Day Fifteen

Accelerating Frames and Artificial Gravity

"It is better to have absolutely no idea
Where one is and to know it
Than to believe confidently that one is
Where one is not." – Cassini

By now we have left Iceworld far behind. Our ship drifts through space, far from any planets or stars and their attendant gravitational pulls. We float around inside the cabin, weightless as we were back in the Bubble.

15.1 Artificial gravity in an accelerating frame

We turn on the engines so the ship begins a period of constant acceleration. We find ourselves falling toward the rear of the ship as though someone had turned gravity back on again. We are now able to stand up in the cabin, with an effective gravity acting toward the rear.

What causes this effective gravity? How large is it?

The rest-frame of our ship has the constant acceleration a_F in the x-direction relative to an inertial frame, as shown in Fig. 1. If our accelerating (ship) frame starts from rest at $t = 0$ relative to the inertial frame, at a later time t its velocity in the inertial frame is $v_F = a_F t$. Also, the position of some fixed point in the ship that started at $x = 0$ has now moved to the position $x_F = (1/2)a_F t^2$. These facts can help us find how the position and velocity of an object in the inertial frame are related to those in the accelerating frame. That is, if we know the position x and velocity v of the object in the inertial frame, what are the position and velocity of the object in the accelerating frame? And *vice versa*?

Fig. 1. An inertial (unprimed) frame and a uniformly accelerating (primed) frame

Let $\mathbf{r} = (x, y, z)$ be the position of a particle in the inertial frame, and $\mathbf{r}' = (x', y', z')$ be its position in our uniformly accelerating (spaceship) frame. Then

$$x' = x - x_F = x - (1/2)\, a_F t^2 ,$$

$$y' = y , \qquad \text{and} \qquad z' = z. \tag{1}$$

That is, the y- and z-coordinates of the particle are the same in both frames, but the x-coordinates differ by the distance the primed frame has moved relative to the unprimed frame, as shown in Fig. 1. In vector form, Eqs. 1 are simply

$$\mathbf{r}' = \mathbf{r} - (1/2)\, \mathbf{a}_F t^2 , \tag{2}$$

where $\mathbf{a}_F$ is in the positive x-direction. The three vectors are illustrated at the right.

Sample Problem 1. The uniform acceleration of our spaceship is $a_F = 9.8$ m/s^2. Inside the ship, we drop a ball from height $h = 2.0$ m above the floor. Find the ball's subsequent position in the spaceship as a function of time.

Solution: Pick an inertial (i. e., unaccelerating) frame that is instantaneously at rest relative to the spaceship when the ball is dropped, and for which the initial position of the ball is $x = h$. When we remove our hand holding the ball, no forces act on the ball, so its position in the *inertial* frame will *remain* at $x = h$. Assuming the spaceship accelerates in the x direction, Eq. 1 tells us that the position of the ball in the spaceship frame is

$$x' = h - (1/2)\, a_F t^2 = 2.0\text{ m} - (1/2)(9.8\text{m/s}^2) \cdot t^2 ,$$

decreasing with time. That is, the ball "falls" toward the floor as we watch it in the ship. The other coordinates y' and z' stay the same as they were when the ball was dropped.

Differentiating Eq. 2 with respect to time, we find that the velocities in the two frames are related by

$$\mathbf{v}' = \mathbf{v} - \mathbf{a}_F t, \qquad (3)$$

as illustrated. Differentiating again, we find that the accelerations $\mathbf{a}$ and $\mathbf{a}'$ of a particle observed in the two frames are related by

$$\mathbf{a}' = \mathbf{a} - \mathbf{a}_F. \qquad (4)$$

Suppose for example that an object is at rest in our accelerating (primed) frame, so that its acceleration $\mathbf{a}' = 0$. Then according to Eq. 4, its acceleration as seen by observers at rest in the inertial frame is $\mathbf{a} = \mathbf{a}_F$, the acceleration of the primed frame itself relative to the inertial frame. That is just what we would expect. When we are at rest in the accelerating spaceship, we will have the same acceleration as the ship itself when observed by people at rest in an inertial frame.

Sample Problem 2. A skyscraper elevator on Earth descends with constant acceleration $a_F = 3.0\ \text{m/s}^2$ relative to the ground. If a passenger drops her keys from height $h = 1.2$ m above the elevator floor, what will be the acceleration of the keys from her point of view, and how fast will they be moving when they strike the floor?

Solution: Let positive acceleration directions be downward. From her point of view, the acceleration of the keys is $\mathbf{a}' = \mathbf{a} - \mathbf{a}_F = \mathbf{g} - \mathbf{a}_F$, whose magnitude is

$$a' = 9.8\ \text{m/s}^2 - 3.0\ \text{m/s}^2 = 6.8\ \text{m/s}^2,$$

since $g = 9.8\ \text{m/s}^2$. *The keys fall with a smaller acceleration in the elevator that they would if the elevator were at rest relative to the ground.* Using the constant-acceleration formula linking a' with position and velocity in the elevator frame, the velocity v of the keys when they strike the floor obeys the equation

$$v^2 = v_0^2 + 2a'h = 0 + 2\,(6.8\ \text{m/s}^2)\,(1.2\ \text{m}) = 16.3\ \text{m}^2/\text{s}^2,$$

giving $v = 4.0$ m/s.

15.2 Newton's law $F = ma$ in uniformly accelerating frames

How can we do physics in accelerating frames? How can we predict motion within an accelerating spaceship, in an elevator accelerating up or down, or in a car accelerating away from a stoplight? We need to have a law we can use to predict how things move.

