts for the difference in their velocities. The second term is the kinetic energy of a hypothetical particle of mass μ moving at speed v_{rel}.

This decomposition into two kinds of kinetic energy is quite useful. For example, it is often used for double-star systems, for the Earth-Moon system, and for molecules. The kinetic energy of a carbon monoxide (C O) molecule is the kinetic energy of the total molecular mass $m_C + m_O$ moving at the speed of the center of mass (i.e., pretending the molecule is a single particle moving at the

CM velocity), plus the kinetic energy of a particle having the reduced mass $\mu = \dfrac{m_C m_O}{m_C + m_O}$ and velocity which is the relative velocity between the two atoms. This second term is the internal kinetic energy of the molecule, which can be in the form of oscillation or rotation.

State of the Search

1. A collision that conserves kinetic energy is *time reversible*. If a video film is made of such a collision and then played back, viewers cannot be certain whether the film is being played forward or backward. These kinetic-energy conserving, time-reversible collisions are called *elastic* collisions.

2. In one-dimensional elastic collisions, the relative velocity of the two colliding objects has the same magnitude before and after the collision.

3. Other collisions are *inelastic*, in that kinetic energy is not conserved. Some of the kinetic energy is converted into internal energy of the colliding objects. This energy can be an energy of deformation, in which the shape or structure of an object is changed, or it can be heat energy, in which molecules in an object gain a larger random energy of motion. Momentum is of course conserved whether the collision is elastic or inelastic.

4. Overall energy is still conserved in inelastic collisions, even though the "mechanical energy" (macroscopic kinetic plus potential energy) decreases as a result of the collision. (In a simple model of an inelastic collision, for example, we will show in the *Special Topic* that energy is absorbed by one of the colliding objects, and stored in the form of internal kinetic energy.) In a *totally* inelastic collision the colliding objects stick together afterwards. In such a collision there is a maximal loss of kinetic energy. In fact, in the center-of-mass frame *all* the initial kinetic energy is lost in this case.

5. The total kinetic energy K of two objects a and b can be separated into two parts:

$$K = \frac{1}{2}MV_{\mathrm{CM}}^2 + \frac{1}{2}\mu v_{\mathrm{rel}}^2 .$$

The first term is the "external" kinetic energy where M is the total mass and V_{CM} is the velocity of the center of mass of the objects. The second term is the "internal" kinetic energy, the kinetic energy of the two objects in the center-of-mass frame. Here $\mu \equiv \dfrac{m_a m_b}{m_a + m_b}$ is the *reduced mass* of the system and v_{rel} is the relative velocity of the two objects.

PICTURE QUIZ 13

1. In the laboratory, a 2-kg ball moving with initial speed v_o makes a head-on elastic collision with a 3-kg ball initially at rest. Relative to the lab frame, the velocity of the center-of-mass frame is

(A) $v_o/5$ (B) $2v_o/5$ (C) $3v_o/5$ (D) $4v_o/5$ (E) v_o

2. In question 1 the velocities of the 2-kg ball and the 3-kg ball in the CM frame *after* the collision are, respectively,

(A) $3v_o/5$, $-2v_o/5$ (B) $4v_o/5$, $-v_o/5$ (C) $-v_o/5$, $4v_o/5$ (D) $-3v_o/5$, $2v_o/5$
(E) none of these

3. In question 1 the velocities of the 2-kg ball and the 3-kg ball after the collision in the *lab* frame are, respectively,

(A) $-4v_o/5$, $v_o/5$ (B) $-2v_o/5$, $3v_o/5$ (C) $-v_o/5$, $4v_o/5$ (D) $-3v_o/5$, $2v_o/5$
(E) none of these

4. In each of the scenarios shown, car L is set to have a head-on collision. If car L is at rest after the collision, in which scenario(s) was the collision totally inelastic?

(A)

(B)

(C)

PROBLEM QUIZ 13

1. A carbon nucleus with velocity v_O makes a head-on elastic collision with an oxygen nucleus initially at rest. Find the final velocities of each nucleus after the collision. The masses of the two nuclei have the ratio $M_C/M_O = 12/16$.

2. An alpha particle with initial velocity v_O makes an elastic collision with a nitrogen nucleus at rest. Afterwards the alpha particle moves at 90° to its original direction of motion. Find the final speeds of (a) the alpha particle (b) the nitrogen nucleus, and (c) find the recoil angle of the nitrogen nucleus relative to the incident direction of the alpha particle. The ratio of the nuclear masses is $M_\alpha/M_N = 4/14$.

3. A neutron of mass m and initial velocity v_O strikes a beryllium isotope of mass $9m$ moving with velocity v_O in the opposite direction. The neutron is absorbed, to form a beryllium isotope of mass $10m$. Find (a) the velocity of the new isotope (b) the fraction of the initial kinetic energy converted into internal energy in the collision.

4. A six-ball moving with speed 4 m/s strikes the eight-ball of equal mass, initially at rest on the pool table. After the collision, if the eight-ball moves at an angle of 25° to the initial velocity direction of the six-ball, at what angle does the six-ball move relative to its initial velocity direction?

5. Both stars in a particular double-star system have the same mass m. They are a distance R apart, and each rotates in a circular orbit about their mutual center of mass.

(a) Find the speed of each star in the CM frame, in terms of G, m, and R.

(b) Using these speeds, sum the total kinetic energies of the two stars in the CM frame.

(c) Find the reduced mass of the two stars, and find the magnitude of their relative velocity; that is, find $|\mathbf{v}_{rel}| = |\mathbf{v}_1 - \mathbf{v}_2|$.

(d) Find the total kinetic energy of the two stars in the CM frame using the formula $K' = \frac{1}{2}\mu v_{rel}^2$, and compare with the result of part (b).

Special Topic: Inelastic Collisions and Internal Energy

Could "internal" energy account for the missing kinetic energy in inelastic collisions? We can construct a crude model in which the target "particle" is a hollow cylinder capped at both ends, with a ball inside that can roll back and forth without friction. This allows some internal motion, as shown in Fig. S1.

Fig. S1. A ball about to strike a "particle" made of a hollow cylinder with a ball free to roll inside

Suppose that the incident ball a and the target "particle" as a whole each has mass m, except that the cylinder has mass $m/2$ and the ball inside it also has mass $m/2$. Now if a with initial velocity v_0 hits the cylinder, afterwards it will move forward with $\frac{1}{3}v_0$ and the cylinder will move forward at $\frac{4}{3}v_0$, as in *Sample Problems* 1 and 2, valid if the incident object has twice the mass of the target, and if the collision is perfectly elastic. Here the cylinder has only half the mass of ball a, and the ball inside the cylinder doesn't start moving right away. Later, when the left end of the cylinder hits the ball inside, the cylinder stops and the inside ball gets the speed $\frac{4}{3}v_0$, as shown in Fig. S2. That is because the cylinder and the inside ball have the same mass, $m/2$.

Fig. S2. Collision of a ball with the (cylinder + ball) "particle".

A bit later, the inside ball reaches the right-hand endcap; it hits it and stops, giving the cylinder all of its velocity $\frac{4}{3}v_0$. This whole process repeats itself; the sequence of events is depicted in Fig. S3. The incident ball a keeps moving forward at constant speed $\frac{1}{3}v_0$ after the original collision, but the cylinder keeps starting and stopping. When the ball inside is moving, the cylinder is at rest; when the cylinder is moving, the ball inside is at rest.

Fig. S3 Subsequent events after the initial collision.

Is energy conserved throughout this sequence? It is. The collisions are elastic every time. Before the first collision, ball a had energy $\frac{1}{2}mv_0^2$. After the collision, the energy of ball a has dropped to

$$\frac{1}{2}m\left(\frac{v_0}{3}\right)^2 = \frac{1}{2}\left(\frac{mv_0^2}{9}\right),$$

and the cylinder plus internal ball has kinetic energy

$$\frac{1}{2}\left(\frac{m}{2}\right)\left(\frac{4v_0}{3}\right)^2 = \frac{1}{2}\left(\frac{8mv_0^2}{9}\right),$$

since *either* the cylinder or the inner ball is moving at any given time, with speed $\frac{4}{3}v_0$. Therefore the total final energy is $\frac{1}{2}mv_0^2$, the same as it was before. Energy is conserved.

Now suppose instead we consider the cylinder plus inner ball to be a *single* "particle" of mass m. Its CM moves at constant speed $\frac{2}{3}v_0$ to the right, because half of it is at rest and half of it is moving at $\frac{4}{3}v_0$ at any instant. So the kinetic energy of this final "particle" is

$$\frac{1}{2}mv_{CM}^2 = \frac{1}{2}m\left(\frac{2v_0}{3}\right)^2 = \frac{4}{9}[\frac{1}{2}mv_0^2].$$

The incident particle has energy $\frac{1}{9}[\frac{1}{2}mv_0^2]$ after the collision, so looked at this way the kinetic energy has dropped to

$$[\frac{4}{9} + \frac{1}{9}]\frac{1}{2}mv_0^2 = \frac{5}{9}[\frac{1}{2}mv_0^2] ,$$

only *five-ninths* of what it was before the collision. *Four-ninths* of the kinetic energy seems to have been lost.

Can we show the $\frac{4}{9}$ x $[\frac{1}{2}mv_0^2]$ is hidden *inside* the composite "particle" of the cylinder plus inner ball? First, note that the CM of the composite "particle" moves to the right at speed $\frac{2}{3}v_0$. So in a CM frame moving to the right at the speed $\frac{2}{3}v_0$ relative to the lab, half of the "particle" (the cylinder, say) is moving to the *right* at $\frac{2}{3}v_0$, and the other half (the inner ball, say) is moving to the *left* at $\frac{2}{3}v_0$. Therefore the sum of the cylinder and inner-ball kinetic energies in that frame is

$$\frac{1}{2}\left(\frac{m}{2}\right)\left(\frac{2v_0}{3}\right)^2 + \frac{1}{2}\left(\frac{m}{2}\right)\left(\frac{2v_0}{3}\right)^2 = \frac{4}{9}[\frac{1}{2}mv_0^2] .$$

It works. We have found the four-ninths of the initial kinetic energy that seemed to be lost. All of the energy is *still there*, but some of it is "hidden" inside the particle in this model.

In a more realistic model, both the incident and target particles might have many ways to distribute the energy inside them, by oscillations, for example. The internal energy would then become randomized, as it ordinarily would rather quickly.

In summary, the total energy of our composite particles is always conserved, including the energies of all the "microscopic" parts. In *elastic* collisions the macroscopic kinetic energy is conserved, where $\frac{1}{2}mv_{CM}^2$ is the kinetic energy of each composite particle. Here v_{CM} is the center-of-mass velocity of the particle. In *inelastic* collisions, the macroscopic kinetic energy decreases, and the energy "lost" goes into internal energy of one or both of the colliding particles.

Questions and Problems from Day Thirteen

13Q1. What are some feasible techniques for measuring the initial and final velocities in the collision experiments described during *Day Thirteen*?

13Q2. If you put a so-called "jumping bean" on a table, it will jump up and down from time to time. How does it do this? Is momentum conservation violated? Is energy conservation violated?

13Q3. A video recording is made of the (nearly) elastic scattering of two pool balls. When the video is shown for a short period of time, one cannot tell if it is being run forward or backward. But suppose a video is made of a ball striking a triangular array of 15 balls. Even if the collisions are elastic, there can be little doubt whether the video is later run forward or backward. Why the difference?

13Q4. Two particles have masses m_1 and m_2, with $m_1 << m_2$. Is the reduced mass μ closer to m_1 or to m_2? A proton is about 1836 times more massive than an electron. Is the reduced mass more nearly that of the proton or of the electron?

Section 13.1 Elastic collisions

13P1. A beryllium nucleus with velocity v_0 makes a head-on elastic collision with a carbon nucleus initially at rest. Find the final velocities of each nucleus after the collision. The masses of the two nuclei have the ratio $M_{Be}/M_C = 9/12$.

13P2. A golf ball of mass m and velocity v_0 makes a head-on collision with a pool ball of mass $4m$ initially at rest. If the collision were elastic, what would be the final velocities of each ball?

13P3. A spaceship of mass $3m$, equipped with a rubber bumper, moves with speed v_0 relative to a second spaceship of mass $5m$. They undergo a head-on elastic collision. Find the final velocity of each space-ship relative to the initial rest frame of the second ship.

13P4. A toboggan and riders with total mass M starts from rest at the top of a snow-slope of height h and angle θ, as shown, and then slides down with coefficient of kinetic friction μ_k. At the bottom the toboggan runs into an innerspring mattress, and rebounds completely elastically.
(a) Find the toboggan's speed v just before it hits the mattress, in terms of h, μ_k, θ, and g.
(b) Find the maximum altitude h reached by the toboggan as it rebounds up the slope.

13P5. Find general formulas for the final velocities of two colliding balls a (of mass m_a) and b (of mass m_b), with arbitrary initial velocities $(v_a)_i$ and $(v_b)_i$. Assume the collision is elastic and takes place in one dimension.

13P6. Write Eqs. 4 and 5 in the following special cases. Explain why the result is reasonable in each case. (a) $m_a = m_b$ (b) $m_a << m_b$ (c) $m_b << m_a$.

13P7. Two small, perfectly elastic superballs are simultaneously dropped from rest at height h_o above the floor; one is above the other, with a slight separation between them. The lower ball has mass m_1 and the upper ball has mass m_2. When the lower ball strikes the hard floor, it bounces back up elastically, but is almost immediately struck by the upper ball on its way down. After the collision with the upper ball, the lower ball rests on the ground. (a) What is the ratio of masses m_2/m_1? (b) What maximum altitude does the upper ball achieve after the collision? (Neglect the ball radii and the small separation between them.)

13P8. Three perfectly elastic superballs are simultaneously dropped from rest at height h_o above the floor. They are arranged vertically in order of mass, with $M_1 >> M_2 >> M_3$, where M_1 is at the bottom. There is a small separation between them. When M_1 strikes the hard floor it bounces back up elastically, striking M_2 on its way down. M_2 then bounces back up striking M_3 on its way down. What is the subsequent maximum altitude achieved by each ball? (*Hint:* Analyze each collision in the CM frame of that collision, which is essentially the rest-frame of the heavier ball because the other is so much lighter. Neglect the ball radii and the small separation between them.)

13P9. *A proof of momentum conservation for elastic one-dimensional collisions*

Two particles of mass m_1 and m_2 collide head-on in an elastic collision. Assume the collision is elastic in all inertial frames, that the particle masses stay fixed, and that the Galilean transformation is valid between inertial frames (That is, assume the vector addition of velocity equation 4b from *Day Two*.) Prove that momentum is conserved in the collision.

Section 13.2 Exploiting time-reversibility: a pictorial approach

13P10. In the lab, a 1-kg ball moving with initial speed v_o makes a head-on elastic collision with a 2-kg ball initially at rest. Using the pictorial method, find the velocity of each ball after the collision in (a) the CM frame (b) the lab frame.

13P11. Solve 13P10, if the incident ball has a mass of 3 kg, and the target ball has a mass of 1 kg.

13P12. Solve 13P10, if the incident ball has a mass of 2 kg, and the target ball has a mass of 3 kg.

13P13. Solve 13P10, if the incident ball has a mass of 5 kg, and the target ball has a mass of 4 kg.

13P14. A Li_3^6 isotope (atomic mass 6), moving with velocity $v_o = 2.0 \cdot 10^5$ m/s makes a head-on elastic collision with a B_5^9 isotope (atomic mass 9), originally at rest in the lab. Using the pictorial method find the final velocity of each isotope after the collision in (a) the CM frame (b) the lab frame.

13P15. A He_2^3 isotope (atomic mass 3), moving with velocity $v_o = 4.0 \cdot 10^6$ m/s makes a head-on elastic collision with a Be_4^9 isotope (atomic mass 9), originally at rest in the lab. Using the pictorial method find the final velocity of each isotope after the collision in (a) the CM frame (b) the lab frame.

13P16. A He_2^4 isotope (atomic mass 4), moving with speed $v_o = 1.1 \cdot 10^6$ m/s, collides with a target isotope initially at rest in the lab. After the collision the He_2^4 isotope is observed to be moving backward (directly opposite to its initial velocity direction) with speed $v_f = 0.30 \cdot 10^6$ m/s. Using the pictorial method, find the target isotope's (a) atomic mass; (b) final speed.

13P17. A Li_3^6 isotope (atomic mass 6), moving with speed $v_o = 8.8 \cdot 10^6$ m/s, collides with a target isotope initially at rest in the lab. After the collision the Li_3^6 isotope is observed to be moving backward (directly opposite to its initial velocity direction) with speed $v_f = 2.2 \cdot 10^6$ m/s. Using the pictorial method, find the target isotope's (a) atomic mass; (b) final speed.

13P18. (a) A neutron in a nuclear reactor has a head-on elastic collision with a Carbon nucleus, part of the graphite "moderator". The Carbon nucleus is initially at rest, and has 12 times the mass of a neutron. What fraction of the neutron's initial speed is lost in the collision? (b) If a neutron collides head-on with a deuteron ($m_d = 2m_n$) instead, a nucleus within a heavy-water (D_2O) molecule used as a moderator in some reactors, what fraction of the neutron's initial speed is lost? (Moderators are used to slow down neutrons, i.e. to moderate their speeds. Slower neutrons are more likely to cause nuclear fission.)