Unfortunately, we already know that $\mathbf{F} = m\mathbf{a}$ *doesn't work* to observers in accelerating frames! For example, if we place a ball at rest in our accelerating spaceship and then take our hand away so that nothing touches the ball, then no forces act on it, so its acceleration should be zero. However, its acceleration from our point of view in the ship is *not* zero; the ball accelerates toward the floor. Newton's second law $\mathbf{F} = m\mathbf{a}$ is valid to observers in inertial frames, but not to those in accelerating frames.

Is there any way we can *rescue* $\mathbf{F} = m\mathbf{a}$? Can we find a way of "fixing it up" so it still works?

Consider the motion of a particle. We know that Newton's second law $\mathbf{F} = m\mathbf{a}$ governs the motion of a particle to observers in an *inertial* frame, where $\mathbf{F}$ is the total force acting on the particle. Multiplying Eq. 4 by the particle's mass m,

$$\begin{aligned} m\mathbf{a}' &= m\mathbf{a} - m\mathbf{a}_F \\ &= \mathbf{F} - m\mathbf{a}_F \,. \end{aligned} \tag{5}$$

This equation can be made to *look* like Newton's second law $m\mathbf{a}' = \mathbf{F}_{\text{Total}}$ if we define a new "pseudoforce"

$$\mathbf{F}_{\text{pseudo}} = -\, m\mathbf{a}_F \,, \tag{6}$$

opposite in direction to $\mathbf{a}_F$, that we have to add to the real forces. Then Newton's law

$$\mathbf{F}' \equiv \mathbf{F} + \mathbf{F}_{\text{pseudo}} = m\mathbf{a}' \tag{7}$$

is obeyed in the uniformly accelerating non-inertial frame *if we add this pseudoforce* $\mathbf{F}_{\text{pseudo}} = -\, m\mathbf{a}_F$ *to all the real forces when we find the total effective force* $\mathbf{F}'$ *in the accelerating frame.*

Now it is clear why there seems to be gravity within a uniformly accelerating spaceship. The ship has a constant acceleration $\mathbf{a}_F$, so the pseudoforce acts just like an effective gravity in the opposite direction, with $\mathbf{g}_{\text{eff}} = -\,\mathbf{a}_F$. This "effective gravitational force" is

$$\mathbf{F}_{\text{pseudo}} = -m\mathbf{a}_F = m\mathbf{g}_{\text{eff}}, \tag{8}$$

arising from the fact that on the ship we are observing motion from a non-inertial frame.

Sample Problem 3. A spaceship moving through gravity-free space has acceleration $\mathbf{a}_F$ in the y direction relative to inertial frames. The rest-frame of the ship is the primed frame. A ball is "dropped" from rest by a passenger on the ship, from a point a height y'_0 above the floor. Find the subsequent velocity v' and altitude y' of the ball as a function of time. Neglect air resistance.

Solution: No real forces act on the ball. However, in the accelerating (ship) frame, there is a pseudoforce $-m\mathbf{a}_F$, where m is the ball's mass and $\mathbf{a}_F$ is the acceleration of the ship as observed in an inertial frame. In the ship's frame Newton's law is $\mathbf{F}' = m\,\mathbf{a}'$, which in this case becomes simply $-m\mathbf{a}_F = m\,\mathbf{a}'$. The ball's acceleration relative to the ship is therefore $\mathbf{a}' = -\mathbf{a}_F$, a constant vector in the negative y' direction. Now $a' = dv'/dt$, so integrating $dv' = a'\,dt$ in the y' direction gives

$$v' = v'_0 + a'\,t = -\,a_F t$$

since $a' = -a_F$ is a constant and the ball starts with $v'_0 = 0$. Also $v' = dy'/dt$, so integrating once again gives

$$y' = y'_0 - (1/2)a_F t^2.$$

Working entirely in the accelerating frame, the pseudoforce behaves just like a real gravitational force. That is, if we define the magnitude of an *effective gravity* by $g_{\text{eff}} = a_F$, the above equations become $v' = -\,g_{\text{eff}}\,t$ and $y' = y'_0 - (1/2)g_{\text{eff}}\,t^2$, the usual equations for the velocity and altitude of a freely-falling particle in uniform gravity.

It is easy to understand the motion by viewing the ship and ball from the point of view of an inertial frame that was at rest relative to the ship at the instant the ball was dropped. According to observers in the inertial frame, the ball stays at rest while the "floor" of the spaceship accelerates up to meet it. It is obvious that shipboard passengers will see the ball "fall".

Sample Problem 4. Find the *weight* of a passenger of mass m standing in a spaceship whose acceleration is $\mathbf{a}_{\text{ship}}$ measured in an inertial frame.

Solution: The magnitude of the passenger's weight is the reading on the scale. *To an observer in the ship*, the scale and the passenger remain at rest. Two forces act on the passenger, the normal force N due to the scale (a *real* force) and the *pseudo*force $ma_{\text{F}} = mg_{\text{eff}}$, which behaves like an effective gravity acting toward the rear of the ship

The acceleration of the passenger is zero in this frame, so the two forces must sum to zero; that is,

$$F' = N - ma_{\text{F}} = ma' = 0,$$

so $N = ma_{\text{F}}$. By Newton's third law, if the force on the passenger due to the scale is N, the force of the observer on the scale is also N, in the opposite direction. Therefore the scale reads $N = ma_{\text{F}}$. The scale reading is the weight of the traveler, $W_{\text{traveler}} = ma_{\text{F}}$.

Alternatively, we can find the weight of the traveler by working in an *inertial* frame. To an observer in an inertial frame the ship, scale, and traveler all have the acceleration a_{F} upwards, as shown.

Only a single force acts on the traveler according to inertial observers: the normal force N of the scale, as shown. (In the inertial frame there is no pseudoforce behaving like an effective gravity; only real forces exist.) Therefore $F = ma$ gives

$$N = ma = ma_{\text{F}}.$$

The force on the scale due to the traveler is also N by the force-counterforce rule (Newton's third law), so the scale reads $N = ma_{\text{F}}$. This is the weight of the traveler, the same result we got by working in the accelerating frame of the ship.

15.3 Inertial and gravitational mass

Life inside our uniformly accelerating spaceship seems the same as life at rest on Earth or Iceworld. As long as we don't look outside to see ourselves rocketing through empty space rather than sitting at rest on the ground, we can't tell the difference. It feels like gravity when we walk around. A ball dropped in our accelerating ship "falls" as we see it, because to us there is a pseudoforce mg_{eff} acting downward, where g_{eff} has the magnitude of the ship's acceleration. If the ship were standing upright at rest on a planet, the ball would fall due to the *real* gravitational force mg.

Is there no way to tell whether the spaceship is at rest on the ground or accelerating uniformly in gravity-free space, assuming we don't look outside? There *is* a *potential* way to tell.

This potential way to tell involves a very interesting conceptual difference between two kinds of mass we can assign to every object. On the one hand, an object has the property of *inertial* mass m_{I}. Inertial mass represents the "sluggishness" of the object, the mass in ma. According to Newton's second law, the acceleration of the object is $a = F_{\text{net}}/m_{\text{I}}$, so that objects with a large inertial mass are harder to accelerate than objects with a small inertial mass. *Inertial mass is that intrinsic property of an object that resists acceleration.*

There is a quite different kind of mass. The *gravitational* mass m_{G} of an object is the mass in $F_{\text{grav}} = m_{\text{G}}g$, the gravitational force on the object when it is in a gravitational field g. If the mass m_{G} of one object is twice that of another object, gravity pulls on the first object twice as hard. *Gravitational mass is the intrinsic property of an object that is pulled upon by gravity.* Note that gravitational mass is conceptually quite different from inertial mass. There is no obvious reason why they should be the same.

Now we can see a potential way to tell whether we are living in a uniformly accelerating spaceship or are at rest on the ground, without peeking. Suppose first we drop a ball inside our accelerating ship. Then the pseudoforce $m_{\text{I}}\mathbf{g}_{\text{eff}} = -m_{\text{I}}\mathbf{a}_{\text{F}}$ acts downward on the ball. The mass in this case is the *inertial* mass m_{I}, because the pseudoforce actually originates on the ma side of Newton's law, as can be seen from Eqs. 5 and 6 where all the masses are inertial masses. Newton's law $\mathbf{F}' = m_{\text{I}}\mathbf{a}'$ applied to the ball in the accelerating ship then becomes $\mathbf{F}' = -m_{\text{I}}\mathbf{a}_{\text{F}} = m_{\text{I}}\mathbf{a}'$, so the acceleration $\mathbf{a}'$ of the ball relative to the ship is $\mathbf{a}' = -\mathbf{a}_{\text{F}}$. The masses cancel out, so any dropped ball accelerates downward relative to the ship with the same acceleration, whatever its inertial and gravitational masses may be.

But now suppose a ship is at rest on the ground, in uniform gravity g. If we dropped a ball in this ship, the real gravitational force $m_G g$ would act upon it. Newton's second law then becomes

$$F = m_G g = m_I a, \tag{11}$$

so the acceleration of the falling ball would be

$$a = \frac{m_G}{m_I} g. \tag{12}$$

Only if $m_G = m_I$ for all objects will everything fall with the same acceleration $a = g$.*

This provides a potential way to tell whether we are in our accelerating ship or are at rest on the ground. If for some object $m_G \neq m_I$, that object will fall differently in a ship at rest on the ground than it will in an accelerating spaceship. So we can figure out which kind of ship we are living in.

The experimental evidence obtained to date shows that the two kinds of mass are equal to one another, within experimental error.* If the equality is only *approximate*, or if there is some special substance with $m_G = 2\, m_I$, for example, then we could test whether real or artificial gravity were acting upon us. If a spaceship were at rest on the ground the special substance would fall twice as fast as anything else inside the ship. But if the ship were accelerating in empty space, the special substance would "fall" just like everything else, because m_G wouldn't make any difference in that case.

* More generally, we could allow the inertial masses of all objects to differ from their gravitational masses by a constant factor. That is, the ratio of m_G/m_I would have to be the same for all objects. If so, an accelerating frame could still be equivalent to a frame at rest in a gravitational field if the magnitude of the gravitational field g were adjusted appropriately.

Box 1. Newton's pendulums

Isaac Newton was well aware of the conceptual difference between the inertial and gravitational masses m_I and m_G, and sought to discover experimentally if there was any measurable difference between them. More precisely, he sought to find if there was any difference in the ratio m_G/m_I between one object and another. He hit upon side-by-side pendulums as a way to do it.

Consider a box suspended from the ceiling by a long cord of length R. Fill the box with some material, so that the box plus contents has inertial mass m_I and gravitational mass m_G. Pull the box aside and let it swing back and forth. What is the *period* (the time for one complete oscillation) of this *pendulum*?