Section 13.3 Elastic collisions in three dimensions

13P19. A neutron with initial velocity v_o makes an elastic collision with a boron isotope initially at rest. After the collision the neutron moves at 90° relative to its incident direction. Find the final speeds of (a) the neutron, and (b) the boron isotope. (c) Find also the angle at which the boron nucleus moves relative to the incident direction of the neutron. The mass ratio of these particles is $M_n / M_B = 1/10$.

13P20. An alpha particle makes an elastic collision with another alpha particle, initially at rest in the lab. After the collision, the incident alpha particle moves with velocity v, at a 30° angle relative to its incident direction. In terms of v, find (a) the final velocity of the target alpha, and the angle it moves relative to the incident direction. (b) the initial speed of the incident alpha particle. (Both particles have the same mass.)

13P21. In 1911, scattering experiments carried out by Rutherford and coworkers showed that gold atoms contain a very small nucleus having nearly all the mass of the atom. In a general collision of an α–particle with a gold nucleus at rest, the α-particle has momentum p_O before and p_α after, and emerges at angle θ to the initial forward direction. The gold nucleus recoils with momentum p_{AU} at angle ϕ to the forward direction.

(a) Conserving momentum in the x-direction and in the y-direction, find two equations relating the momentum and angle variables introduced above.

(b) In vector form, momentum conservation is $\mathbf{p}_O = \mathbf{p}_\alpha + \mathbf{p}_{AU}$, giving (by vector addition) the triangle shown. Which angle in the triangle is θ? Which angle is ϕ?

(c) Using the *law of cosines*, find the magnitude of the gold nucleus' momentum p_{AU} in terms of θ and the magnitudes p_O and p_α. (d) Using *conservation of energy*, find the magnitude of the gold nucleus' momentum p_{AU} in terms of p_O, p_α, and the masses.

(e) Eliminate p_{AU} between (c) and (d), and so find an expression for the final momentum of the α-particle in terms of p_O and θ.

(Note that the three conservation laws, for p_x, p_y, and E, allow us to solve for any three of the original unknowns θ, ϕ, p_α, and p_{AU} in terms of the fourth. So in particular if we fix θ, the angle of the emerging α-particle, we can determine ϕ, p_α, and p_{AU}.)

13P22. In Rutherford scattering experiments, α-particles are fired at stationary nuclei. Some of the experiments carried out in Rutherford's laboratory in Manchester, England, used α-particles emitted by the radioactive Radium isotope Ra C, with initial velocity $v_O = 2.06 \times 10^7$ m/s. The α-particles collided elastically with stationary gold nuclei in a gold foil. Geiger and Marsden found that α-particles emerged with scattering angles θ over the full range from 0° to 180°. Consider a particular α-particle that bounces directly backwards, i.e. with a scattering angle $\theta = 180^\circ$. Find the speeds v_α of the α-particle and v_{AU} of the gold nucleus after the collision. The masses of α-particles and gold nuclei are 4.00 u and 197 u, respectively, where 1 u = 1 "atomic mass unit" = 1.66×10^{-27} kg.

13P23. Consider a Rutherford scattering experiment, as described in 13P21, in which the α-particle is scattered at 90° relative to its incident direction. Find the speed v_α of the α-particle and the speed v_{AU} and recoil angle ϕ of the gold nucleus (relative to the forward direction) after the collision.

13P24. The bubble-chamber photograph reproduced on p. 13-11 shows that in the collision of two (equal-mass) protons, the final velocities are close to being perpendicular to one another. This is expected for scattering in a plane, but almost certainly some of the scattering events shown are not in the plane of the photograph itself. Why then does the photo show nearly perpendicular velocities, even though it is only projections we see?

13P25. A ball of mass M strikes a target ball of mass $M/2$, originally at rest, in a perfectly elastic collision. Show that the maximum angle through which M can be scattered from the forward direction is $\theta = 30^{\circ}$.

13P26. A ball of mass M strikes a ball of mass m, originally at rest, in a perfectly elastic collision, where $M > m$. Show that the maximum angle through which M can be scattered from the forward direction is $\theta = \sin^{-1}(m/M)$ *Hint*: First view the collision in the CM frame before and after, and then transform the final picture back to the lab frame.

13P27. *A proof of momentum conservation for elastic collisions in general.*

Using the same assumptions as given in 13P9, except that the collision need not be one-dimensional, prove that momentum is conserved in the collision.

Section 13.4 Inelastic collisions

13P28. (a) A 1-kg ball with speed v_0 collides with a 2-kg ball initially at rest. If the collision is completely *inelastic*, find the speed v of the balls after the collision. (b) What fraction of the initial kinetic energy is lost in the collision? (c) Repeat parts (a) and (b) for a completely inelastic collision of two balls with arbitrary masses m_a and m_b. Initially ball b is at rest, and ball a has speed v_0.

13P29. A truck of mass M and speed v_0 comes up from behind and collides with a car of mass $m = M/2$ moving in the same direction with speed $v_0/3$. The two are stuck together as a consequence of the collision. (a) What is the velocity of the system immediately after the collision? (b) What fraction of the initial kinetic energy of the two vehicles is lost in the collision?

13P30. A bullet of mass m is fired at speed v_0 along the symmetry axis of two washers, each of mass $m/4$, as shown. (a) After picking up the washers, how fast is the system bullet + washers moving? (b) What fraction of the initial kinetic energy is lost in the collision?

13P31. A neutron (n) with initial velocity v_0 collides with a carbon isotope (C^{14}) and is absorbed. (a) What is the velocity of the resulting isotope C^{15}? (b) What fraction of the initial kinetic energy is converted into internal energy in the collision? The mass ratios are $M_{C14}/M_n = 14$, $M_{C15}/M_n = 15$.

13P32. A Λ (Lambda) -particle of mass M_Λ decays into a proton of mass M_p and a negative pion of mass M_π:

$$\Lambda \rightarrow p + \pi^-.$$

(a) Suppose the initial Λ is at rest, and that momentum is conserved in the decay. Find $(K.E.)_p / (K.E.)_\pi$ in terms of M_p and M_π. (b) If a proton and pion of just the right velocities were fired at one another, the reaction $p + \pi^- \rightarrow \Lambda$ could happen. Is such a collision of the p and π^- *elastic* or *inelastic*?

13P33. A neutron collides with a U_{92}^{235} nuclear isotope, originally at rest. If the neutron is completely absorbed, forming U_{92}^{236}, what fraction of the original kinetic energy is converted into internal energy? The masses are $M_{\text{neutron}} = 1$ u and $M_{U_{92}^{236}} = 236$ 236 u, where "u" stands for *atomic mass unit.* (1 u = 1.66 x 10^{-27} kg) The internal energy of the U_{92}^{236} isotope goes partly into oscillations which will likely cause the isotope to break up (fission) into two lighter isotopes and several neutrons).

13P34. A neutron collides with a U_{92}^{238} nuclear isotope, originally at rest. (a) If the neutron is completely absorbed, forming U_{92}^{239}, what fraction of the original kinetic energy is converted into internal energy? (a) Find the answer in terms of the neutron mass M_n and U^{239} mass M_{U239}. (b) Find the numerical answer, using the measured masses $M_n = 1.0087$ u and $M_{U239} = 239.05$ u, where "u" stands for *atomic mass unit.* (1 u = 1.66 x 10^{-27} kg)

13P35. Starting from rest, a block of mass m slides without friction down a quarter-circle ramp of radius r, as shown. At the bottom of the ramp it strikes a second block of mass $2m$, initially perched at rest a distance R above the floor. If the collision between the two blocks is perfectly elastic, i.e. if the total kinetic energy is conserved in the collision, (a) Find the distance ℓ_a from the base of the cliff where the mass $2m$ strikes the floor. (b) Find the distance ℓ_b where the mass m ultimately strikes the floor. (c) If instead the collision is perfectly inelastic, i.e. if m and $2m$ stick together upon colliding, find the distance ℓ_c where the together strike the floor.

13P36. A block of mass M is suspended from a string of length ℓ. A bullet of mass m is then fired with speed v_o into the block, as shown. (a) If the bullet becomes lodged in the block, what is the maximum angle θ achieved by the string, measured from the vertical? b) What fraction of the initial kinetic energy of the bullet is converted into internal energy?

13P37. When two pool balls collide head-on, the ratio of their relative speeds *after* the collision to their relative speeds *before* the collision is $|v_2 - v_1|_a / |v_2 - v_1|_b \equiv R$, called the *coefficient of restitution.* ($R = 1$ if the collision is elastic, as explained in Section 13.1.) (a) In such a collision of equal-mass balls in which one ball is initially at rest and the other has velocity v_o, find the final velocities v_1 (of the incident ball) and v_2 (of the target ball) in terms of v_o and R. (b) Find the fractional change in the total kinetic energy, $(K_o - K_f)/K_o$ in terms of R, where K_f is the sum of the final kinetic energies. (c) Check your results in the limiting cases $R = 0$ and $R = 1$.

13P38. A ball dropped from rest at height h_0 above the floor bounces back to height h_1. (a) Find the ratio h_1/h_0 in terms of the coefficient of restitution R when the ball bounces. (The coefficient of restitution is the ratio of the relative speeds of two colliding objects just after a collision to their relative speeds just before the collision.) (b) If R is the same for each bounce, how high will the ball rise after N bounces, in terms of h_0 and R? (c) What fraction of the ball's initial kinetic energy just before a bounce is lost in each collision with the floor?

13P39. A particle of mass m traveling along the x-axis with speed v_0 collides with a particle of mass $5m$ initially at rest. After the collision, the first particle moves at a right angle to the incident direction, while the target particle moves off at angle θ, with $\sin\theta = 3/5$, as shown. Find (a) the speeds v_m and v_{5m} of the two particles after the collision. (b) the fraction of the initial energy absorbed in the collision.

13P40. An 1812-ton 10-car passenger train moving at 32 miles/hour strikes a 2-ton automobile stalled on the tracks. How much is the train slowed if the collision is (a) completely elastic? (b) totally inelastic? (c) If the collision is totally inelastic, what fraction of the train's initial kinetic energy is converted into internal energy of the auto and train?

13P41. A massless spring is compressed by an amount Δx between two blocks of masses m_1 and m_2 sitting on a frictionless surface, as shown. The spring is unattached to either block. The blocks are released from rest, and the spring pushes them apart. Immediately thereafter m_1 travels to the left on a surface with kinetic

coefficient of friction μ_K, and comes to rest after traveling a distance d. Block m_2 travels to the right on a frictionless surface, and then strikes and sticks to a third block of mass m_3 hung from a ceiling by a massless rope of length ℓ. The two blocks swing together up to a maximum angle θ, as shown. In terms of gravity g and given parameters, find (a) the angle θ (b) the spring constant k.

13P42. *A proof of momentum conservation for elastic or inelastic collisions*

In an inelastic collision of two particles, the final kinetic energy of the particles is ΔE less that their initial kinetic energy. If the internal energy ΔE is the same in all inertial frames, if the particle masses stay fixed, and if the Galilean transformation is valid between inertial frames, prove that momentum is conserved in the collision.

13P43. A ball of mass $m_1 = 52$ g and velocity $v_1 = 13$ cm/s makes a head-on collision with a ball of mass $m_2 = 76$ g and velocity $v_2 = -37$ cm/s. (a) Find the total kinetic energy of the system by summing the individual kinetic energies. (b) Find the "external" kinetic energy of the system, using the total mass and the velocity of the center of mass. (c) Find the "internal" kinetic energy, using the reduced mass and the relative velocity of the two balls. (c) Show that the sum of the "external" and "internal" kinetic energies is equal to the kinetic energy found in part (a).

13P44. (a) Show that the reduced mass μ of two objects with masses m_1 and m_2 obeys the equation $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$. (b) Prove that μ is smaller than either mass, as long as neither mass is zero. (c) Find the reduced mass of the NO molecule in atomic mass units (u), where the atomic masses are $m_N = 14$ u and $m_O = 16$ u.

13P45. The gravitational potential energy of two stars with masses m_1 and m_2 is $U(r) = -Gm_1m_2/r$, where r is the distance between them. Show that this potential energy can also be written $U(r) = -GM\mu/r$, in terms of their total mass M and reduced mass μ. Therefore the total internal (CM frame) energy of a double-star system is $E = \frac{1}{2}\mu v^2 - \frac{GM\mu}{r}$ (where v is the relative velocity of the two stars), as though the system were a single star of mass μ orbiting a fixed star of mass M. This shows that the "two-body problem" of two stars, both moving in response to the other, can be reduced to an equivalent "one-body problem".

13P46. A planet and its moon have masses m_P and m_M; they are a distance r apart. Each moves in a circular orbit about their mutual center of mass. (a) Find the distance r_P of the CM from the center of the planet, and the distance r_M of the CM from the center of the moon, in terms of the masses and the total distance r. (b) Using $F = ma$ and the inverse-square-law Newtonian gravitational force between them, find the speed of each body about the CM. (c) Find the total kinetic energy of the objects in the CM frame, by summing the kinetic energies of the two bodies. (d) Find the system's reduced mass μ and relative velocity v_{rel}. (e) Find the CM-frame kinetic energy of the single abstract object μ moving with speed v_{rel}, and compare with the result found in part (c).

Special topic: Inelastic Collisions and Internal Energy

13P47. An incident ball of mass m and velocity v_0 strikes a target "particle" at rest. The "particle", of mass $2m$, consists of a hollow cylinder of mass m containing a ball of mass m. A similar system is illustrated in Fig. S1.
(a) Draw a set of pictures analogous to Figs. S2 and S3, showing what happens following the collision. (b) What fraction of the initial kinetic energy is converted into "internal" energy, considering the cylinder and contained ball to be a single composite particle of mass $2m$, moving at the CM speed of the composite?

Day Fourteen

What have we learned about motion here on Iceworld?

On the surface of the planet there is a gravitational force $\mathbf{F} = m\mathbf{g}$ on an object of mass m, where $\mathbf{g}$ is the *gravitational field.* The field $\mathbf{g}$ points downward. Unlike other forces caused by something *touching* the object, like string tension, springs, normal (surface) forces, and friction, the gravitational pull is somehow able to reach out through empty space.

We found the behavior of projectiles in uniform gravity. If gravity is the *only* force acting, the acceleration of the projectile is downward and has a constant magnitude. Therefore the projectile moves with constant *velocity* in the *horizontal* direction, with

$$v_x = v_{ox} \qquad \text{and} \qquad x = x_o + v_{ox}t\,,$$

and with constant *acceleration* $a = -\,g$ in the *vertical* direction (where the positive direction is upward), so that

$$v_y = v_{oy} - gt \qquad\qquad y = y_o + v_{oy}\,t - \frac{1}{2}gt^2$$

$$y = y_o + \left(\frac{v + v_o}{2}\right)t \qquad\qquad v^2 = v_o^2 - 2g(y - y_o)\,.$$

One consequence of these equations is that the projectile moves in a parabolic path.

We found approximate empirical rules for surface friction, both static and sliding, and thought a bit about air resistance. Static friction and sliding (or kinetic) surface friction on an object both depend upon the object and the surface in which they are in contact, through coefficients of friction μ_s and μ_k. Both forces are proportional to the normal force N between the object and surface. Static friction F_s cannot exceed $\mu_s N$, i. e.,

$$F_s \leq \mu_s N,$$

and sliding (kinetic) friction obeys

$$F_k = \mu_k N,$$

directed opposite to the direction of motion of the object. The air resistance on an object depends upon the object's velocity, and is directed opposite to its direction of motion.

We defined the *work* W done by a force F on a particle, as the particle moves in one dimension from x_a to x_b, to be

$$W[a \to b] = \int_{x_a}^{x_b} F\,dx,$$

the integral of the force over the path. If the force happens to be constant (a constant gravitational force $F = mg$, for example), the work integral becomes simply

$$W[a \to b] = F\,\Delta x,$$

the product of the force and the displacement $\Delta x = x_b - x_a$. Work has the dimensions of Newtons · meters, or *Joules*.

If a particle has a general path in two or three dimensions (or even in one dimension), the work done by a force $\mathbf{F}$ on the particle as it moves from point a to point b is

$$W[a \to b] = \int_a^b \mathbf{F} \cdot d\mathbf{s}$$

where the infinitesimal vector $d\mathbf{s}$ has a *length* equal to an infinitesimal distance moved along the path, and a *direction* in the direction of motion of the particle. The quantity $\mathbf{F} \cdot d\mathbf{s}$ is the *dot product* of the vectors $\mathbf{F}$ and $d\mathbf{s}$.

The dot product of *any* two vectors $\mathbf{A}$ and $\mathbf{B}$ is a scalar quantity,

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta,$$

the product of the magnitudes of the two vectors multiplied by the cosine of the smaller of the two angles between the two vectors, as illustrated. Note that $\mathbf{A} \cdot \mathbf{B}$ is the length of $\mathbf{A}$ multiplied by the component of $\mathbf{B}$ that is parallel to $\mathbf{A}$, and also the length of $\mathbf{B}$ times the component of $\mathbf{A}$ parallel to $\mathbf{B}$. In terms of the components of $\mathbf{A}$ and $\mathbf{B}$, the dot product can also be written in the useful form

$$\mathbf{A} \cdot \mathbf{B} = A_X B_X + A_X B_X + A_X B_X.$$

Starting with $\mathbf{F} = m\mathbf{a}$, we derived a relationship between the total work done on a particle and the change in the particle's *kinetic energy*. The result is the *work-energy theorem*

$$W_{\text{Total}}[a \to b] = K_{\text{b}} - K_{\text{a}},$$

where W_{Total} is the total work done by *all* forces acting on a particle of mass m as it moves from a to b, and K_{a} and K_{b} are the kinetic energies of the particle at a and at b. The kinetic energy at any point is given by

$$K = \tfrac{1}{2}mv^2,$$

where v is the speed of the particle at that point.