An isolation diagram of the box plus contents is shown. Two forces act on the box (if air resistance is neglected: gravity $m_G g$ and the cord tension T. The box obeys Newton's second law $\mathbf{F} = m_I a$. In the tangential direction (perpendicular to the cord) the only force is the component of gravity $m_G g \sin\theta$, where θ is the angle of the cord from the vertical. The arc-length distance of the box from its lowest point is $s = R\theta$, so the tangential velocity and acceleration of the box are

$$v_{\text{tangential}} = \frac{ds}{dt} = R\frac{d\theta}{dt} \quad \text{and} \quad a_{\text{tangential}} = \frac{d^2 s}{dt^2} = R\frac{d^2\theta}{dt^2}.$$

The box should therefore obey $F_{\text{tangential}} = m_G g \sin\theta = m_I\, a_{\text{tangential}} = m_I\, R\dfrac{d^2\theta}{dt^2}$, so the angular acceleration $\dfrac{d^2\theta}{dt^2}$ of the box is

$$\frac{d^2\theta}{dt^2} = \left(\frac{m_G}{m_I}\right)\frac{g}{R}\sin\theta.$$

The angular acceleration obviously depends upon the ratio m_G/m_I. So if two boxes of the same size and shape (to make air resistance the same for each) swing back and forth side by side, they should stay swinging *together* if the ratio m_G/m_I is the same for each. If the ratio is *less* for one than the other, the acceleration of the first will be *less*, so it will *lag* the second in its oscillations back and forth. Newton tried this for many materials, and found that the pendulums swung together, neither one lagging behind the other.

> . . . I tried the thing in gold, silver, lead, glass, sand, common salt, wood, water, and wheat. I provided two equal wooden boxes. I filled the one with wood, and suspended an equal weight of gold (as exactly as I could) in the centre of oscillation of the other. The boxes, hung by equal threads of 11 feet, made a couple of pendulums perfectly equal in weight and figure, and equally exposed to the resistance of air: and, placing the one by the other, I observed them to play together forwards and backwards for a long while, with equal vibrations. And therefore (by Cor. I and VI, Prop. XXIV, Book II) the quantity of matter in the gold was to the quantity of matter in the wood as the action of the motive force upon all the gold to the action of the same upon all the wood; that is, as the weight of the one to the weight of the other.
>
> And by these experiments, in bodies of the same weight, one could have discovered a difference of matter less than the thousandth part of the whole.

I. Newton in *Principia* (1686), "The System of the World", paragraph 19. Originally published by the University of California Press.

15.4 The principle of equivalence

The equivalence of uniformly accelerating frames with frames at rest in uniform gravity is intimately tied up with the equality of inertial and gravitational mass. However, the equivalence of uniformly-accelerating frames and unaccelerated frames with gravity goes much farther, according to a *Principle of Equivalence* proposed by Albert Einstein.

Einstein believed that for experiments carried out entirely within a closed box (such as the cabin of a spaceship), there is *no way* to tell whether the box is at rest on a planet with uniform gravity or is in empty space undergoing uniform acceleration. Einstein proposed that the Principle of Equivalence should hold not only for experiments in mechanics, but for any experiments at all, including experiments in optics, nuclear or particle physics, thermodynamics, you name it. So far, no exceptions have been found.

A symbolic picture of the principle is shown in Fig. 3.

Fig. 3. The Principle of Equivalence (figure by A.Mittelstaedt, Am. J. Phys., 65 318 (1997)

What can we predict from the principle of equivalence?

(1) We already know we can predict that $m_G = m_I$ for all objects. If any discrepancy from this rule were observed, however small, that would disprove the principle.

(2) We know that there must be an effective gravity in the accelerating ship, which behaves just like real gravity.

(3) *Any* experiment we do entirely within a frame at rest on the ground must behave the same way in an accelerating frame without gravity, and vice versa. Sometimes one frame is easier to work in, or understand physics in, than the other, but we can firmly predict what will happen in the other frame, based on the principle of equivalence.

Suppose for example someone had never before been in a car accelerating forward when the light turns green. How might they predict what it would feel like?

The car is an accelerating frame, accelerating forward. But that is equivalent to a car at *rest*, with an effective gravity *backward*, i.e. in the opposite direction to the car's acceleration, and with a magnitude equal to the magnitude of the car's acceleration relative to the ground, as shown in Fig. 4. This effective gravity would pull a person backward, pressing the person harder into the back of the seat. (The effective gravity is of course in *addition* to the real gravity pulling downward.)

Fig. 4. The principle of equivalence applied to an accelerating car.

Sample Problem 5. A pair of "fuzzy dice" hangs from a string attached to the rear-view mirror inside a drag-racing car. After accelerating for some time, the driver notices that the string has the steady angle 17° from the vertical. What is the acceleration of the dragster?

Solution: In the accelerating frame of the car, there is an effective gravity $\mathbf{g}_{eff} = -\mathbf{a}_{car}$ directed toward the rear, in addition to the real gravity $\mathbf{g}$ directed vertically downward. The net gravity is therefore as shown, at angle 17° to the vertical so the string will hang in that direction. It follows from the triangle that $a_{car}/g = \tan 17^o$, so $a_{car} = g\tan 17^o = 9.8\ \text{m/s}^2 \cdot 0.306 = 3.0\ \text{m/s}^2$.

Note: Artificial gravity may turn out to be important for long spaceflights because there is evidence that prolonged exposure to "zero gee" can contribute to a loss of muscle tone and bone mass. As we saw in Section 15.1, uniformly accelerated straight-line motion could in principle be used to provide an artificial gravity within a spaceship. However, it is not possible for spaceships to undergo substantial uniform accelerations for very long, at least if they use conventional chemical fuels. As shown in the *Special Topic* at the end of *Day Six*, the mass M of a ship (including onboard fuel) with uniform acceleration a is $M = M_0 e^{-at/u}$, dropping off exponentially with time t, where M_0 is its initial mass and u is the ex- haust velocity. If $u = 3$ km/s (typical for chemical fuel rockets) and the ship acceleration is $a = 10\ \text{m/s}^2$ (about one Earth gee), the mass remaining is already only about 1% of the initial mass in 23 minutes after blast-off. So it is not possible for such chemically fueled uniformly-accelerating rockets to provide artificial gravity for very long. A better way is needed to provide artificial gravity for prolonged spaceflights or for living in space.

15.5 A rotating space station

Following a period of uniform acceleration while we enjoyed the artificial gravity, our ship has now been drifting for several hours while we float around weightless inside.

A giant cylinder looms into view dead ahead. It slowly rotates about its axis of symmetry.

We know that the rotating frame of the cylinder is a non-inertial, accelerating frame. Is there artificial gravity within the cylinder, just as there is in the uniformly accelerating frame of a spaceship?

We enter the cylinder through an air-lock at one end, very near the rim. We find ourselves pressed against the rim just as though gravity was acting radially outward from the cylinder's rotation axis. This is similar to what happened during *Day One* in the Bubble; when the Bubble rotated we were pulled outward as though by gravity.*

We are pressed against the rim because, from the standpoint of an external inertial observer, we naturally tend to move in a straight line at constant speed, as shown in Fig. 5(a). This straight-line motion is obstructed by the rim, which pushes us toward the axis of rotation so we move in a circle from the external observer's point of view, accelerating always toward the center of the circle. To the outside inertial observer the only force on us is this normal force due to the rim, so our isolation diagram is as shown in Fig. 5(b).

Fig. 5. A rotating space station. (a) Standing on the inside rim, we are forced to move in a circle from the point of view of an external inertial observer. (b) Our isolation diagram. Note that it is the normal force due to the rim that is responsible for deflecting us from a straight-line path.

* Rotating centrifuges use this type of artificial gravity to separate chemical and biological samples.

We have defined the weight of an object to be the force exerted by the object on a scale that supports it. A scale exerts a normal force on the object to keep it in place; the object exerts an equal but opposite force on the scale. This is the weight of the object. What then is our weight within the rotating space station?

If we stand on a scale, as shown, we move in a circle according to inertial observers, so we accelerate toward the cylinder axis with acceleration $a = v^2/R = R\omega^2$, where v is the rim speed, R is the radius of the circle, and ω is the angular speed of the station.

The only force F on us is due to the scale, so from Newton's second law this force is $F = ma = mv^2/R = mR\omega^2$, directed toward the cylinder axis, where m is our mass. By Newton's third law (the force-counterforce rule), we exert an equal but opposite force on the scale, directed radially outward, away from the cylinder axis.

By definition, this force F is our weight

$$\mathbf{W} = \frac{mv^2}{R}\hat{\mathbf{r}} = mR\omega^2\,\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}$ is a unit vector (i.e., a vector of unit length) pointed radially *outward*, in the direction of increasing r. The weight of any object in the station is proportional to its mass, so we can write $\mathbf{W} = m\mathbf{g}_{\text{eff}}$, where the magnitude of the effective gravity vector $\mathbf{g}_{\text{eff}}$ is

$$g_{\text{eff}} = \frac{v^2}{r} = r\omega^2$$

(r is the distance from the center of the cylinder), and the direction of $\mathbf{g}_{\text{eff}}$ is outward, away from the rotation axis. At a given position, $\mathbf{g}_{\text{eff}}$ is the same for all objects.

The weight of an object depends upon its location within the station. The angular velocity ω of the station is everywhere the same, so the weight of an object near the center is less than it is near the rim, since the effective gravity has the magnitude $g_{\text{eff}} = r\omega^2$.

Colonists in a huge rotating space colony could build a small mountain whose base is at the rim and whose summit is at the center of the colony, so that mountaineers would experience less and less gravity as they ascended, a rewarding way to climb! If zero-gee experiments or manufacturing were required, they could be carried out in the center of the colony, while residents could live for the most part on the rim.

Even though the artificial gravity felt by colonists living on the inside rim of a rotating colony could feel the same as ordinary gravity on the Earth, there are some obvious differences in the environment. For example, in a rotating colony people could look straight overhead and see other colonists walking around upside down on the opposite side of the rim.

The behavior of rapidly moving objects would also be different than on Earth. For example, suppose an object is thrown straight up in the air, above the thrower. On Earth the object falls back pretty much straight down. What happens in the colony?

Sample Problem 6. A cylindrical space colony of radius R rotates with period T. A colonist standing on the rim throws a ball straight "up" toward the rotation axis, with speed $v = 2\pi R/T$.

(a) Sketch the trajectory of the ball as seen by an outside inertial observer.

(b) Find the time required for the ball to return to the rim, as a multiple of T.

(c) How far along the rim must the colonist run, relative to the rim itself, to catch the ball?

(d) Sketch the path of the ball as seen by observers at rest in the colony.

Solution: (a) As seen by an inertial observer, the rim of the colony moves with speed $v = \dfrac{\text{distance}}{\text{time}} = \dfrac{2\pi R}{T}$, the circumference divided by the period. If the colonist throws the ball toward the rotation axis with this same speed, then from the point of view of an outside inertial observer the ball moves at a 45° angle as shown. In the inertial frame the ball subsequently moves in a straight line at constant speed, because there is no force on it. The ball then intersects the rim one-quarter of the way around, as shown.