If the work done by a force $\mathbf{F}$ on a particle as it moves between two given points is independent of the path between those two points, then the force $\mathbf{F}$ is *conservative*. Examples include the gravitational force and the spring force. (Frictional forces are *non*conservative.) The work done by any conservative force depends *only* on the endpoints a and b(and not on the path taken between them) and can be written

$$W[a \to b] = \int_a^b \mathbf{F} \cdot d\mathbf{s} = -U_{\text{b}} + U_{\text{a}}$$

where U_{a} and U_{b} are the *potential energies* at the endpoints. It follows that the potential energy for a particular force $\mathbf{F}$ can be found by calculating the indefinite integral

$$U(\mathbf{r}) = -\int^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{s}$$

which in one dimension becomes simply $U(x) = -\int^x F\,dx$.

An example is

$$U_{\text{grav}} = mgh$$

for uniform gravity, where g is the magnitude of the gravitational field and h is the altitude above some arbitrary point, chosen for convenience. Another example is the

potential energy of a Hooke's-law spring (a spring that exerts a linear restoring force $F = -kx$, where k is the *force constant* of the spring and x is the spring stretch away from equilibrium); in this case

$$U_{\text{spring}} = \tfrac{1}{2}kx^2.$$

In one dimension, all forces that depend *only* on position (and not on velocity or time) are conservative, so a potential energy function exists is such cases. Conversely, if the potential energy $U(x)$ for some force is known, the force can be found from

$$F(x) = -\frac{dU(x)}{dx}.$$

An example of a three-dimensional conservative force is the gravitational force of one spherically symmetric body of mass M on another of mass m. It is given by the inverse-square law

$$\mathbf{F}_{\text{grav}} = -\frac{GMm}{r^2}\hat{\mathbf{r}},$$

where G is Newton's gravitational constant, r is the distance between the centers of the two objects, and $\hat{\mathbf{r}}$ is a unit vector pointing away from M. The minus sign means that the force is attractive, in the $-\hat{\mathbf{r}}$ direction. The corresponding gravitational potential energy of the bodies is

$$U_{\text{grav}}(r) = -\frac{GMm}{r}.$$

There is no potential energy for friction, because friction is a nonconservative force.

For conservative forces, for which the work is $W[a \to b] = \int_a^b \mathbf{F} \cdot ds = -U_b + U_a$, the work-energy theorem $W_{\text{Total}}[a \to b] = K_b - K_a$ shows that

$$K_a + U_a = K_b + U_b$$

so the *energy* (or *mechanical energy*)

$$E = K + U$$

is *conserved*, in that it stays constant in time. Here U is the sum of all the potential energies if the particle is subject to more than one conservative force. Conservation of energy is a very important and useful law. If a nonconservative force like friction acts on the particle (or on systems of particles) then E is not conserved. Likewise if we do outside work on a system of particles, by a force like our hand for which we cannot write a potential energy, then energy is not conserved. Note that *all* energies have the dimensions of Joules.

The concept of mechanical energy $E = K + U$ can be broadened to include other kinds of energy so that the total energy of an isolated system is always conserved.* For example, if friction acts on a system (so that the mechanical energy typically decreases with time) we find that heat is produced by the rubbing. If we assign a certain amount of energy to heat, then total energy, including heat energy, is conserved. Other kinds of energy include energy of deformation, or the energy in sound waves or light waves. It is interesting and useful to figure out how to transform one kind of energy into another; in many respects our civilization and life itself is based on such transformations.

Elastic collisions are those in which the total kinetic energy of the colliding objects is conserved before and after. In the one-dimensional elastic collision of two objects the relative velocity of the objects has the same magnitude before and after. Elastic collisions are time reversible. Inelastic collisions are those in which some of the initial kinetic energy is converted to heat, or deformation, or some other form of energy. Totally inelastic collisions are those in which there is a maximum loss of kinetic energy consistent with momentum conservation. In totally inelastic collisions the two objects stick together after the collision.

The center-of-mass frame of a collision is the frame in which the total momentum of the colliding objects is *zero*. The total kinetic energy of two objects with masses m_1 and m_2 (in any frame) can be separated into two parts,

$$K = \frac{1}{2} M V_{\mathrm{CM}}^{\ 2} + \frac{1}{2} \mu v_{\mathrm{rel}}^{\ 2}$$

where $M = m_1 + m_2$ is the total mass, $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ is the "reduced mass", V_{CM} is the speed of the CM frame relative to our laboratory frame, and v_{rel} is the magnitude of the relative velocity of the two objects.

*According to *quantum mechanics*, measurements can show that E is not conserved, but only for very short time intervals $\Delta t \sim \hbar / \Delta E$, where $\hbar$ is Planck's constant and ΔE is the energy nonconservation.

SELF QUIZ 14

This is a review quiz. Solutions are at the end of the text.

1. A projectile is fired at speed v_0 with elevation angle θ_0 in uniform gravity g. Use dimensional arguments alone to answer the following questions:
(A) If θ_0 and g are kept constant while v_0 is doubled, by what factor is h_{max} increased, where h_{max} is the maximum height attained by the projectile?
(B) If θ_0 and v_0 are kept constant while g is cut in half, by what factor is h_{max} increased?

2. A block of mass m slides on a horizontal frictionless surface, under the influence of a spring of force-constant k, as shown. The block oscillates back and forth, alternately stretching and compressing the spring. A, B, and C are the points where the spring is most compressed, in equilibrium, and most stretched, respectively. Find the instantaneous power input to the block from the spring at point (1) A (2) B (3) C.

3. A particle of mass m = 0.50 kg moves with the potential energy shown.
(a) The particle is placed at rest at $x = 0$. What is its total energy in Joules?
(b) The particle is then released at time $t = 0$. What is its speed v when in reaches point b, located at $x = 2.0$ m?
(c) At what time does the particle reach point d, located at $x = 6.0$ m?
(d) At what time does the particle *next* reach point d?

4. A woman of mass m_0 is standing inside an elevator of mass $2m_0$, supported by a light cable, as shown. The cable hangs over a frictionless pulley of negligible mass, and the other end of the cable is attached to a counterweight of mass $2m_0$. The counterweight is held in place until time $t = 0$, when it is released, allowing the elevator and counterweight to move.
(a) Draw isolation diagrams of (1) the counterweight; (2) the elevator plus passenger as a unit; (3) the passenger by herself.
(b) Find the acceleration of the elevator, as a multiple of g.
(c) Find the weight of the passenger in the falling elevator. That is, if she were standing on a calibrated scale inside the elevator, what would the scale read, in terms of m_0 and g?

5. A small sled of mass m starts from rest at A in the figure, which is at altitude h above the bottom of an icy frictionless track. At the bottom, the sled enters a frictionless circular track of radius R, and slides at least up to B as shown, located 30° above the center of the circular arc.

(a) As the sled slides from A to B, what is the total work done on it due to the normal force? (b) What is the total work done on it due to the gravitational force? (c) Find an expression for the speed v of the sled at point B, in terms of any or all of m, g, h, and R. (d) Draw a complete isolation diagram of the sled when it is at point B. (e) Find the value of the initial altitude h, expressed as a multiple of R, which will cause the sled to start coming off the track at point B.

6. Suppose that instead of being frictionless, there is a coefficient of kinetic friction μ_k = 0.50 between the sled and the straight portion of the initial slope shown in problem 5, and that because of friction the sled's kinetic energy at the bottom of the straight portion, an altitude ℓ below the starting point, is half what it would have been without friction.

(a) What is the angle θ of this slope relative to the horizontal, in degrees?
(b) After leaving the frictional portion, the sled enters a frictionless circular arc of half-angle ϕ, as shown. Prove that $\phi = \theta$, if the slope of the arc at the left matches that of the straight initial slope. (c) The sled then projects into space, landing a distance D from the lip of the circular arc. Find D as a multiple of ℓ.

7. A neutron of mass m moves with speed v_o within a nuclear reactor. It collides with a deuteron of mass $2m$, originally at rest in the reactor, so that the neutron emerges from the collision with velocity v_n at right angles to its incident direction, and the deuteron recoils at angle θ to the forward direction with speed v_d, as shown. (The deuteron is within a heavy-water molecule D_2O; heavy water is used to slow down neutrons.)

(a) Show that $\tan\theta = v_n/v_o$, by conserving momentum in (i) the x-direction (ii) the y-direction.
(b) Eliminate θ between the two momentum-conservation equations to find v_d in terms of v_o and v_n alone.
(c) The collision is also elastic. Find (i) v_n, as a multiple of v_o; (ii) v_d, as a multiple of v_o; (iii) θ, in degrees.

8. A motor pulls a car of mass M vertically upward against gravity by means of an attached cable. Find the power input from the motor to the car as a function of time, if the car has (a) constant upward speed v; (b) constant upward acceleration a, with velocity $v = 0$ at time $t = 0$. (c) When the car has reached height h, the cable breaks. How much work is done by gravity as the car falls back to the ground?

What's Next?

The explorers have already discovered for themselves many, even most of the *fundamental* laws of classical particle mechanics. This includes the three laws of Newton, namely the law of inertia, the dynamical law $\mathbf{F} = d\mathbf{p}/dt$, and the law that for every force there is an equal but opposite counterforce. They have also discovered the conservation laws of momentum and of energy. What remains?

The next important step is to find the laws of rotational motion. What causes rotation, how is it described, and what are its effects? Wheels, planets, stars, gyroscopes all rotate. Can we understand and predict their behavior?

Also worth investigating is the important case of oscillating objects, such as a mass oscillating on the end of a spring, and hundreds of applications to mechanical systems and analogous electrical circuit oscillations. Waves involve oscillations of vast numbers of particles, from particles in a rope to water molecules in a water wave to air molecules in a sound wave. Wave motion is also important in understanding the behavior of light and even of individual particles like electrons and protons.

It is also fascinating to explore the *limits* of classical mechanics. Are its laws always valid, or do they break down in extreme cases? For example, are they valid for particles moving close to the speed of light? Are they valid for submicroscopic particles like electrons in a hydrogen atom? Are there limitations to the laws that were discovered from collisions and other experiments involving objects about our size, and which don't move very fast, when we try to extend them to domains far from common experience?

It turns out that there *are* limitations, and the laws of relativity and quantum mechanics discovered in the twentieth century have to be used to understand many physical systems. They are necessary in atomic, nuclear, and particle physics, for example, and in many astrophysical systems, the study of the large-scale history and structure of the universe, and in solids, including electronic devices and computer chips.

Even though the laws of classical mechanics are not absolute, they are still valid for a wide range of situations, and they serve as the paradigm of what physical laws are all about.

Trigonometry

For a *right* triangle,

the Pythagorean Theorem is $a^2 + b^2 = c^2$.

Define: $\sin\theta = a / c$, $\cos\theta = b / c$, $\tan\theta = \sin\theta / \cos\theta = a / b$,

$\cot\theta = \cos\theta / \sin\theta = b / a$, $\sec\theta = 1 / \cos\theta = c / b$, $\csc\theta = 1 / \sin\theta = c / a$,

from which follow the identities:

$\sin^2\theta + \cos^2\theta = 1$ $\qquad$ $\sec^2\theta - \tan^2\theta = 1$ $\qquad$ $\csc^2\theta - \cot^2\theta = 1$.

Given one trig function, what are the others? For example, if $\sin\theta = A$, what is $\cot\theta$?

Method 1: Use identities: e.g. $\cos^2\theta = 1 - \sin^2\theta = 1 - A^2$, so $\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{\sqrt{1-A^2}}{A}$.

Method 2: (a) Draw a triangle, labeling sides to make $\sin\theta = A$: (b) Find the third side from Pythagoras: (c) Read off $\cot\theta = \dfrac{\sqrt{1-A^2}}{A}$.

Other identities:

$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$ $\qquad$ $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

$\sin^2\theta = (1/2)(1 - \cos 2\theta)$ $\qquad$ $\cos^2\theta = (1/2)(1 + \cos 2\theta)$

For an *arbitrary* triangle, with $\alpha + \beta + \gamma = 180°$,

law of sines: $(\sin\alpha) / a = (\sin\beta) / b = (\sin\gamma) / c$

law of cosines: $a^2 = b^2 + c^2 - 2bc\cos\alpha$

Solutions to Quiz Problems

Day One

PICTURE QUIZ 1

1. In (a) and (e) the Bubble is nonrotating; in (b) and (c) the Bubble is rotating counterclockwise; in (d) and (f) the Bubble is rotating clockwise.

2. (a) The ball is at rest when the velocity $v = dx/dt = 0$, where the slope of the position curve is zero. This includes points B and F. (b) The slope is positive at A, G, and H. (c) The slope is negative at C, D, and E. (d) The acceleration is zero where there is no curvature in $x(t)$, because the acceleration is the second derivative of $x(t)$. These points appear to include C, D, and F. (e) The acceleration $a = d^2x/dt^2$ is positive at E, G, and H. (f) The acceleration is negative at A and B. That is where the ball accelerates toward the left.

3. The answer is (b). The displacement vector is the vector that extends from the initial location to the final location. It has components $\mathbf{D} = (13 - (-27))\,\hat{\mathbf{x}} + (-18 - 32)\,\hat{\mathbf{y}} + (-49 - (-79))\,\hat{\mathbf{z}}$ light- years $= 40\,\hat{\mathbf{x}} - 50\,\hat{\mathbf{y}} + 30\,\hat{\mathbf{z}}$ light-years. The length of this vector is $D = \sqrt{30^2 + 40^2 + 50^2} = 71$ light-years. Note that we moved along a path of distance $vt = (c/3)(360 \text{ years}) = 120$ light-years, which must be a circuitous path, because the displacement is only 71 light-years.

4. Both (c) and (d) have the correct sum $\mathbf{S}$, but only (c) has the correct difference $\mathbf{D}$. So the correct answer is (c).

PROBLEM QUIZ 1

1. (a) The velocity $\Delta x/\Delta t$ at $t = 2$ s is 1 m/s. (b) 0 (c) $-(2/3)$ m/s (d) $(1/2)$ m/s

2. The vector (a) $\hat{\mathbf{x}} + \hat{\mathbf{x}} + \hat{\mathbf{x}}$ has magnitude 3, the vector (b) $\hat{\mathbf{x}} + \hat{\mathbf{y}}$ has magnitude $\sqrt{1^2 + 1^2} = \sqrt{2}$, and the vector (c) $\hat{\mathbf{x}} + \hat{\mathbf{y}} - \hat{\mathbf{z}}$ has magnitude $\sqrt{3}$. Therefore (a) has the greatest magnitude and (b) the least.

3. If $\mathbf{A}$ and $\mathbf{B}$ are aligned:

the magnitude of $\mathbf{A} + \mathbf{B}$ is 5; if they are antialigned:

the magnitude is 1. Any other orientation:

gives an intermediate value. The range of possible magnitudes of $\mathbf{A} - \mathbf{B}$ is the same.

4. (a) The displacement vector is a diagonal of the cube. The length of the vector is

$$\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} = \sqrt{L^2 + L^2 + L^2} = \sqrt{3}L.$$

The path taken to get to the destination doesn't matter, as far as the displacement vector is concerned. (b) The displacement vector is zero in this case.

Day Two

PICTURE QUIZ 2

1. The answer is (a), 140 km/hr. The teacup is thrown toward the rear from the point of view of passengers, so it will not be traveling as fast relative to the ground as the train itself.

2. The answer is (c). The vertical component of the velocity is the same in both frames, $200\sqrt{2}\cos 45^\circ$ km/hr = 200 km/hr. The horizontal component is 200 km/hr relative to the ground, but zero relative to the engineer, because the train itself is moving in the same direction at 200 km/hr. So the engineer will see the bounced hailstone moving vertically upward with speed 200 km/hr.

3. The answer is (e). There is no wind, so the snowflakes are falling vertically in the ground frame of reference. The vertical component of velocity v_y of the snowflakes (which is the same in both the ground frame and the bus frame) must be such that $\tan 30^\circ = 30$ mi/hr / v_y, as shown. Therefore $v_y = 30$ mi/hr / $\tan 30^\circ = 30\sqrt{3}$ mi/hr. The speed of snowflakes relative to the bus is the square root of the sum of squares of velocity components v_x and v_y, $\sqrt{30^2 + (30\sqrt{3})^2}$ mi/hr = $\sqrt{4\cdot(30)^2}$ mi/hr = 60 mi/hr.

4. (a) to the right (b) to the left (c) constant velocity (d) to the left

PROBLEM QUIZ 2

1. Train A moves to the right relative to train B, so A's frame is primed, and B's frame is unprimed. The velocity between the two frames is $V = 50$ m/s. The flashbulb flashes at $x' = 40$ m at time $t = 10$ s, so the transformation $x' = x - Vt$ gives

$$x = x' + Vt = 40 \text{ m} + 50 \text{ m/s} \cdot 10 \text{ s} = 540 \text{ m},$$

which is the location of the flash in B's frame

2. The spaceship is the primed frame (since it moves to the right), so

$$\mathbf{v}' = \mathbf{v} - \mathbf{V} = \mathbf{v} + (-\mathbf{V})$$

as shown, where $V = v_s$. The rocks approach the spaceship with speed $\sqrt{v_r^2 + v_s^2}$, and the angle relative to the forward direction is $\theta = \tan^{-1}(v_r/v_s)$.