(b) The ball's speed in this frame is $v = (v_x^2 + v_y^2)^{1/2} = (2\, v_x^2)^{1/2} = \sqrt{2}\,\dfrac{2\pi R}{T}$, so the time required for the ball to reach (i.e., "fall to") the rim is

$$t = \frac{\text{distance}}{\text{speed}} = \frac{\sqrt{2}\,R}{\sqrt{2}\,(2\pi R/T)} = \frac{T}{2\pi}.$$

(c) During this time the rim itself turns a distance $d = \text{speed} \cdot \text{time} = \dfrac{2\pi R}{T}\left(\dfrac{T}{2\pi}\right) = R$, so the ball strikes the rim a distance

$$\left(\frac{2\pi R}{4} - R\right) = \frac{\pi}{2}R - R = 0.57\,R$$

from the colonist. This is how far the colonist has to run to catch the ball, even though the ball was thrown vertically upward in the colony frame.

(d) The trajectory of the ball as seen by the colonist is as shown. Note that it rises vertically at the beginning, that its highest point is a distance $R/\sqrt{2}$ from the colony center (the same as in the inertial frame) and that as it arrives at the rim it falls vertically (perpendicular to the rim itself at that point), a distance $0.57\,R$ around the rim from where it started.

If a ball is thrown upward only a short distance, i. e., if its initial speed in the colony is only a small fraction of the colony's rim speed, the ball will behave more nearly like a ball thrown on Earth, rising almost straight up and falling almost straight down.

Sample Problem 7. A rotating cylindrical resort hotel is constructed, featuring a low-gravity swimming pool extending around much of the rim, as shown. (The illustration on the facing page is by Bob McCall, in an article by Timothy Ferris in *The New York Times Magazine*, Nov 28, 1999. The background for this problem was taken from Ferris' article.) The hotel has radius R and rotation period T, and a diving board is placed a distance $R/2$ from the rim. A guest steps off the board.

(a) Sketch the path of the guest in the frame of an outside inertial observer.

(b) How long does it take the guest to reach the pool?

(c) A friend is treading water beneath the diving board. How far from the friend will the diving guest enter the water, measured around the rim?

Solution: (a) As seen by an outside inertial observer looking in through one end of the hotel, the rim moves with speed $v_{\text{rim}} = \dfrac{2\pi R}{T}$, and the diver (before stepping off) has speed $v_{\text{O}} = \dfrac{\pi R}{T}$, since the diver is at distance $R/2$ from the axis of rotation. After stepping off, no forces act on the diver, so the diver subsequently moves in a straight line at constant speed, as shown.

(b) As seen in the sketch above, in the inertial frame the diver moves along one leg of a right triangle, a distance $d = \dfrac{\sqrt{3}}{2}R$, using the Pythagorean theorem. The speed of the diver is $v = v_{\text{O}} = \dfrac{\pi R}{T}$, so the time to reach the pool is $t = d/v = \dfrac{\sqrt{3}}{2}R \,/\, \dfrac{\pi R}{T} = \dfrac{\sqrt{3}}{2\pi}T$.

(c) The diver strikes the pool at a point an angle $\theta = 60^{\circ} = \pi/3$ radians around the rim from the point initially directly below the board, as can be seen in the triangle above. Measured around the rim, the distance is therefore $s_{\text{diver}} = R\theta = \pi R/3$. This is not the distance from the friend in the pool, however, because in the inertial frame the friend has been moving around a distance $s_{\text{friend}} = v_{\text{rim}}t$, where $t = \dfrac{\sqrt{3}}{2\pi}T$ is the duration of the dive. That is, the friend in the water has moved a distance $s_{\text{friend}} = \dfrac{2\pi R}{T}\dfrac{\sqrt{3}}{2\pi}T = \sqrt{3}R$. As the diver enters the water, the distance between the diver and friend is therefore $\Delta s = s_{\text{friend}} - s_{\text{diver}} = \left(\sqrt{3} - \dfrac{\pi}{3}\right)R = 0.685\,R$. The friend travels farther than the diver, so the diver enters the pool to the "left" of the friend.

State of the Search

1. Artificial gravity is achieved in accelerating reference frames. If a spaceship is uniformly accelerating in a straight line, for example, there is a backward effective gravity within the ship, numerically equal to the acceleration of the spaceship. That is, $\mathbf{g}_{\text{eff}} = -\mathbf{a}_{\text{ship}}$. The motion of an object inside the ship can be found in two ways:

   (*i*) We can solve $\mathbf{F} = m\mathbf{a}$ from the point of view of an observer in an external inertial frame, using only real forces to find the net force $\mathbf{F}$ on the object of mass m. Having found the motion from the point of view of the inertial (unprimed) observer, we can then translate the positions into the accelerating (primed) frame using the position, velocity, and acceleration transformations

   $$\mathbf{r}' = \mathbf{r} - (1/2)\,\mathbf{a}_F t^2, \qquad \mathbf{v}' = \mathbf{v} - \mathbf{a}_F t, \qquad \text{and} \qquad \mathbf{a}' = \mathbf{a} - \mathbf{a}_F.$$

   as explained in Section 15.1.

   (*ii*) We can solve $\mathbf{F}' = m\mathbf{a}'$ directly in the accelerating frame, if we add to all the real forces the pseudoforce $\mathbf{F}_{\text{pseudo}} = -m\mathbf{a}_F = m\mathbf{g}_{\text{eff}}$.

2. Newton's second law, $\mathbf{F} = m\mathbf{a}$ (where $\mathbf{F}$ is the sum of real forces acting on an object) may *only* be used by observers at rest in an inertial frame of reference. However, we can use $\mathbf{F} = m\mathbf{a}$ in a uniformly accelerating frame if we add to all the real forces the pseudoforce $\mathbf{F}_{\text{pseudo}} = -m\mathbf{a}_F = m\mathbf{g}_{\text{eff}}$, where $m\mathbf{g}_{\text{eff}}$ is an "effective gravitational force" acting in a direction opposite to the acceleration direction of the accelerating frame.

3. We can define two kinds of mass possessed by an object, the *inertial mass* m_I and the *gravitational mass* m_G. The inertial mass measures the "sluggishness" of the object, the mass in $m_I a$. *Inertial mass is that intrinsic property of an object that resists acceleration.* The gravitational mass of an object is the mass in $F_{\text{grav}} = m_G g$, the gravitational force on the object when it is in a gravitational field g. *Gravitational mass is the intrinsic property of an object that is pulled upon by gravity.* It is remarkable that these two very different kinds of mass are nevertheless equal to one another, according to all experiments done to date. More precisely, experiments show that the *ratio* of the two kinds of mass is the same for all bodies tested, whatever their composition, within experimental error.

4. Einstein's principle of equivalence states that physics in a uniformly accelerating frame, with no gravity, is the same as physics in an inertial frame with uniform gravity. That is, if an observer is inside a spaceship cabin, the observer cannot be sure, without looking outside, whether the ship is at rest in uniform gravity g or is accelerating in outer space in the absence of gravity, with an acceleration numerically equal to g. One consequence of the principle of equivalence is that the inertial and gravitational masses of an object are equal to one another.

   [The principle of equivalence was a step toward finding the theory of General Relativity, Einstein's 1915 theory of gravity. Using the principle of equivalence with light rays, one can show that if there is gravity, high-altitude clocks run fast compared with low-altitude clocks. One can then show that Euclidean (plane) geometry is invalid in four-dimensional spacetime, the space in which the single dimension of time is coupled to the three dimensions of space. The failure of Euclidean geometry in spacetime means that spacetime is "curved" in the presence of gravity. In general relativity the effects of gravity are due to this spacetime curvature. The theory is consistent with all confirmed experimental tests performed so far.]

5. Artificial gravity can be achieved in linearly accelerating frames; it can also be achieved in *rotating* reference frames, such as rotating space stations or space colonies. People living on the inside rim of a rotating cylindrical colony would experience an effective gravity directed outward, away from the axis of rotation. The magnitude of the effective gravity is $g_{\text{eff}} = r\omega^2$, where r is the radius of the colony and ω is the angular velocity of rotation.

PICTURE QUIZ 15

Test yourself; answers are given at the end of the text.

1. Tie a helium-filled balloon on a string attached to the floor of a car. Accelerate away from a stop sign. While the acceleration remains constant, does the balloon move

(A) forward (B) backward

(C) stay vertically above the place where it is tied?

2. An elevator on Earth can be made to accelerate up or down. We want to perform an experiment inside the elevator which requires that the effective gravity inside be 2.3 m/s^2. Relative to Earth's surface, the elevator could accelerate

(A) Upward, with $a = 12.1$ m/s^2

(B) Downward, with $a = 12.1$ m/s^2

(C) Upward, with $a = 7.5$ m/s^2 (D) Downward, with $a = 7.5$ m/s^2

(E) Upward, with $a = 2.3$ m/s^2 (F) Downward, with $a = 2.3$ m/s^2.

Mark all the above answers which apply.

3. A rotating cylindrical space station of radius R_0 and rotation period T_0 produces an effective gravity g_0 for people living on the inside rim. A second rotating station has radius $R_0/2$ and provides the same effective gravity g_0. What is its rotation period as a multiple of T_0? *Hint*: Use dimensional reasoning.

(A) $2\,T_0$ (B) $\sqrt{2}\,T_0$ (C) T_0

(D) $T_0/\sqrt{2}$ (E) $T_0/2$

PROBLEM QUIZ 15

Test yourself; answers are given at the end of the text.

1. Two blocks with masses m_1 and m_2 are connected by a string, which is placed over a pulley connected to a cart of mass M sitting on the Earth, as shown. The cart is pushed from the left with a constant force F_0. The pulley and all of the surfaces are frictionless. What force F_0 is needed to keep the entire system in relative equilibrium, so the two blocks will remain in their same positions relative to the cart?

(a) First, draw isolation diagrams and reason from the point of view of an inertial frame.
(b) Repeat the analysis in the accelerating frame of the cart, taking account of an effective gravity in this frame.

2. A rotating cylindrical space colony has radius R and angular speed ω. A motorcyclist of mass m drives around the inside rim at speed $v = R\omega/2$, in the plane of the picture. Find the two possible weights W the motorcyclist can have, depending upon whether she is driving in the same or in the opposite direction to the rotation direction of the colony.

3. A colonist stands on the inside rim of a rotating cylindrical space colony of radius R and angular speed ω. In what direction, and with what velocity in the plane of the picture, could the colonist throw a ball so that it would never strike the rim? Sketch the path of the ball as seen by an outside inertial observer who sees the colony rotate, and also the path as seen by colonists within the rotating frame.