3. The water frame is primed, since it is moving to the right. Hence the kayak's speed is $v' = v_k = 3$ m/s. The relative speed between the water frame and the shore is $V = v_r = 1$ m/s. Relative to the shore, $\mathbf{v} = \mathbf{v}' + \mathbf{V}$, which must be straight across the river, as shown. Therefore $\sin\theta = 1/3$, so $\theta = \sin^{-1}(1/3)$.

Day Three

PICTURE QUIZ 3

1. (B) is the correct answer. Conserve momentum in the frame of the tracks. The initial momentum is

$$p_o = (M + m)v_o,$$

and the final momentum is

$$p_f = mv_c + M(-v_o)$$

where v_c is the velocity of the cannonball relative to the tracks afterwards, and the minus sign comes from the fact that the cannon and car are moving to the left afterwards. So

$$(M + m)v_o = mv_c - Mv_o, \text{ giving } v_c = (2M + m)v_o/m.$$

2. The answer is (a), *Yes*. Conserving momentum with A initially at rest,

$$(p_B)_i = (p_A)_f + (p_B)_f, \quad \text{so} \quad (p_A)_f = (p_B)_i - (p_B)_f.$$

Therefore $(p_A)_f > (p_B)_i$ for any collision in which $(p_B)_f$ is negative, i.e., for any collision in which the incident ball B bounces back after the collision, as shown.

When does this happen? When ball B is less massive than ball A. If a ping-pong ball is thrown at a basketball in a head-on collision, the ping-pong ball bounces back and the basketball gains a forward momentum that is larger than the initial momentum of the ping-pong ball. *Moral:* Momentum has a *sign* as well as a magnitude for 1-D collisions.

3. Equation (a) can't be correct, because the right-hand side has the dimensions of velocity, not momentum. Equation (b) can't be correct either, because the second term on the right has different dimensions than the other terms in the equation. The dimensions are consistent for both (c) and (d), so these equations could be the correct answer to some problem.

4. Set (a) cannot be an actual collision, because the initial momentum of the system is zero and the final momentum isn't. (We are assuming here that the initial velocity vectors really are equal but opposite.) Set (b) could be an actual collision, if the sum of the velocity vectors after is equal to the initial velocity vector of ball a. Set (c) cannot be real, because the total momentum is zero before, but not afterward. Set (d) could be real, if the vector sum of the final velocities is equal to the initial velocity of ball a.

PROBLEM QUIZ 3

1. (a) Conserve momentum. That is, $mv_0 = mv_f + (1.5m)(2v_f) = 4mv_f$ so $v_0 = 4v_f$.
   (b) Again conserve momentum. Thus $mv_0 = (m + M)v_f = 2.5\, mv_f$, so $v_0 = 2.5\, v_f$.

2. A picture of the collision is shown. Conserving momentum in the x- direction and in the y-direction,

x: $\quad mv_0 = Mv\cos\theta$

y: $\quad 0 = mv_f - Mv\sin\theta$

so $mv_0 = Mv\cos\theta$ and $mv_f = Mv\sin\theta$, where v is the final speed of the lead ball. Divide the second of these latter equations by the first (to eliminate v), giving

$$\tan\theta = v_f/v_0, \quad \text{so} \quad \theta = \tan^{-1}(v_f/v_0).$$

Square the equations and add them together, using the identity $\sin^2\theta + \cos^2\theta = 1$, to eliminate θ. This gives $v = (m/M)\sqrt{v_0^2 + v_f^2}$.

3. $v = 11.2\dfrac{\text{km}}{\text{s}} = 11.2\left(\dfrac{\text{km}}{\text{s}}\right)\left(\dfrac{1\text{ mi}}{1.61\text{ km}}\right)\left(\dfrac{3600\text{ s}}{1\text{ hr}}\right) = 25{,}000\text{ mi/hr}.$

4. (a) Use dimensional reasoning. The period P is a time, so we need a combination of m, ℓ, and g with the dimensions of time. The dimensions of these quantities are

$$[m] = M \text{ (i.e., a mass)}, \ [\ell] = L, \ [g] = L/T^2.$$

The only combination with the dimensions of time is $\sqrt{\ell / g}$. (b) The period of a pendulum is proportional to $\sqrt{\ell / g}$, and does not depend upon the mass of the bob.

5. (a) $\ell n\ 8 = \ell n\ 2^3 = 3\ \ell n\ 2 = 3(0.693) = 2.08$

(b) $e^{0.6} = e^{0.3 + 0.3} = e^{0.3}\ e^{0.3} = (1.35)^2 = 1.82$

Day Four

PICTURE QUIZ 4

1. The answer is (C). Point c is closest to the CM.

2. (a) The answer is (b). Modeling the jumper as a uniform stick, the CM of the jumper is 3 ft above the ground, so the CM must be raised only about 3 ft for the jumper to get over a 6-ft bar. (b) If the jumper bends as shown, it is possible for her/his CM to be a bit beneath the bar.

3. The answer is (B). The CM of the stick is at its center. So the overall CM is a distance

$x_{CM} = [2m(L/2) + mL]/3m = 2L/3$ from the left end of the stick.

4. (a) The answer is (D). The momentum gained by the ship is its mass times the change in its velocity. (The mass is constant during this time interval to several significant figures.) (b) The answer is (D). The average force is the momentum change divided by the time. (c). The answer is (C). The momentum of the exhaust, 25 kg · $\langle v \rangle$, must equal the momentum given to the ship.

5. The answer is (E). By Newton's third law, the forces are equal but opposite, so have the same magnitude.

PROBLEM QUIZ 4

1. (a) $x_{CM} = \dfrac{\Sigma m_i x_i}{M} = \dfrac{2\text{ g} \cdot 0 + 3\text{ g} \cdot 10\text{ cm} + 4\text{ g} \cdot 20\text{ cm}}{(2+3+4)\text{ g}} = \dfrac{110\text{ g cm}}{9\text{ g}} = 12.2\text{ cm}$ from the 2-gram bead. (b) No. There are 5 g to the left and 4 g to the right of the CM.

(c) $V_{CM} = \dfrac{\Sigma m_i V_i}{M} = \dfrac{2 \cdot 1 + 3 \cdot 3 + 4 \cdot 4}{9} \dfrac{\text{cm}}{\text{s}} = 3\text{ cm/s}.$

(d) $v_{2g} = (1 - 3)$ cm/s $= -2$ cm/s, $v_{3g} = (3 - 3)$ cm/s $= 0$, $v_{4g} = (4 - 3)$ cm/s $= 1$ cm/s.

2. (a) $3v_0/2$, by vector addition of velocities. (b) $V_{CM} = [mv_0 - 3m(v_0/2)]/(m + 3m) = -\ v_0/8$; i.e., $v_0/8$ to the left. (c) $v_0/8$ to the left (d) zero.

3. (a) $I = p_f - p_i = p_f$, since the initial momentum is zero. So $I = p_f = mv = m \cdot 2v_0 = 2mv_0$.
(b) $I_{ab} = -I_{ba}$, so $I_{\text{to pool ball}} = 2mv_0$ also. (c) $F = \Delta I/\Delta t = 2mv_0/\Delta t$

4. The momentum change of each absorbed photon is $\Delta p = p_f - p_i = 0 - p_i = -p_\gamma$
The force exerted on these photons by the mirror is

$$F_{abs} = \frac{\Delta\text{Impulse}}{\Delta t} = \frac{\text{Impulse}}{\text{photon}} \cdot \frac{\text{no. of photons}}{\Delta t} = \Delta p \cdot \frac{N}{2} = -p_\gamma \frac{N}{2} = -\frac{1}{2} p_\gamma N.$$

using the fact that half the photons are absorbed. The momentum change of each reflected photon is $\Delta p = p_f - p_i = -p_\gamma - p_\gamma = -2p_\gamma$, twice as much as for an absorbed photon. The force exerted on these photons by the mirror is

$$F_{ref} = \Delta p \cdot \frac{N}{2} = -2p_\gamma \frac{N}{2} = -p_\gamma N.$$

The total force on photons by the mirror is $F = F_{abs} + F_{ref} = -\frac{3}{2} p_\gamma N$. The total force on the mirror due to photons is equal but opposite to this, $F_{\text{on mirror}} = \frac{3}{2} p_\gamma N$, by Newton's third law. [The momentum of a photon is $p_\gamma = h\nu/c$, where h is Planck's constant, c is the speed of light, and ν is the frequency of the photon. The momentum of a single visible-light photon (whose frequency is $\nu \sim 5 \cdot 10^{14}\ \text{s}^{-1}$) is very small, since $h = 6.6 \cdot 10^{-34}$ kg m^2/s and $c = 3 \cdot 10^8$ m/s. Powerful lasers can generate large numbers of photons, however, so the force of radiation can be significant.]

Day Five

PICTURE QUIZ 5

1. The answer is (C). The force constant of each half-spring is *twice* the force constant of the full spring. The reason is that a half-spring stretches only half as much as the full spring for the same pulling force. Therefore from the relationship $F = k\Delta x$, if Δx is half as much, k must be twice as much for the same force. Said another way, a half-spring is twice as *stiff* as a full spring.

2. The answer is (C). The force on the block is kx_0, so the block's acceleration is $F/m = kx_0/m$. This acceleration is constant, so the distance moved in time t (starting from rest) is $d = (1/2)at^2$. Therefore $t = \sqrt{2d/a} = \sqrt{2dm/kx_0}$.

3. The block's acceleration is the second derivative $a = d^2x/dt^2$ From the graph, the second derivative of $x(t)$ is seen to be positive at A and E, zero at B and C, and negative at D. So the force is (i) to the right at A and E, (ii) to the left at D, and (iii) zero at B and C.

4. The answer to the first question is (B); it takes the same time to stop each ball. The reason is that they start with the same momentum and each loses momentum at the same rate, since the retarding force $F = dp/dt$ is the same for each. The answer to the second question is (C). It may seem surprising that the ping-pong ball, the lighter of the two balls, requires a longer *distance* to stop. The force on each ball is the same, so the time rate of change of momentum $dp/dt = F$ is the same for each. Since the balls begin with the same momentum, it takes the same *time* to stop each. The ping-pong ball is traveling much faster than the bowling ball, however (since it is much lighter yet has the same momentum), so in the same time it will travel farther than the bowling ball.

5. (C) is the right answer. After each collision the puck attached to the string will have the same speed v. The angular speed is $\omega = v/r$, so if r is doubled while keeping v the same, the angular speed will be cut in half.

6. (C) is again the right answer. The centripetal acceleration is v^2/r. Since the speed v of the attached puck after the collision is the same in each case, if the radius is doubled the centripetal acceleration is halved.

PROBLEM QUIZ 5

1. (a) The acceleration $a = dv/dt = bC\cos bt$

(b) $x = x_0 + \int_0^t v\,dt = 0 + \int_0^t C\sin bt\,dt = C\left.\frac{-\cos bt}{b}\right|_0^t = \frac{C}{b}(1-\cos bt)$

(c) (i) Yes, it will be found at $x = C/b$ at time $t = \pi/2b$, since $\cos[b(\pi/2b)] = 0$. (ii) No, it can never be found at $x = 2Cb$; the quantity Cb has the wrong dimensions. (iii) No, it can never be found at $x = -C/b$. The quantity $\cos bt$ can never exceed unity, so x is never negative.

2. (a) The positions are $x_M = \frac{1}{2}at^2$ and $x_P = vt$. They intersect when $x_M = x_P$; that is, when

$$\frac{1}{2}(2.0\text{ m/s}^2)\,t^2 = (20\text{ m/s})\,t$$

with solution $t = 0$ (at the green light) and $t = \frac{20\text{ m/s}}{1.0\text{ m/s}^2} = 20$ s.

The distance is $x = x_P = (20\text{ m/s})(20\text{ s}) = 400$ m from the green light. (b) $t = 20$ s, as noted in part (a).

(d)

(c) $v_M = v_0 + a_M t = 0 + 2.0\text{ m/s}^2 \cdot 20\text{ s} = 40$ m/s.

3. (a) The time for one revolution is $T = \frac{\text{distance}}{\text{speed}} = \frac{12\text{ km}}{0.9999(3\cdot 10^8\text{ m/s})} = 4\cdot 10^{-5}\text{ s} \equiv 40\ \mu\text{s}$, so the number of revolutions per second is $\frac{1}{\#\text{ sec/revolution}} = \frac{1}{4\cdot 10^{-5}\text{ s}} = 2.5\cdot 10^4$ rev/s.

(b) $\omega = v/R \cong \frac{3\cdot 10^8\text{ m/s}}{12\text{ km}/2\pi} = \frac{\pi}{2}\cdot 10^5\text{ s}^{-1}$ (c) $a = v^2/R \cong c^2/R = \frac{3\pi}{2}\cdot 10^{13}\text{ m/s}^2$

4. (a) $F_A = ma_{\text{centripetal}} = mv^2/R = mR\omega^2$, directed inward to the center of the circle.

(b) $x_B(t) = x_A(t) = R\cos\theta(t) = R\cos\omega t$. (c) $v_B(t) = dx_B(t)/dt = -\omega R\sin\omega t$

(d) $F_B(t) = ma_B(t) = m\,dv(t)/dt = m(-\omega R)\,\omega\cos\omega t = -mR\,\omega^2\cos\omega t$.

(e) $x_B(t) = R\cos\omega t$, so $F_B(t) = -m\omega^2 x_B(t)$

(f) $F_B(t) = -m\omega^2 x_B(t) = -kx_B(t)$ (a Hooke's-law spring force) where the force constant is $k = m\omega^2$.

Day Six

PICTURE QUIZ 6

1. (A). The net forward force is $2T\cos 30^\circ = ma_0$, so $T = ma_0/2\cos 30^\circ = ma_0/\sqrt{3}$.

2. Diagrams E and H are correct.

Diagram E shows two forces acting on M, the force F_0 and the normal force N caused by m. There is only one force acting on m, the contact force N caused by M. Note that the normal forces on M and m are equal but opposite, as required by Newton's third law.

Newton's second law then gives $F_0 - N = Ma$ for block M, and $N = ma$ for block m. The accelerations have to be equal, because the blocks are moving together. Adding these equations gives $F_0 = (M + m)a$, so the acceleration of each ball is

$$a = \frac{F_0}{M + m}.$$

The force of contact between the blocks is therefore

$$N = ma = \left(\frac{m}{M + m}\right)F_0.$$

Diagram H shows one force exerted on the entire system of M + m, the external force Fo. (The nordmal forces between the blocks are internal forces, which cancel by Newton's third law.) Therefore Newton's second law is $F_0 = (M + m)a$, and so the acceleration is the result given above. The isolation diagram for m then gives N, the same result given above.

The other diagrams are either missing a force on one of the blocks, or show N in the wrong direction on one of the blocks, or else add ma, which is *not a force*. Diagrams A and D incorrectly show F_0 acting on m.

3. Diagram C is correct. The only thing touching the ball is the string, so the string tension is the only force exerted on the ball. The net force on the string is therefore T, directed inward, so Newton's second law is $F_{net} = ma_{in} = mv^2/r$, because an object moving in a circle has inward acceleration v^2/r.

Diagrams A, B. D, and E are incorrect, because they show a non-existent force mv^2/r. There is nothing touching the ball that could exert such a force. Diagrams D and F are wrong because ma is shown as a force. There is no such force. Only the string touches the ball, so only the string (through its tension) exerts a force on the ball.

4. The answer is (B). In (A), the point O is not the CM of the stars, so will not stay fixed in an inertial frame. In (C), the point O is not the CM of the stars in either the second or third picture, so cannot remain at rest in an inertial frame.

PROBLEM QUIZ 6

1. The normal force is the only force on the container, so $N = ma = 25 \cdot 10^3 \text{ kg} \cdot 0.60 \text{ m/s}^2 = 15 \cdot 10^3 \text{ N}$.

2. (a) The isolation diagram of the two containers as a unit is as shown, so (b) the net force on the system is N_o. (c) The acceleration of the system is therefore $a = N_o/(m_1 + m_2)$. (d) The isolation diagram of the front container is given at the right, showing that the only external force on it is N_1, the normal force due to the rear container. Therefore (e) $N_1 = m_1 a = m_1 N_o/(m_1 + m_2)$.

(f) The isolation diagram of the rear container is shown below. The net force on this container is $N_o - N_2$, so $N_o - N_2 = m_2 a = m_2 N_o/(m_1 + m_2)$. (g) Solving for N_2, the magnitude of the force of the front container on the rear container, $N_1 = N_o - m_2 N_o/(m_1 + m_2) = m_1 N_o/(m_1 + m_2)$, the same answer we found in part (e). (h) The results must be the same according to Newton's third law.

3. The isolation diagram of the outer ball is shown. It has only one force T_1 acting upon it (note that only string 1 touches the outer ball), so (a) $F_{net} = T_1 = ma = mr\omega^2 = m(2\ell)\omega^2$. The isolation diagram of the inner ball is also shown at the right. From this diagram, note that $F_{net} = T_2 - T_1 = ma = mr\omega^2 = m\ell\omega^2$. (b) Solving for T_2, $T_2 = T_1 + m\ell\omega^2 = 3m\ell\omega^2$ using the result of part (a).

Day Seven

1. The plane's velocity with respect to the *ground* has to be oriented due south, as shown. Since the wind is blowing toward the east, the plane must have a velocity relative to the air which has a component west of 60 m/s (to cancel the effects of the wind) as well as a component south, as shown. By the law of Pythagoras the component of velocity south is therefore 80 m/s. Since the plane has to travel 80 km south, this will take 1000 seconds.

2. From dimensional reasoning, the only way to combine an acceleration g and a distance h to get the dimensions of velocity v is to let $v \sim \sqrt{gh}$. Therefore if the depth doubles, the velocity increases from v_o to $\sqrt{2}\,v_o$.

3. The distances must have a four-to-one ratio, with the heavier mass closer to the center of mass. Therefore m_1 is a distance $(4/5)L$ from the CM, and $4m_1$ is a distance $(1/5)L$ from the CM.

4. (a) The rock is has velocity $-5v_o$ relative to her, so has velocity $-4v_o$ relative to the inertial frame of the ship, minus meaning toward the left. In this frame the total momentum before she catches the rock is $mv_o + (m/3)(-4v_o) = -(1/3)mv_o$. After she catches the rock the total momentum is $(m + m/3)v_f$, where v_f is their mutual final velocity. Conserving momentum, we find $v_f = -(1/4)v_o$. Her momentum change is $\Delta p = p_f - p_o = m[-(1/4) - 1]v_o = -(5/4)mv_o$. The impulse exerted upon her by the rock is therefore $I = -(5/4)mv_o$. (b) Her final speed of return is only one-quarter as much as her speed of departure, so it will take four times as long to get back, a time $4T$.

5. Differentiating, the velocity of the particle is $v = c/t$, and so its acceleration is a second derivative, $a = -c/t^2$. The net force on the particle is therefore $F = ma = -m\,c/t^2$.

6. (a) The net force on all three sleds taken as a unit is $F = 100$ N, so their common acceleration is $a = F/M_{\text{total}} = 100\text{ N}/\ 500\text{ kg} = 0.2\text{ m/s}^2$.

(b) Isolating the last two sleds, the only external force on them is T_1, so $T_1 = (m_2 + m_3)a = (400\text{ kg})(0.2\text{ m/s}^2) = 80$ N.

(c) Isolating m_3, the only external force is T_2, so $T_2 = (m_3)a = (250\text{ kg})(0.2\text{ m/s}^2) = 50$ N.

(d) With constant acceleration $a = 0.2\text{ m/s}^2$ during the first minute of the trip, the dogs move a distance $x = \frac{1}{2}at^2 = \frac{1}{2}(0.2\text{ m/s}^2)(60\text{ s})^2 = 360$ m, ending the minute with velocity $v = at = (0.2\text{ m/s}^2)(60\text{ s}) = 12$ m/s. During the second minute the dogs move at the constant velocity 12 m/s, so they cover an additional distance $vt = (12\text{ m/s})(60\text{ s}) = 720$ m. Therefore the total distance covered by the dogs is 360 m + 720 m = 1080 m.

Day Eight

PICTURE QUIZ 8

1. The answer is (d). The vertical component of velocity must be the same in each case, so $v\cos 45^\circ = v_0$. Therefore $v = v_0/\cos 45^\circ = \sqrt{2}\,v_0$.

2. The answer is (e). The height h depends upon v_0 and g, with dimensions L/T and L/T^2, respectively. The only way to eliminate time and get the dimensions of the height h (length) is to square v_0 and divide by g. Thus h depends on v_0^2, so h increases by a factor of four if v_0 is doubled.

3. The answer is (c). None of the balls is accelerating, so string 2 supports the lower two balls against gravity, as shown in the isolation diagram of balls A and B. Therefore $T_2 = 4\text{ kg} \cdot g = 40$ N.

4. The answer is (d). Again no ball is accelerating, so the isolation diagram of ball A shows that $T_1 = mg = 2\text{ kg} \cdot g = 20$ N.

5. The answer is (b). The isolation diagram of all three balls, taken as a unit, is shown. The net force on the three balls is $3mg - T_3 = 3ma$, so $T_3 = (3m)(g - a) = 6\text{ kg} \cdot 6.0\text{ m/s}^2 = 36\text{ N}$.

PROBLEM QUIZ 8

1. The distance fallen from rest is $s = (1/2)at^2$, where $a = g$ in this case. So with $g = 32\text{ ft/s}^2$, a rock falls 16 ft in the first second. At the end of two seconds, the rock will have fallen 64 ft, of which 48 ft was covered during the *second* second alone. If the mineshaft is not extremely deep, the time it takes for the sound of a dropped rock hitting the bottom of the shaft to arrive at the top is negligible, since sound travels about 1000 ft/s. One can measure the time it takes to hear the sound after the rock is dropped, and calculate the depth using $(1/2)gt^2 = 16\,t^2$ feet, where t is in seconds.

2. The feather falls in the same time as the hammer, related to the velocity at the surface and to g_m by $v_h = g_m t$, so $t = v_h/g_m$.

3. The single spot is the center of mass of the racquet, which moves along a parabolic arc, as though the whole tennis racquet were shrunk to a single particle located at the CM. That is because $\mathbf{F}_{net} = \mathbf{F}_{grav} = M_{tot}\,\mathbf{a}_{CM}$.

4. The isolation diagram of the person in the elevator is as shown, where N is the normal force due to a scale the person is standing on. Then $F_{net} = N - mg = ma = mg/2$ (since $a = g/2$). Therefore $N = \frac{3}{2}mg$ upward. The weight of the person is mg, downward.

5. Calculate the height y when $x = 60\text{ yds} = 180\text{ ft}$. We have

$$x = v_{ox}t \qquad \text{and} \qquad y = v_{oy}t - (1/2)gt^2.$$

Eliminate t between these two equations, to give

$$y = v_{oy}\left(\frac{x}{v_{ox}}\right) - \frac{1}{2}g\left(\frac{x}{v_{ox}}\right)^2 = x - \frac{gx^2}{v_0^2} = 180\text{ ft} - \frac{32\text{ ft/s}^2 \cdot (180\text{ ft})^2}{(80\text{ ft/s})^2} = 18\text{ ft.}$$

We have used the fact that $v_{ox} = v_{oy} = v_0/\sqrt{2}$, since it was kicked at a 45^0 angle. The football would sail over the 10 ft crossbar with about 8 ft to spare.

6. Answer E is correct. There is an upward force T on m, due to the string, and a downward gravitational force mg. There is an upward force T on M, due to the string, and the downward gravitational force Mg. We expect M will fall and m will rise; that is why we chose the positive acceleration direction to be down for M and up for m.

From $F = ma$ for each block, using the diagrams for m and M, we find that

$$(1)\ T - mg = ma \qquad \text{and} \qquad (2)\ Mg - T = Ma.$$

Note that $F_{\text{net}} = T - mg$ for m, because the positive acceleration direction is upward, and that $F_{\text{net}} = Mg - T$ for M, because the positive acceleration direction is downward. Adding (1) and (2) eliminates T, and so

$$a = \left(\frac{M-m}{M+m}\right)g.$$

In particular, $a = 0$ if $m = M$, as expected, and $a = g$ if $m = 0$, also as expected.

We can then solve for the string tension,

$$T = m(g+a) = \frac{2mM}{M+m}g,$$

reducing to $T = Mg$ if $m = M$, as expected, and $T = 0$ if $m = 0$, also as expected.

Answer A is incorrect, because it is the string that touches the blocks, and so it is the string tension that exerts forces on m and M, and not the gravity of the other block. The string tension is neither Mg nor mg.

The diagram for m in answer B is correct, but not the diagram for M. The upward force on M is due to the string through its tension. Also, it is inconsistent to have both positive-acceleration directions upward: If one block accelerates upward, the other must acceleration downward if the string is to keep a constant length.

Answer C is incorrect because neither ma nor Ma is a force. Also the force due to the string is not shown.

Answer D would be correct except for the fact that it is inconsistent to have both positive-acceleration directions upward, as explained under answer B.

Answer F is incorrect because the diagram for m shows an upward force Mg instead of T. The string tension is not equal to Mg.

The string tension is the same on both sides of the peg because we have assumed that the string has negligible mass, and that the peg is frictionless. If the string had a nonnegligible mass it could not accelerate, as it must, unless the force were unbalanced with a slightly higher tension on one side than the other. If there were friction between the string and the peg, there would have to be more tension on one side than the other just to keep the string moving. Real strings, with some mass and some friction with the peg, have tensions which are somewhat different on the two sides. The difference goes to zero in the limits of a zero-mass string and zero friction with the peg.

[This device is known as *Atwood's Machine*, after the English mathematician and physicist George Atwood, who in 1784 used it to demonstrate uniformly accelerated motion. Note that the acceleration a is small if M and m are nearly equal, so it is easy to observe the uniform acceleration.]

Day Nine

PICTURE QUIZ 9

1. (A) Forward. Friction is the only horizontal force, so only friction can accelerate the unicyclist.
(B) backward. Friction is the only horizontal force, so only friction can decelerate the unicyclist.

2. The answer is (b). Friction is the only horizontal force on the penny, so $F_f = ma = mv^2$, since the penny is moving in a circle at constant speed. Therefore

$$v = \sqrt{F_f\, r/m} \le \sqrt{\mu_s N\, r/m} = \sqrt{\mu_s mg\, r/m} = \sqrt{\mu_s g\, r} = \sqrt{0.11 \cdot 10\ \text{m/s}^2 \cdot 1.6\ \text{m}} = 1.33\ \text{m/s}.$$

3. The answer is (c). The frictional force is proportional to the mass m (since $N = mg$ in this case for each skier) and so the m's cancel in $F = ma$. Their decelerations are equal, so they come to rest at the same time.

PROBLEM QUIZ 9

1. The downward force increases the normal force N between the car and track, which increases the maximum possible static friction force $F_{f\,max} = \mu_s N$. This makes it less likely that the car will spin out on curves.

2. Suppose the rider's mass is m. Then her inward (centripetal) acceleration is $a_{in} = r\omega^2$, so the inward normal force on her due to the wall is $N = ma = mr\omega^2$ (since N is the only inward force). Static friction is the only vertical force that can counteract gravity, so $F_f = mg$ and also $F_f \le \mu_s N = \mu_s mr\omega^2$. So $\omega \ge \sqrt{\dfrac{g}{\mu_s r}}$,

and $\omega_{min} = \sqrt{\dfrac{g}{\mu_s r}}$.

3. Sliding (kinetic) friction is the only horizontal force, and is constant, so the acceleration is

$$a = F/m = -\mu_k N/m = -\mu_k\, mg/m = -\mu_k\, g,$$

a constant. The uniform acceleration equation $v^2 = v_o^2 + 2ad$ therefore gives

$$v_o^2 = v^2 - 2ad = 0 + 2\,\mu_k\, gd,$$

since the final velocity $v = 0$. The initial speed of the block is therefore $v_o = \sqrt{2\mu_k gd}$. Momentum of the bullet + block is conserved during their collision (because we assume the effects of the external frictional force is negligible during the very brief collision), so

$$m_1 v_{bullet} = (m_1 + m_2)\, v_o;$$

the speed of the bullet just before impact was $v_b = \left(\dfrac{m_1 + m_2}{m_1}\right) v_o = \left(\dfrac{m_1 + m_2}{m_1}\right)\sqrt{2\mu_k gd}$.

4. (a) In equilibrium the net force on each block is zero. We choose coordinates parallel and perpendicular to the slope for each block. The force-balancing equations are

(1) $N_1 = m_1 g\cos 30^o$ (2) $T = m_1 g\sin 30^o + F_s$ (3) $N_2 = m_2 g\cos 60^o$ (4) $T = m_2 g\sin 60^o$

Eliminating T between Eqs. (2) and (4) gives $F_s = (m_2 g)\sin 60^o - (m_1 g)\sin 30^o$. The force of static friction obeys $F_s \le \mu_s N_1 = \mu_s(m_1 g)\cos 30^o = (2/3)(m_1 g)\cos 30^o$ using Eq. (1) and the given value $\mu_s = 2/3$.

That is, $(m_2 g)\sin 60^\circ - (m_1 g)\sin 30^\circ \le (2/3)(m_1 g)\cos 30^\circ$. Substituting in $\sin 60^\circ = \cos 30^\circ = \sqrt{3}/2$ and $\sin 30^\circ = 1/2$ and solving for m_2, we find that

$$(m_2)\max = 1.244\ m_1.$$

If m_2 is any larger than this, friction can no longer keep m_1 in place, and it will start to slide up the hill.

(b) If $m_2 = 2m_1$, the blocks will certainly slide to the right, as shown by the result of part (a). The isolation diagrams are as shown, with a force of kinetic friction acting on m_1.

(c) The $F = ma$ equations for the two blocks are

$$(1)\ N_1 - m_1 g\cos 30^\circ = 0 \qquad (2)\ T - m_1 g\sin 30^\circ - F_k = m_1 a$$

$$(3)\ N_2 - m_2 g\cos 60^\circ = 0 \qquad (4)\ m_2 g\sin 60^\circ - T = m_2 a = 2m_1 a$$

where $F_k = \mu_k N_1 = \mu_k m_1 g\cos 30^\circ$. Summing (2) and (4) to eliminate T gives

$$m_2 g\sin 60^\circ - m_1 g\sin 30^\circ - \mu_k m_1 g\cos 30^\circ = 3m_1 a.$$

Using $\mu_k = 1/3$, $a = [2\sin 60^\circ - \sin 30^\circ - (1/3)\cos 30^\circ]\, g/3 = 0.314\, g$. Then (4) gives $T = 1.104\, m_1 g$.

Day Ten

PICTURE QUIZ 10

1. The answer is (E). According to conservation of energy, if the valley is the same vertical distance below the ridge in each case, the kinetic energy in the final valley must be the same, and therefore the speed must be the same in each case. (Note that in (D) the skier's speed will be higher in the depression than in the final valley.)

2. The answer is (E), none of the given answers. Block M loses gravitational potential energy Mgh as it falls a distance h, while block m does not change its potential energy. The system as a whole therefore loses potential energy Mgh, which must show up as an increase in kinetic energy, since energy is conserved. The total kinetic energy is

$$\frac{1}{2}mv^2 + \frac{1}{2}Mv^2 = \frac{1}{2}(m+M)v^2 = Mgh, \text{ so } v = \sqrt{\frac{2Mgh}{M+m}}.$$

Both blocks have the same speed since we assume that the cord doesn't stretch.

3. (a) The kinetic energy $K = E - U$ is a maximum at point A. (b) The kinetic energy is zero at the turning point E. (c) The object cannot be at F, because $K = E - U$ would be negative there, impossible because $K = (1/2)mv^2$ cannot be negative.

PROBLEM QUIZ 10

1. (a) At the top of the crest the car is weightless if there is no normal force, as shown in the isolation diagram. The only remaining force on the car is gravity mg, acting downward. The car is going over the crest in a circular arc of radius R, so its centripetal acceleration is v^2/R.

Newton's second law therefore gives $mg = mv^2/R$, so $mv^2 = mgR$.
Conserving energy between the beginning and the crest, we have $mgh = (1/2)mv^2 + mgR$. Eliminating mv^2 between these two equations yields $mgh = (1/2)mgR + mgR$, so $h = 3R/2$. (If h is any larger than this, the car will not be in contact with the ground when it reaches the crest.)
(b) If $h = 3R/2$, conserving energy between the beginning and the valley gives $mgh = (1/2)mv^2$, so in the valley $mv^2 = 2mgh = 3mgR$. The centripetal acceleration of the car is upward, equal to $v^2/R = 3g$. The isolation diagram for the car in the valley therefore gives $F_{net} = N - mg = mv^2/R$, so the normal force is $N = mg + 3mg = 4mg$, which is also the car's weight.

2. (a) The energy is all kinetic at the equilibrium point, so $E = (1/2)mv_m^2$; the energy is all potential at maximum stretch (or compression), so also $E = (1/2)kA^2$. According to energy conservation these are equal to one another, so we can solve for k to get $k = m(v_m/A)^2$. (b) Again by energy conservation, $E = (1/2)mv_m^2 = (1/2)kx^2 + (1/2)m(v_m/2)^2$. Solving for x, we find that $x = (3m/4k)^{1/2}\, v_m$. An alternative answer is found by substituting the formula found in part (a) for k: $x = (\sqrt{3}/2)\, A$.

Day Eleven

PICTURE QUIZ 11

1. The answer is (c), $+mgh$. The barbell is at rest before and after, so the kinetic energy has not changed. From the work-energy theorem, this means that no net work was done on the barbell.
Gravity (a constant force) does work $-mgh$ on the barbell (negative because the force is opposite to the displacement.) Therefore the weightlifter must supply work $+mgh$ to make the overall work zero.
2. The answer is (B). As the barbell falls, only gravity does work on it, $W_{grav} = +mgh$. The kinetic energy must increase this much, according to the work-energy theorem. Therefore $mgh = (1/2)mv^2$, so $v = \sqrt{2gh}$.

3. The answer is (C). The work done by friction must be negative, because the force direction is opposite to the displacement. Therefore the answer must be either (C) or (D). If the angle θ were increased, the normal force would increase (since the normal force must counteract the vertical component of the custodian's force F_o.) And if the normal force is increased, the force of kinetic friction $F_f = \mu_k N$ increases as well, increasing the work done by friction as the mop moves the distance s. Note that the denominator of (C) decreases as θ increases, so the expression in (C) increases with increasing θ, whereas (D) behaves in the opposite (and incorrect) way.

4. The answer is (E). The kinetic energy of the mop is unchanging, so the net power input is zero.

PROBLEM QUIZ 11

1. (a) We assume there is no friction between the sled and snow, so all the work on the sled and rider is done by the dogs. The change in kinetic energy of the sled and rider is $(1/2)mv^2 = (1/2)(220\text{ kg})(7.5\text{ m/s})^2 = 6200$ J, so this is the total work done by the dogs. (b) The work done is $10(\langle F\rangle)s = 6200$ J, so $\langle F\rangle = 620\text{ J}/17\text{ m} = 36$ N.

2. The work-energy theorem is $W_T = W_{grav} + W_{air} = \Delta K.E. = K_f - K_i = \frac{1}{2}mv_f^2 - 0$. Therefore

$$W_{air} = \frac{1}{2}mv_f^2 - W_{grav} = \frac{1}{2}mv^2 - mgd = -85.6 \text{ J}$$

3. The maximum work (for constant force) is $W = F \cdot d \sim$
$10 \cdot 10^{-12}$ N $\cdot\ 8 \cdot 10^{-9}$ m $= 8 \cdot 10^{-20}$ J

4. (a) $W_{grav} = \int \mathbf{F}_{grav} \cdot ds = -mg\int dy = -mgy$, where y is the vertical coordinate. The reason is that

$$\mathbf{F}_{grav} \cdot ds = F_{grav}ds\cos(90^\circ + \theta) = -F_{grav}ds\sin\theta = -F_{grav}dy$$

as can be seen from the sketch. Measured up from the bottom, $y = R(1 - \cos\theta)$, and also $\theta = \omega t$ for constant-speed circular motion, starting at the bottom. Therefore $W_{grav} = -mgR(1 - \cos\omega t)$.
(b) $W_T = \Delta K.E. = 0$, so $W_{seat} = +mgR(1 - \cos\omega t)$, by the work-energy theorem. (c) $W_T = 0$.

5. (a) $\mathbf{A} \cdot \mathbf{B} = (2\hat{\mathbf{x}} - 3\hat{\mathbf{y}} - a\hat{\mathbf{z}}) \cdot (3\hat{\mathbf{x}} - a\hat{\mathbf{y}} + 4\hat{\mathbf{z}}) = 6 + 3a - 4a = 6 - a = 0$ if $\mathbf{A}$ is perpendicular to $\mathbf{B}$.
Therefore $a = 6$. (b) $\mathbf{A} = 2\hat{\mathbf{x}} - 3\hat{\mathbf{y}} - 6\hat{\mathbf{z}}$, so $|\mathbf{A}| = \sqrt{2^2 + (-3)^2 + (-6)^2} = \sqrt{49} = 7$

$\mathbf{B} = 3\hat{\mathbf{x}} - 6\hat{\mathbf{y}} + 4\hat{\mathbf{z}}$ so $|\mathbf{B}| = \sqrt{3^2 + (-6)^2 + 4^2} = \sqrt{61}$

6. Shown is a triangle whose sides are given by the vectors **A**, **B**, and **C**, oriented so that **A** = **B** + **C**. Then

$$\mathbf{C} \cdot \mathbf{C} = C^2 = (\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = A^2 + B^2 - 2\mathbf{A} \cdot \mathbf{B} = A^2 + B^2 - 2AB\cos\theta,$$

which is the law of cosines.

7. (a) The work done by the spring is $W = \int F dx = \int_0^A -k\sqrt{x}dx = -k\frac{x^{3/2}}{3/2}\Big|_0^A = -2kA^{3/2}/3$.

(b) When the block is released, only the spring does work on it, an amount of work $W = +2kA^{3/2}/3$. According to the work-energy theorem, this is also the kinetic energy increase $(1/2)mv^2$ of the block. The speed of the block is therefore $v = \sqrt{4kA^{3/2}/3m}$.

8. (a) At constant speed v, $P_{grav} = mgv$ and $P_{Total} = 0$ (since there is no change in kinetic energy.) Therefore $P_{rope} = P_{Total} - P_{grav} = -mgv$. (b) At constant acceleration $a = g/10$, $v = at = gt/10$, so $P_{grav} = mg^2t/10$, and $P_{rope} = F_{rope}v$. What is F_{rope}? We find it using Newton's second law:

$$F_{grav} + F_{rope} = F_{net} = ma = mg/10,$$

so $F_{rope} = mg/10 - mg = -(9/10)mg$. Therefore

$$P_{rope} = F_{rope}\,v = -(9/10)mg \cdot gt/10 = -(9/100)\,mg^2t$$

$$P_T = (mg^2/10)t[1 - 9/10] = (mg^2/100)t.$$

9. The speed v is constant, so there is no net power input. The power input from the normal force is $P_N = \mathbf{N} \cdot \mathbf{v} = 0$. since $\mathbf{v} \perp \mathbf{N}$, and the power input from gravity is

$P = m\mathbf{g} \cdot \mathbf{v} = mgv \cos(90^{\circ} - \theta) = mgv \sin\theta,$

where θ is the slope angle shown. The power input from friction must therefore be $P_{fric} = -P_{grav} = -m\mathbf{g} \cdot \mathbf{v} = -mgv \sin\theta = -P$.

Day Twelve

PICTURE QUIZ 12

1. (a) The slope is negative (and therefore the force $F = -dU/dx$ is positive) at points A, B, C, and G. (b) The speed is greatest where the kinetic energy $K = E - U$ is greatest, namely at point D. (c) The force is zero where the slope is zero, at points D, F, and H. (d) The particle accelerates toward $x = 0$ (which we take to be between points F and G, as represented by the vertical axis), at points A, B, C, I, and J. These are the points where the force is directed toward $x = 0$. (e) The particle's speed is zero at points A and J. There is no kinetic energy at these two turning points.

2. (a) To reach $r = 0$ the particle must surmount the potential energy barrier at $r = 2$ m, so (since the barrier has height a bit more than 2 J) the particle must have energy a bit over 2 J. (b) to reach arbitrarily large radii it must have at least zero Joules of energy. A negative energy particle would be trapped inside the well. (c) There is no force at $r = 2$ m and at $r = 5$ m, where the slope of $U(r)$ is zero. (d) The kinetic energy is a maximum at $r = 5$ m, where the potential energy is a minimum.

3. The answer is (C). By the vector addition of velocities, if the asteroid is moving at speed $v_0/2$ to the right relative to the Sun, then as it approaches the asteroid the spacecraft is moving with speed $v_0 + v_0/2 = 3v_0/2$ in the Sun's frame, and on departure is moving with speed $v_0 - v_0/2 = v_0/2$. (By energy conservation in the asteroid's frame, the final speed of the spacecraft is v_0, just as it was initially.)

PROBLEM QUIZ 12

1. In equilibrium $kd = mg$, so $d = mg/k$. If dropped, energy is conserved. Measure x positive downward. We can set all energies initially zero, so

$$E = 0 = U_{grav} + U_{spring} + K.E. = -mgx + (1/2)kx^2 + (1/2mv^2.$$

The speed v is maximal when $U = -mgx + (1/2)kx^2$ is a minimum. That is, $\dfrac{dU}{dx} = -mg + kx = 0$ when $x = mg/k$. (a) Therefore $x = d$ when v is a maximum. (b) The weight is instantaneously at rest when $v = 0$. That is, $mgx + \frac{1}{2}kx^2 = U = 0$, so $x(-mg + \frac{1}{2}kx) = 0$. This gives $x = 2mg/k = 2d$.

2. (A) is conservative, because it is a one-dimensional force that depends only on position. (B) is conservative also; its potential energy is $(1/2)Cr^2$. (C) is nonconservative, because the work done by $\mathbf{F}$ in a closed circular counterclockwise path at constant radius r is $W = -C(2\pi r)$, which is not zero. (D) is nonconservative because it depends upon velocity. (E) is nonconservative because the work done around a closed path is not necessarily zero. If you begin at the origin, move a distance along the positive x-axis, then move vertically in the y-direction, then turn and move in the negative x-direction to the y axis, and then turn down along the y-axis back to the origin, the work done is zero along every section except the third, where it is positive.

3. (a) According to the work-energy theorem, the total work done on the skier is zero, since the skier's kinetic energy is not changing. (b) The work done by gravity is $mgL\sin 30^{\circ} = mgL/2$. The work done by the normal force is zero, since the force and displacement are perpendicular to one another. (d) friction must do work $-mgL/2$ to cancel the work done by gravity.

4. Conserve energy: $K_0 + U_0 = 0 + \dfrac{-GMm}{R + R/2} = \dfrac{1}{2}mv^2 + \dfrac{-GMm}{R}$ where m is the spaceship's mass.

So $\dfrac{1}{2}mv^2 = GMm\left[\dfrac{1}{R} - \dfrac{1}{3R/2}\right] = \dfrac{GMm}{3R}$, which gives $v = \sqrt{\dfrac{2GM}{3R}}$.

5. (a) $\rho = \dfrac{\text{mass}}{\text{volume}} = \dfrac{M}{\frac{4}{3}\pi R^3} = \dfrac{3M}{4\pi R^3} = \dfrac{3(2\cdot 10^{30}\text{ kg})}{4\pi(10^7\text{ m})^3} = \dfrac{15}{\pi}\cdot 10^8\text{ kg/m}^3$

(b) $\dfrac{1.5\text{ tonne}}{\pi\text{ cm}^3}$ (c) $g = \dfrac{GM}{R^2} = \dfrac{(2/3)\cdot 10^{-10}(2\cdot 10^{30})}{(10^7)^2}\dfrac{\text{m}}{\text{s}^2} = 1.33\cdot 10^6\text{ m/s}^2$

(d) $v_{esc} = \sqrt{\dfrac{2GM}{R}} = \sqrt{\dfrac{2\cdot\frac{2}{3}\cdot 10^{-10}\,\frac{\text{m}^3}{\text{kg s}^2}(2\cdot 10^{30}\text{ kg})}{10^7\text{ m}}} = 5.2\cdot 10^6\,\dfrac{\text{m}}{\text{s}} = 460\, v_{esc}$ Earth

Day Thirteen

PICTURE QUIZ 13

1. Answer (B) is correct. The CM frame moves with velocity $(m_1v_1 + m_2v_2)/(m_1 + m_2) = (2v_0 + 0)/5 = 2v_0/5$.

2. Answer (D) is correct. By vector addition of velocities, going into the CM frame that moves to the right at speed $2v_0/5$, the initial velocities of the 2-kg ball and the 3-kg ball are $v'_{2i} = (3/5)v_0$ and $v'_{3i} = -(2/5)v_0$. Their final velocities are just the opposite, $v'_{2f} = -(3/5)v_0$ and $v'_{3f} = (2/5)v_0$.

3. Answer (C) is correct. Using the pictorial method, the sequence of pictures is

(note that the momenta are equal but opposite and the relative velocity is still v_0.)

4. All three answers are correct. Note that momentum is conserved in each case if car L is finally at rest, as are the right-hand car in (A), the truck in (B), and the wall in (C). The wall is attached to the ground, so can be taken to be extremely massive.

PROBLEM QUIZ 13

1. From Eq. 4, $(v_{ox})_f = \dfrac{2m_c}{m_c + m_{ox}} v_o = \dfrac{6}{7} v_o$. From Eq. 5, $(v_c)_f = \dfrac{m_c - m_{ox}}{m_c + m_{ox}} = -\dfrac{1}{7} v_o$

2.

Conserve momentum in the x-direction: $m_\alpha v_o = m_N v_N \cos\theta$

y-direction: $m_\alpha v_\alpha = m_N v_N \sin\theta$

Square the two equations and sum them to eliminate θ: (1) $m_\alpha^2(v_o^2+v_\alpha^2) = m_N^2 v_N^2$.

Conserve kinetic energy (elastic collision):

$$\frac{1}{2} m_\alpha v_o^2 = \frac{1}{2} m_\alpha v_\alpha^2 + \frac{1}{2} m_N v_N^2 \quad \text{so} \quad (2)\ m_N v_N^2 = m_\alpha (v_o^2 - v_\alpha^2)$$

Combine Eqs. (1) and (2) : $m_\alpha^2(v_o^2 + v_\alpha^2) = m_N^2 v_N^2 = m_N m_\alpha (v_o^2 - v_\alpha^2)$

(a) Solve for v_α: $v_\alpha = \sqrt{\dfrac{m_N - m_\alpha}{m_N + m_\alpha}}\, v_o = \dfrac{\sqrt{5}}{3} v_o$ (b) Then $v_N = \dfrac{m_\alpha}{m_N}[v_o^2+v_\alpha^2]^{1/2} = \dfrac{4}{3\sqrt{14}} v_o$

(c) $\tan\theta = v_\alpha/v_o$ (by taking the ratio of the two momentum-conservation equations), so $\theta = \tan^{-1}\dfrac{\sqrt{5}}{3}$.

3. (a) Conserving momentum, $mv_o + (-9mv_o) = 10mv_f$, so $v_f = -(4/5)\, v_o$. The final composite particle moves in the direction of the heavier of the two initial particles.

(b) $K_o = \dfrac{1}{2} mv_o^2 + \dfrac{1}{2} 9mv_o^2 = \dfrac{1}{2} 10mv_o^2 = 5m\, v_o^2$ and $K_f = \dfrac{1}{2} 10mv_f^2 = \dfrac{16}{5} mv_o^2$

so $$\frac{K_o - K_f}{K_o} = \frac{\text{internal energy}}{\text{initial energy}} = \frac{\left(5 - \frac{16}{5}\right)}{5} = \frac{9}{25}$$

4. The balls have equal mass and one starts at rest, so as shown in the text their final velocity vectors must be perpendicular to one another. Therefore $\theta = 65^o$.

5. The orbits are as shown in the CM frame; since the stars have equal mass, each is a distance $R/2$ from the center of mass. (a) Applying Newton's law of gravity and Newton's second law to either one of the two stars,

$$F = \frac{Gmm}{R^2} = ma = m\frac{v^2}{r} = m\frac{v^2}{(R/2)}$$

where we can use $a = v^2/r$, because each travels in a circle. Solving for v, $v = \sqrt{\frac{Gm}{2R}}$.

(b) The total kinetic energy is $2 \cdot (1/2)mv^2 = Gm^2/2R$. (c) Their reduced mass is $\mu = mm/(m+m) = m/2$, and their relative velocity is $2v = 2\sqrt{\frac{Gm}{2R}}$, since they have the same speed v in opposite directions (i.e., the velocity of one as seen by the other is 2v). (d) The total CM kinetic energy is

$\frac{1}{2}\mu v_{rel}^2 = \frac{1}{2}\left(\frac{m}{2}\right)\left(2\sqrt{\frac{Gm}{2R}}\right)^2 = \frac{Gm^2}{2R}$, the same result found in part (b).

Day Fourteen

1. The angle θ_0 is dimensionless, while the dimensions of the other quantities are $[v_0] = L/T$, $[g] = L/T^2$, and $[h_{max}] = L$. We can't tell how h_{max} depends upon θ_0 using dimensions alone, but we do know that $h_{max} \propto v_0^2/g$. Therefore (A) if v_0 is doubled, h_{max} increases by a factor of four, and (B) if g is cut in half. h_{max} is doubled.

2. The instantaneous power input is $P = Fv$, the product of the spring force on the block and the block's velocity. Therefore $P = 0$ at A and C, because the velocity is zero at these points, and $P = 0$ at B because the spring force is zero at that point.

3. (a) The particle is at rest at $x = 0$, so all of its energy is potential energy, which is 4 J at that point, according to the graph. (b) At point b the potential energy is zero, so the total energy of 4 J is all kinetic; that is, $\frac{1}{2}mv^2 = 4$ J, so $v = \sqrt{\frac{8\text{ J}}{0.5\text{ kg}}} = 4$ m/s (c) Between points a and b the force $F = -dU/dx =$ constant, so the acceleration is $a = -\frac{dU/dx}{m} = -\frac{\Delta U/\Delta x}{m} = -\frac{(-4\text{ J}/2\text{ m})}{0.5\text{ kg}} = 4\text{ m/s}^2$. Therefore the time it takes to travel from a to b can be found from the uniform-acceleration formula $x = (1/2)at^2$. This gives $t = \sqrt{2x/a} = \sqrt{2(2\text{ m})/4\text{ m/s}^2} = 1$ s. Between points b and c the potential energy is constant, so there is no force on the particle; the particle therefore moves at constant velocity 4 m/s (from part (b)) and so requires 1/2 s to go the distance 2 m. It requires another 1 s to go from c to d, since the acceleration is equal but opposite to the acceleration from a to b. The total time to reach point d from point a is therefore 2.5 s. (d) It then requires an additional 2.5 s to return to a, and then another 2.5 s to return to d. The total time is therefore 7.5 s.

4. (a) The isolation diagrams are as shown.

(b) The $F_{net} = ma$ equations for the counterweight and the elevator (with rider) are

$$T - 2m_0 g = 2\, m_0\, a \qquad \text{and} \qquad 3\, m_0 g - T = 3\, m_0\, a.$$

Adding these equations to eliminate T and solving for a gives $a = g/5$. (c) The $F_{net} = ma$ equation for the rider is $m_0 g - N = m_0\, a = m_0 g/5$. Therefore the normal force is $N = m_0 g - (1/5)\, m_0 g = (4/5)\, m_0 g$, which is the rider's weight.

5. (a) The normal force N is everywhere perpendicular to the motion from A to B, so the work $W_N = 0$. (b) The work done by gravity is positive as the sled slides downhill and negative as it slides uphill. The net work done by gravity is $mg\Delta y$, where Δy is the net downward vertical displacement, equal to $\Delta y = h - R - R\sin 30^{\circ}$ (from the problem figure). Therefore the net work done by gravity is $W_{grav} = mg(h - (3/2)R)$. (c) According to the work-energy theorem, the total work done equals the change in kinetic energy. Only gravity does work, and there is no initial kinetic energy, so the net work done by gravity will be the kinetic energy $\frac{1}{2} m v^2$ at point B. Therefore, $v_B = \sqrt{2g(h - (3/2)R)}$.

(d) The sled's isolation diagram when at point B is shown at the right.

(e) The normal force vanishes as the sled starts to leave the track. The only inward force on the sled is therefore the gravity componen $mg\cos 60^{\circ}$, which by Newton's second law is related to the inward acceleration $a_{centripetal} = v^2/R$ by $F = mg\cos 60^{\circ} = ma = mv^2/R$. Therefore the squared speed at B must be $v_B{}^2 = gR\cos 60^{\circ} = gR/2$. We can eliminate v_B between this equation and the result found in part (c) to solve for h, giving $h = 7R/4$.

6. (a) If there were no friction, energy would be conserved, so the kinetic energy at the bottom of the straight section would be $mg\,\ell$, the potential energy at the top. The kinetic energy at the bottom is only half that large, however, so the energy removed by friction must be $mg\,\ell/2$; that is, the work done by friction must be $W_{fric} = -mg\,\ell/2$, negative because it has removed energy. The work done by friction is also the product of the frictional force and the distance d moved (there is no need to integrate, since the frictional force is constant). That is,

$$W_{fric} = F_{fric} \cdot d = -\mu_k N \cdot d = -\mu_k mg\cos\theta \cdot d$$

where θ is the slope angle relative to the horizontal, as shown. The height $\ell = d\sin\theta$, so the frictional work is

$$W_{fric} = -\mu_k mg\cos\theta \cdot \ell/\sin\theta = -\mu_k mg\,\ell\cot\theta = -mg\,\ell/2,$$

as explained above. Therefore since $\mu_k = 0.50$, it follows that $\cot\theta = 1$, corresponding to $\theta = 45^{\circ}$.

(b) Construct the triangle shown. The angle at the lower left must be $90^{\circ} - \theta$, so $\phi = \theta$.

(c) The velocity at the bottom of the straight portion of the initial frictional slope can be found from $\frac{1}{2}mv^2 = \frac{1}{2}mg\ell$, which gives $v = \sqrt{g\ell}$. This is also the take-off speed at the right, since energy is conserved within the circular arc. The projectile range at 45° is therefore $D = v_0^2/g = \ell$.

7. (a) The momentum conservation equations are

$$p_x: \quad mv_0 = 2mv_d \cos\theta \qquad\qquad p_y: \quad mv_n = 2mv_d \sin\theta$$

Dividing the second equation by the first gives $\tan\theta = v_n / v_0$.
(b) Square the two equations and add them; then use the identity $\sin^2\theta + \cos^2\theta = 1$. This gives $v_d^2 = (v_0^2 + v_n^2)/4$. (c) Kinetic energy is conserved for elastic collisions, so

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_n^2 + \frac{1}{2}(2m)\, v_d^2 .$$

(i) Using the result of (b) to eliminate v_d, we find $v_n = v_0/\sqrt{3}$. (ii) It then follows from either equation that $v_d = v_0/\sqrt{3}$ as well. (iii) From the p_x equation above we find that $\cos\theta = \sqrt{3}/2$, corresponding to $\theta = 30^{\circ}$.

8. (a) The net force on the car must be *zero*, since it is not accelerating. Therefore the cable tension must be Mg. The power input from the motor is then $P = Fv = Mgv$. (b) The net force on the car is $T - Mg = Ma$, so the cable tension is $T = M(g + a)$. The upward velocity of the car is $v = at$ since a is constant and $v_0 = 0$. The power input is therefore $P = Tv = M(g + a)at$. (c) The gravitational force is constant, so the work done by gravity is Mgh, a positive quantity. If the car is initially moving upward as it reaches height h, gravity does negative work in bringing it to rest at some height greater than h, and then an equivalent positive work to bring it back to h. So the work done by gravity if the car starts at h and ends at the ground does not depend on the car's initial velocity.

Answers to some of the odd-numbered problems

1P1. 12:20 pm, 15 mi East of Barstow.

1P3. (a) $v = 3\alpha t^2$ $a = 6\alpha t$ (b) $v = \dfrac{-\alpha x_o}{(1+\alpha t)^2}$ $a = \dfrac{2\alpha^2 x_o}{(1+\alpha t)^3}$

1P5. (a) $v = (x_1/t_o)\sec^2(t/t_o)$ $a = (2x_1/t_o^2)\sec^2(t/t_o)\tan(t/t_o)$

(b) $v = -(x_1/t_o)\csc^2(t/t_o)$ $a = (2x_1/t_o^2)\csc^2(t/t_o)\text{cotan}(t/t_o)$

1P7. (a) $[0 \rightarrow 1)$ s & $(3 \rightarrow 4]$ s (b) $(1 \rightarrow 3)$ s (c) $(2 \rightarrow 4)$ s (d) $(0 \rightarrow 2)$ s

1P9. (a) $\mathbf{r} = (8\hat{\mathbf{x}} + 8\hat{\mathbf{y}} - 36\hat{\mathbf{z}})$ in (b) 38 in

1P11. (c) $\mathbf{D} = (4, 3, 0) = 4\hat{\mathbf{x}} + 3\hat{\mathbf{y}}$ (d) 5 m

1P13. (c) $\mathbf{r}_2 = (7, 0, 0) = 7\hat{\mathbf{x}}$ $\mathbf{r}_3 = (7, 2, 0) = 7\hat{\mathbf{x}} + 2\hat{\mathbf{y}}$

1P15. (a) thrown from any point on the rotation axis, along the rotation axis

1P17. slightly to the right, approximately 3 mm

1P19. (a) 4.4 m (b) 35°

1P21. (a) $\mathbf{D} = (-2\hat{\mathbf{x}} + 5\hat{\mathbf{y}} - 5\hat{\mathbf{z}})$ (b) 7 m (c) no. It moves 9 m (to 1 sig. fig.)

1P25. (a) $(1/2)x^{-1/2}$, 0.500 (b) 0.475

1P27. (a) $\mathbf{v} = \left(-\alpha x_o e^{-\alpha t},\ -2\beta y_o t e^{-\beta t^2},\ -3\gamma z_o t^2 e^{-\gamma t^3}\right)$

(b) $\mathbf{a} = \left(\alpha^2 x_o e^{-\alpha t},\ -2\beta y_o\left(1-2\beta t^2\right)e^{-\beta t^2},\ -3\gamma z_o t\left(2-3\gamma t^3\right)e^{-\gamma t^3}\right)$

1P31. (a) $\sqrt{2}D$ (b) $\left(\dfrac{\pi}{2} - \sqrt{2}\right)D$

2P1. (a) $v' = \dfrac{\sqrt{5}}{2\sqrt{2}}v_o$ $\theta' = 63°$

2P3. (a) $v\cos\theta$, to the right (b) 3.5 m/s, to the right

2P5. (a) $t = 30$ s, $d = 15$ m (b) $\theta = 120^o$, $t = 35$ s

2P7. $v = 29$ m/s $\theta = 63^o$

2P9. (a) $\theta = \tan^{-1}(v_b/v_r)$ (b) The number of raindrops entering the bus is independent of its speed.

2P11. $\sin^{-1}(4/5)$

2P13. (a) 93 m/s (b) 54°

2P15. (1) $v_1 = 2v_\alpha$ $\theta_1 = 0$ $v_2 = 0$ $\theta_2 = 0$

(2) $v_1 = \sqrt{2}v_\alpha$ $\theta_1 = 45^o$ $v_2 = \sqrt{2}v_\alpha$ $\theta_2 = 45^o$

2P17. (a) 27° (b) 6:00 a.m. (0600 hours)

2P19. (a) Yes, it takes only about an hour to arrive. (b) 0°. It approaches from due south.

2P21. The shape is a parabola.

2P23. (a) 4.0 minutes (b) twice around

2P25. (a) $x' = x\cos\theta + y\sin\theta$ $y' = -x\sin\theta + y\cos\theta$ $z' = z$

(b) $x' = 2.0$ m $y' = -1.0$ m $z' = 3.0$ m

2P27. (a) $t = 1/H$ (b) $1.4 \cdot 10^{10}$ yr $\leq$ age $\leq$ $1.7 \cdot 10^{10}$ yr (c) older

3P1. (a) $(v'_a)_f = -20$ cm/s $(v'_b)_f = 70$ cm/s (b) $(v'_a)_f = -12$ cm/s $(v'_b)_f = 42$ cm/s

3P3. (a) 2.9 m/s (b) $v_{cf} = -0.9$ m/s $v_{8f} = 1.9$ m/s

3P7. (a) $-v_o/2$; bounces *back*. (b) $v_o/6$, *forward*.

3P9. 8.0 mi/hr opposite to its original direction of motion

3P11. (a) $m_a = 1.5$ kg (b) $(v'_b)_i = -100$ cm/s, $(v'_a)_f = -50$ cm/s, $(v'_b)_f = -25$ cm/s.

3P13. (a) 200 m/s (b) No, at least not by conventional means. Their limit is about 100 mi/hr.

3P17. (a) $M_b v_1/(M_p + M_g)$ (b) $M_b v_2/(M_p + M_g + M_b)$

3P19. (a) $v_f = [(M + m)v_o - mv_b] / M$ (b) 38 m/s

3P21. (a) $(2/\sqrt{3})$ m/s (b) $(\sqrt{3} + 1/\sqrt{3})$ m/s

3P23. (a) $v_{HC} = 63$ mi/hr (b) $v_{both} = 29$ mi/hr

3P25. 4.4 kg m/s

3P27. Two quantities must be measured.

3P29. (b) $M_N/M_p = 16$ (d) $v_{pi} = 2.45 \cdot 10^7$ m/s

3P31. (c) $(p_b)_f^2 = (p_a)_i^2 + (p_a)_f^2 - 2(p_a)_i (p_a)_f \cos\theta$

3P33. (a) $9.1 \cdot 10^{-28}$ g (b) $h = 6.627 \cdot 10^{-27}$ g cm^2/s

3P35. v increases by factor $\sqrt{2}$

3P37. (a) ℓ_p would be reduced by the factor $1/(2\sqrt{2})$

3P39. $2\sqrt{2}\,T$

3P41. (a) $v_2 = 2V + v$, and moves to the right. (b) $v_1' = V + v$ to the left,

$v_2' = V + v$ to the right. (c) $\theta = \tan^{-1}(v_o/V)$

3P43. 117°, $1.12\,p_N$

4P1. (a) $x_{CM} = 16\frac{2}{3}$ cm (b) $x_{CM} = 25$ cm

4P3. (a) $y_{CM} = 6.5$ m (b) No (c) $V_{CM} = 0.88$ m/s *downward*

4P5. 4.7 m

4P7. (a) $L/6$ (b) $L/6 + v_o L/u$

4P9. (a) $7L/12$ (b) $3L/4$

4P11. $\frac{1}{3}\sqrt{4d^2 - D^2}$

4P13. (a) 0.941 Å (b) 0.302 Å

4P15. $4R/3\pi$ from the center of the bottom edge of the semicircle

4P17. $L/\sqrt{3}$ from any vertex

4P19. (a) $V_{CM} = v_o/4$ (b) α moves to the right at $3v_o/4$, β to the left at $v_o/4$. (c) It's not.

4P21. (b) the speeds are $v_A' = (\sqrt{5}/3)\cdot 10^4$ km/s $v_B' = (2\sqrt{5}/3)\cdot 10^4$ km/s

4P23. (a) 0.2 m/s (b) 14 kg m/s (c) 0 (d) – 0.17 m/s (e) – 14 kg m/s (f) 0 (g) 0

4P25. (a) $2.5 \cdot 10^6$ kg m/s (b) – $2.5 \cdot 10^6$ kg m/s (c) 6.36 km/s (d) 6.30 km/s

4P29. $F_{max} = 3 \cdot 10^3$ N

4P31. (a) $I = 0$ (for $0 \le t < t_o$); $I = \frac{I_f}{(t_1 - t_o)}(t - t_o)$ (for $t_o \le t < t_1$); $I = I_f$ $(t \ge t_1)$

4P33. (a) $At^3/3$ (b) $(A/\alpha)(1 - e^{-\alpha t})$

4P35. (a) 39 N/m^2 (b) 78 N/m^2 (c) 68 N

4P37. (a) $3.3 \cdot 10^3$ cm/s (b) 7.9 cm^3/s (c) 3.9 N (d) less than

4P39. (a) $Nv/6\ell$ (b) $Nmv^2/3\ell^3$

4P41. (a) 25 m/s (b) 390 N

4P43. (a) $mnAv^2$ (b) $1.5 \cdot 10^{-10}$ N (c) roughly 10^9 years

5P1. (a) $d/2$ (b) $3d/2$

5P3. (a) $\Delta x/2$ (b) $\Delta x/3$ (c) $11\Delta x/6$

5P5. $k/2$. The double-spring is half as stiff as a single spring, so it has a smaller force constant.

5P11. (a) $a = \frac{F_0}{m} e^{\alpha t}$ (b) $v = v_o + \frac{F_0}{m\alpha}(e^{\alpha t} - 1)$ (c) $x = x_o + v_o t + \frac{F_0}{m\alpha^2}[e^{\alpha t} - 1 - \alpha t]$

5P13. $(-0.88 + 18t^2) \cdot 10^{-3}$ N (t in sec)

5P15. (a) $\langle F \rangle = 7.5 \cdot 10^{10}$ N (b) $\langle a \rangle = 1.9$ m/s^2 (c) $\langle F \rangle$ and $\langle a \rangle$ would each increase by a factor of ten, but the velocity gained would not be changed.

5P17. (a) $v_o t_1 + [mv_o^2 / 2F_o]$ (b) $t = t_1 + [2mv_o/F_o]$

5P19. (a) 53 m/s^2 (b) 5.3 s

5P21. (a) 540 m (b) 46 s

5P23. (a) 125 s (b) 4.5 km

5P25. Yes, 12 light-hours from the base

5P27. (a) 10 s (b) 0.4 m/s (c) 800 m

5P29. (a) (1/60) s^{-1} (b) 0.10 s^{-1} (c) 0.37 cm/s (d) 0.038 m/s^2

5P31. (a) $L\omega \cos\theta$ (b) $L\omega^2 \cos\theta$

5P33. (a) 34 ft/s (b) 0.063 s (c) 100 s^{-1} (d) $3.3 \cdot 10^3$ ft/s^2 (e) 10^2 g's

5P35. (a) $v = 470$ m/s $a = .034$ m/s^2 (b) $v = 10^6$ m/s $= c / 300$ $a = 10^8$ m/s^2

5P37. $n \cong -2$

5P39. (a) $v_1 = \sqrt{2F_0\ell / m}$ (b) $v_N = \sqrt{2NF_0\ell / m}$ (c) $r_N = \sqrt{2NF_0\ell m} / qB$

6P1. $\frac{2}{3}D - \frac{1}{3}\ell$

6P3. $\Delta x/2$

6P5. $\theta = 140^o$ $T = 3.2 \cdot 10^{-2}$ N

6P7. (a) $\omega = v/\ell$ (b) $v_o = \sqrt{T_0\ell/m}$

6P9. (a) $gP^2/2\pi^2$ (b) $Mg + mg/4 - 4\pi^4 mx^2/gP^4$

6P11. (a) $T = N$ (b) $\Delta x = N/k$

6P13. (a) $a = \dfrac{k\Delta x}{M+m}$ (b) $N = \left(\dfrac{M}{M+m}\right)k\Delta x$

6P17. $\Delta x_1 = (m_1 + m_2 + m_3)a/k_1$ $\Delta x_2 = (m_2 + m_3)a/k_2$ $\Delta x_3 = m_3 a/k_3$

6P19. $a = \dfrac{F_o}{m_1 + m_2}$ The tension in the right string is F_o, and the left string is $T = \dfrac{m_2}{m_1 + m_2} F_o$

6P21. $m_3 = m_2 = T_o/4a$ $m_1 = T_o/2a$

6P23. (a) 10 N, to the right (b) 15 N, to the left

6P25. (a) $4.5 \cdot 10^{10}$ N (b) $3.0 \cdot 10^{10}$ N (c) $1.5 \cdot 10^{10}$ N (d) 43 km

6P27. (b) $v_m = \dfrac{\dfrac{kd}{m}\left(\dfrac{T}{2\pi}\right)}{\dfrac{k}{\mu}\left(\dfrac{T}{2\pi}\right)^2 - 1}$, where $\mu = \dfrac{mM}{m + M}$

6P29. $\Delta x_1 = \dfrac{\left[m_2(\ell_1 + \ell_2)k_2 + m_1\ell_1(k_2 - m_2\omega^2)\right]\omega^2}{(k_1 - m_1\omega^2)(k_2 - m_2\omega^2) - m_2\omega^2 k_2}$ $\Delta x_2 = \dfrac{\left[k_1 - m_1\omega^2)\right]\Delta x_1 - m_1\ell_1\omega^2}{k_2}$

6P31. distance $= \dfrac{qE_o\tau^2}{6m}$

6P35. (a) 6.9 km/s (b) $2.3 \cdot 10^3$ km/s (c) $6.9 \cdot 10^4$ km/s

6P37. (a) 0.12 (b) 5.2 km/s

6P39. (a) $M_o \sim 10^{22,000}$ kg (b) $M_{universe} \sim 4 \cdot 10^{51}$ kg <<< M_o

6P41. (a) $v = u\ \ell n[M_o/(M_o - \lambda t)]$ (b) valid until all fuel is used.

6P43. $v \cong u\ \ell n[1 + (100m/M)] \cong u[\ 100m/M\ ]$

8P1. (a) 550 N (b) 540 N (c) 200 N Her weights are also 550 N, 540 N, 200 N respectively.

8P3. (a) 4.1 m (b) 9.0 m/s

8P5. 6.1 m

8P7. (a) 10 km (b) 60 km (c) 92 s (d) 550 s

8P9. 50 m

8P11. (a) 27 m (b) 180 m

8P13. 2.5 ft

8P15. (a) 47 m (b) 21°

8P17. $\tan^{-1} 2 = 63.4^{o}$

8P19. (b) 69° and 21°

8P21. (a) 120 m (b) 130 m (c) 26 m/s (d) 27°

8P23. (a) $h_{max} \sim v_o^2/g$ (b) factor of four (c) $(\sin^2\theta_o)\, v_o^2/2g$ (d) $R/4$

8P25. no

8P27. (a) 24 ft/s (b) 60°

8P29. (a) $\theta = (\pi/4) + (\phi/2)$

8P31. $\sqrt{\sqrt{3}\, gd}$

8P33. 36 lbs

8P37. (b) $2m_1g$ (c) $(m_2 - m_1)g$ (d) $2m_2g$ (e) $(m_2 - m_1)g/m_1$ (f) $(m_2 - m_1)g/2m_1$

8P39. (b) $mg/\cos\theta$ (d) $2\pi\sqrt{\ell\cos\theta / g}$

8P41. (a) 3.27 m/s^2 (b) 13.3 lbs

8P43. $(M + m)(g + a)/2$

8P45. (b) $-0.045\, g$ (c) $1.37\, mg$ (c) $1.82\, mg$

8P47. $a = g/3\sqrt{3}$

8P49. 50 s

8P51. (a) 23° (b) 1.09

8P53. (b) $\theta = \pm \cos^{-1}\left(\dfrac{g}{R\omega^2}\right)$, 0, π

8P55. The Sun pulls on Earth as well, so Earth and Moon "fall" together toward the Sun.
That is, they both accelerate together toward the Sun as they orbit the Sun.

8P57. (a) $5.1 \cdot 10^3$ s (1.4 hours)

8P59. $h = 35.8 \cdot 10^3$ km; $h/R_E = 5.6$

8P61. (a) $T = 2\pi / \sqrt{\frac{4}{3}\pi G\rho_{DM}}$

9P1. $a_{min} = g/\mu_s$

9P3. 0.1

9P5. 0.44

9P7. $\mu_s \geq \tan\theta$

9P9. (a) $v_o^2/2\mu_s g$ (b) $v_o^2/\mu_s g$

9P11. (a) $\mu_s(g + v^2/R)$ (b) $\mu_s(g - v^2/R)$

9P13. (a) $\omega_{min} = \sqrt{\frac{g}{r}\left[\frac{\cos\theta - \mu_s \sin\theta}{\sin\theta + \mu_s \cos\theta}\right]}$

9P15. 4

9P17. (b) 0.65

9P19. (a) – 2 m/s^2 (b) 3 m/s

9P21. (a) $\theta = \tan^{-1}\mu_k$ (b) $\mu_k Mg/\sqrt{1+\mu_k^2}$ (c) $1/\sqrt{1+\mu_k^2}$

9P23. (b) $g\left[\frac{(m+M)\sin\theta - (m\mu_m + M\mu_M)\cos\theta}{M+m}\right]$

9P25. (b) 0.27

9P27. No, the car was traveling about 87 km/hr.

9P29. (a) $3mg\sin\theta + 3F_k$ (b) $mg\sin\theta + F_k$ (c) $Mg\cos\theta$ (d) $(3m+M)g\sin\theta + 3F_k$

9P31. (a) $F_o = \frac{\mu_k mg}{\sin\theta - \mu_k \cos\theta}$ (b) $\theta_c = \tan^{-1}\mu_s$

9P33. (a) $6.4 \cdot 10^2$ N (b) $6.9 \cdot 10^2$ N (c) 0.61 m/s^2 (d) $6.0 \cdot 10^2$ N

9P43. $v = \frac{v_o}{\left(1 + \frac{k'}{k}v_o\right)e^{kt/m} - \frac{k'}{k}v_o}$

9P45. (b) M/L^3 (c) 5 %

9P47. High η, F_D is doubled; low η, F_D is quadrupled

9P49. factor of 16

10P1. (a) 17 m/s (b) 21 m

10P3. (a) $\theta_o = \sin^{-1}\left(\frac{1}{2}\sin\theta\right)$ (b) 14°

10P5. $1.3 \cdot 10^3$ ft

10P7. (a) $\sqrt{v_0^2 + 2gh/3}$ (b) $\sqrt{v_0^2 + 4gh/3}$ (c) $\sqrt{v_0^2 + 2gh}$

10P9. (a) $\sqrt{g\ell(2 - \sqrt{2})}$ (b) $\sqrt{2g[h - (\ell/\sqrt{2})]}$

10P11. $\sqrt{\frac{2(M - m\sin\theta)gh}{M+m}}$

10P13. (a) $\sqrt{\frac{2mgy}{2M+m}}$ (b) $\sqrt{\frac{2(2M+m)h}{mg}}$ (c) $\sqrt{\frac{2mgh}{2M+m}}$

10P17. (a) 50 J (1 sig fig) (b) 5 m

10P19. (a) $N = \frac{mv_0^2}{\ell} - 3mg$ down, assuming the block reaches the top (b) $\sqrt{3g\ell}$

10P21. (a) $\sqrt{2g\ell\left(\cos\theta - \cos\theta_0\right)}$ (b) $T = mg(3\cos\theta - 2\cos\theta_0)$

10P23. (a) $v_{max} = \sqrt{gR(2 - \cos\theta_0)}$ (c) he will reach B if and only if $\theta_0 \le \cos^{-1}(2/3)$

10P25. (a) $h_{max} = R/2$ (b) $\theta = \cos^{-1}[(2/3)(1 + \alpha/2)]$

10P27. (a) $v_A = \sqrt{2gh}$ $v_B = \sqrt{2g(h-R)}$ $v_C = \sqrt{2g(h-2R)}$ (b) $g(1 + 2h/R)$ *down*

10P27. $v = \sqrt{\frac{g}{\ell}(x - x_0)(\Delta x_0 + 2(x - x_0))}$

10P29. (a) $\sqrt{\frac{k}{m}}\left(\frac{D}{2}\right)$ (b) 18 cm/s

10P31. (a) $\omega A\cos\omega t$ (b) $(m/2)\omega^2A^2\cos^2\omega t$ (c) $(k/2)A^2\sin^2\omega t$ (d) $kA^2/2$

10P33. $k\Delta x^2/2mg$

10P35. (a) $\sqrt{v_0^2 + gh/2}$ (b) $\sqrt{v_0^2 + gh}$ (c) $\sqrt{v_0^2 + 3gh/2}$

10P37. $3mg$ *upward*

10P39. (a) $R = (2/3)h$ (b) zero

11P1. (a) $9.8 \cdot 10^2$ J (b) $1.5 \cdot 10^2$ m/s

11P3. (a) $Mv/2F_0$ (b) $Mv^2/4$

11P5. $2.1 \cdot 10^{14}$ J

11P7. (a) F_0d (b) zero (c) zero (d) $-F_0d$

11P9. (a, b, c) $\sqrt{v_0^2 - 2gy}$

11P11. (a) $-mgD/2$ (b) TD

11P13. (a) $70{\cdot}5^\circ$ (b) $48{\cdot}2^\circ$ (c) $131{\cdot}8^\circ$ (48.2° relative to the $-z$ axis)

11P15. (a) 2 (b) 60°

11P21. -11 J

11P23. (a) $\mathbf{N} = N(\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}})$ $\mathbf{F}_{grav} = -mg\,\hat{\mathbf{y}}$ $\mathbf{d} = d(\cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{y}})$

(b) $W_N = 0$ $W_{grav} = mgd\sin\theta$ (c) $K = mgd\sin\theta$ (d) $v = \sqrt{2gd\sin\theta}$

11P25. (a) $2C/k$ (b) $-2C/k$ (c) 0 (d) $2C/k$

11P27. (a) $-kA^6/6$ (b) $-(63/6)kA^6$ (c) $-(32/3)kA^6$

11P29. (a) $v = \sqrt{\frac{2}{m}\left(\frac{1}{2}k(\Delta x)^2 - \mu_k mg\Delta x\right)}$ (b) $\frac{k(\Delta x)^2}{2\mu_k mg} - \Delta x$

11P31. (a) $-3kA^2/8$ (b) $A/4$

11P33. (a) 59.4 m/s (b) 37.0 m/s (c) 41,200 J

11P35. $d = d_0 + \frac{1}{k}\left[\sqrt{(\mu_k mg)^2 + 2k\left(\frac{1}{2}mv_0^2 - \mu_k mgd_0\right)} - \mu_k mg\right]$

11P37. (a) $mgD\sin\theta$ $D\sin\theta/\sin\phi$

11P39. (a) $(1/2)kA^2$ (b) $(1/2)kA^2$ (c) $-(1/2)kA^2$ (d) A

11P41. (a) $MgL/8$ (b) $\sqrt{3gL}\ /2$

11P43. (a) (b) (c) zero

11P45. (a) $P_{grav} = -mgv$ $P_{rope} = +mgv$ $P_{tot} = 0$ (b) $P_{grav} = -(1/20)mg^2 t$
$P_{rope} = (21/400)mg^2 t$ $P_{tot} = (1/400)mg^2 t$

11P47. (a) mg/k (b) $g^2\sqrt{m^3/k}$ (c) $-mg^2\sqrt{m/k}$ (d) zero

11P49. 620 tonnes/s

11P51. $mg^2t/8$, $mg^2t/16$

11P53. (a) 28 mi/hr (b) 25%

11P55. (a) 225 W (b) 625 W (c) 850 W

11P57. (a) $\sqrt{2gs\sin\theta}$ (b) $Mgs\sin\theta$ (c) $Mgv\sin\theta$

11P59. (a) $-mg(v_o \sin\theta_o - gt)$

12P1. (a) $m^2g^2/2k$ (b) m^2g^2/k (d) $-m^2g^2/2k$ (e) m^2g^2/k

12P3. (a), (b) mg/k

12P7. (a) mg/k (b) $2mg/k$

12P9. (a) $-kQq/r$ (b) $kQq\left(\frac{1}{r} - \frac{1}{r_o}\right)$

12P11. (a) B, C, G, H (b) D (c) D, F (d) E (e) B (f) A

12P13. K is a maximum at f, and zero at a and k.

12P15. (a) 9.0 J (b) -3.0 m (c) $+3.0$ m (d) $\pm\sqrt{3}$ m/s

12P19. (b) $5ax^{-6} - 4bx^{-5}$ (c) 5a/4b (d) $\frac{256}{3125}\frac{b^5}{a^4}$

12P21. (a) $r = r_o$, $U = 0$ (b) D_e

12P23. $1.75 \cdot 10^6$ J

12P25. (b) $v = \sqrt{2g\ell(\sin\theta - \mu_k\cos\theta)}$ (c) $\mu_k = \frac{(\ell + x)\sin\theta - x^2/2d}{(\ell + x)\cos\theta}$

12P27. (a) $-1.5 \cdot 10^4$ J (b) 120 m/s

12P29. about $8.6 \cdot 10^{10}$ solar masses

12P31. 19.3 km/s

12P33. (a) 42,300 km

12P35. (b) $\frac{\sqrt{5}}{2}v_o$ (c) $\frac{3}{2}v_o$

12P37. (a) 10^{18} kg/m^3 (b) 10^9 tonnes/cm^3 (c) $2 \cdot 10^{12}$ m/s^2 (d) $2 \cdot 10^8$ m/s

12P39. (a) $\sqrt{G\hbar/c^5}$ (b) $\sqrt{\hbar c^5/G}$

13P1. $(v_C)_f = (6/7)\,v_o$ $(v_{Be})_f = -(1/7)\,v_o$

13P3. $(v_b)_f = (3/4)\,v_o$ $(v_a)_f = -(1/4)\,v_o$

13P5. $(v_a)_f = \left(\frac{m_a - m_b}{m_a + m_b}\right)(v_a)_i + \left(\frac{2m_a}{m_a + m_b}\right)(v_b)_i$

13P7. (a) $m_2/m_1 = 1/3$ (b) $h_{max} = 4\,h_o$

13P11. (a) $v_3 = -v_o/4$ $v_1 = 3v_o/4$ (b) $v_3 = v_o/2$ $v_1 = 3v_o/2$

13P13. (a) $v_5 = -4v_o/9$ $v_4 = 5v_o/9$ (b) $v_5 = v_o/9$ $v_4 = 10v_o/9$

13P15. (a) $v_{f\,He} = -3.0 \cdot 10^6$ m/s $v_{f\,Be} = 1.0 \cdot 10^6$ m/s (b) $v_{f\,He} = -2.0 \cdot 10^6$ m/s
$v_{f\,Be} = 2.0 \cdot 10^6$ m/s

13P17. (a) 10 (b) $6.6 \cdot 10^6$ m/s

13P19. (a) $v_{nf} = 0.9\,v_o$ (b) $v_B = 0.13\,v_o$ (c) $\theta = \tan^{-1}(0.9)$

13P21. (a) $p_o = p_\alpha \cos\theta + p_{AU} \cos\phi$ $p_\alpha \sin\theta = p_{AU} \sin\phi$

(c) $p_{AU} = \sqrt{p_\alpha^2 + p_o^2 - 2p_\alpha p_o \cos\theta}$ (d) $p_{AU} = \sqrt{\frac{M_{AU}}{M_\alpha}}\sqrt{p_o^2 - p_\alpha^2}$

13P23. $v_\alpha = 2.02 \cdot 10^7$ m/s $v_{AU} = 8.34 \cdot 10^4$ m/s $\phi = 44.5^o$

13P29. (a) $(7/9)v_o$ (b) $8/57 = 0.14..$

13P31. (a) $v_o/15$ (b) 14/15

13P33. 0.996

13P35. (a) $\ell_a = \frac{4}{3}\sqrt{Rr}$ (b) $\ell_b = \frac{2}{3}\sqrt{Rr}$ (c) $\ell_c = \frac{2}{3}\sqrt{Rr}$

13P37. (a) $v_1 = (1 - R)\,v_o/2$ $v_2 = (1 + R)\,v_o/2$ (b) $(1/2)(1 - R^2)$

13P39. (a) $v_{5m} = v_o/4$ $v_m = 3v_o/4$ (b) 1/8

13P41. (a) $\theta = \cos^{-1}\left[1 - \left(\frac{m_1}{m_2 + m_3}\right)^2 \frac{\mu_k d}{\ell}\right]$

13P43. (a) $5.6 \cdot 10^4$ g cm^2/s^2 (b) $1.8 \cdot 10^4$ g cm^2/s^2 (c) $3.9 \cdot 10^4$ g cm^2/s^2

13P47. 1/2

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Greek Alphabet

α, A	Alpha	ν, N	Nu
β, B	Beta	ξ, Ξ	Xi
γ, Γ	Gamma	ο, O	Omicron
δ, Δ	Delta	π, Π	Pi
ε, E	Epsilon	ρ, P	Rho
ζ, Z	Zeta	σ, Σ	Sigma
η, H	Eta	τ, T	Tau
θ, Θ	Theta	υ, Y	Upsilon
ι, I	Iota	φ, Φ	Phi
κ, K	Kappa	χ, X	Chi
λ, Λ	Lambda	ψ, Ψ	Psi
μ, M	Mu	ω, Ω	Omega

Some Constants and Physical Properties

speed of light	$c = 3.00 \times 10^{8}$ m/s
gravitational constant	$G = 6.67 \times 10^{-11}$ N m^2/kg^2
Planck's constant	$h = 6.63 \times 10^{-34}$ J s
proton charge	$e = 1.60 \times 10^{-19}$ Coulomb
Boltzmann's constant	$k = 1.38 \times 10^{-23}$ J/molecule K
electron mass	$m_e = 9.11 \times 10^{-31}$ kg
proton mass	$m_p = 1.67 \times 10^{-27}$ kg
dry air at 1 atm and 20° C	density $\rho = 1.21$ kg/m^3;
	viscosity $\eta = 1.8 \times 10^{-5}$ N·s/m^2
water (at 20° C)	density $\rho = 1$ gm/cm^3 = 1000 kg/m^3
	viscosity $\eta = 1.0 \times 10^{-3}$ N·s/m^2
Earth: mass	$m = 5.98 \times 10^{24}$ kg
mean (i.e., average) radius	$r = 6.37 \times 10^{3}$ km
mean Earth-Sun distance	$d = 1.49 \times 10^{8}$ km
Sun: mass	$M = 1.99 \times 10^{30}$ kg
radius	$R = 6.96 \times 10^{8}$ m

Some Conversion Factors

mass:	1 kg = 1000 g 1 tonne = 1000 kg 1 ton = 2000 lbs
	1 u = 1.66×10^{-27} kg (atomic mass unit)
length:	1 m = 100 cm = 39.4 in = 3.28 ft
	1 in = 2.54 cm
	1 mi = 5280 ft = 1.61 km
time:	1 y = 3.16×10^{7} s
speed:	1 mi/hr = 1.47 ft/s = 1.61 km/hr = 0.447 m/s
force:	1 N = 10^5 dyne = 0.225 lb
energy:	1 J = 10^7 erg = 0.738 ft-lb
	1 eV = 1.60×10^{-19} J
	1 kiloton explosion of TNT = $4.2 \cdot 10^9$ J
power:	1 horsepower = 746 W = 550 ft lb/s