Special Topic: The Coriolis Pseudoforce

A ball thrown in a rotating space station behaves strangely. If we stand still on the inner rim, with the station's artificial gravity pressing us "downward", things feel pretty much like they would on Earth. But when we throw a ball up toward the center of the station, it doesn't fall straight down again. The ball's path is curved as shown at the right, and also as pictured in *Sample Problem* 6 of Section 5.5.

Why does the ball move like that?

Suppose we establish a straight-line standard in the station, by tying one end of a string to a point on the rotation axis, and the other end to a point P on the rim, as shown, so that the string is taut. The string therefore defines a straight line from the axis to the rim. We then throw a ball from the axis toward the rim, in a direction parallel to the string. How does the ball subsequently move relative to the string?

Imagine a hypothetical observer floating outside the cylinder, who is at rest in a nonrotating, inertial frame. The observer sees that the station is rotating, and can peer into the station through a glass endcap, observing the motions of the stretched string and thrown ball.

The outside observer sees the ball move in a straight line at constant speed, since there are no forces on it. The position of the ball and a point P on the cylinder rim are shown at three successive times in Fig. S1(a). The ball starts at the center and moves in a straight line toward the right, initially in the direction of the point P. However, the observer sees that the stretched string rotates along with the rotating cylinder rim, so the ball cannot move alongside the string. The ball and string get farther and farther apart as time goes on, and the ball will not strike point P when it reaches the rim.

From our point of view within the rotating frame, the positions of the string and ball are shown in Fig. S1(b). The string stays at rest from our point of view, but we know that the ball and string must get farther and farther apart. Therefore the ball's path must *curve* from our point of view. The curvature can't be caused by the effective gravity within the rotating frame, which is always directly outward, away from the axis of rotation. There must be something else going on.

Fig. S1. A string is stretched taut in a rotating cylinder, extending from the rotation axis to a point P on the rim. A ball is thrown from the axis toward the rim, initially along the string direction.
(a) A sequence of three pictures from the point of view of an external inertial observer
(b) The same pictures from the point of view of cylinder occupants.

Consider the motion of the ball for a short time Δt after it is thrown from the center, seen from our point of view inside the station. The ball's sideways deflection will be $\Delta s = r\Delta\theta$, where r is the distance the ball has reached from the center, and $\Delta\theta$ is the angle through which the cylinder has turned during time Δt, as seen by the external inertial observer. From the external observer's point of view, it is the *string* that has turned through angle $\Delta\theta$ while the ball has gone straight; from our point of view the string has stayed put while it is the *ball* that has suffered a sideways deflection.

The station rotates through an angle $\Delta\theta = \omega\Delta t$ during time Δt, where ω is the station's angular velocity. That is, the sideways deflection of the ball is

$$\Delta s = r\Delta\theta = r(\omega t) = v\omega t^2 \qquad \text{(S1)}$$

since the radial distance moved by the ball is $r = vt$, where v is the ball's outward speed. The outward speed v is essentially constant for a very short time, and ω is constant as well, so the equation has the form of the usual uniform-acceleration equation $\Delta s = (1/2)at^2$, where the acceleration is called the *Coriolis acceleration**

* The properties of this acceleration were worked out by the French scientist G. G. Coriolis (1792-1843)

$$a_{\text{Cor}} = 2\omega v\,, \tag{S2}$$

the sideways acceleration as the ball starts to move away from the rotation axis. The Coriolis acceleration exists only in the noninertial, rotating frame of the station. There is no such acceleration in an inertial frame.

We can think of the Coriolis acceleration as caused by a *Coriolis pseudoforce*

$$F_{\text{Cor}} = ma_{\text{Cor}} = 2m\omega v \tag{S3}$$

on an object of mass m; again, this pseudoforce exists only in the rotating frame. Remarkably, the magnitude of the Coriolis pseudoforce is equal to $F_{\text{Cor}} = 2m\omega v$ *everywhere* along its path (not only near the rotation axis), where v is the speed of the particle at that point. The Coriolis pseudoforce is also always perpendicular to the velocity $\mathbf{v}$ at that point. We have assumed that the motion is confined to the plane of rotation. The proof of these properties will be carried out in Special Topic II at the end of *Day Twenty*. By that time we will have introduced *cross products*, which greatly simplify the derivation.

The direction of the Coriolis acceleration (and of the Coriolis pseudoforce) is as follows: If the rotating frame (the frame of the rotating station) rotates *counterclockwise* as seen by the external observer, moving objects deflect to the right as they move. That is, the Coriolis acceleration makes them bend in a clockwise sense, as in the examples below.

If the rotating frame rotates *clockwise* as seen by the external observer, the moving object deflects to the left, as in the examples shown below; that is, the Coriolis acceleration makes them bend in a counterclockwise sense.

There are exactly two pseudoforces acting on an object of mass m within a frame rotating with constant angular velocity ω. They are

(*i*) the centrifugal pseudoforce $F_{\text{centrifugal}} = m\omega^2 r$, acting radially outward from the rotation axis, where r is the distance of the object from the axis. The centrifugal pseudoforce provides an artificial gravity. Note that $F_{\text{centrifugal}}$ depends on the object's position, but not its velocity.

(*ii*) the Coriolis pseudoforce $F_{\text{Cor}} = 2m\omega v$, where v is the object's velocity within the rotating frame. The pseudoforce acts to deflect the object, because it is always perpendicular to the object's velocity. Note that F_{Cor} depends on the object's velocity, but not it position.

Sample Problem S1. A ball is thrown from the rim of a rotating space station so that it remains always at rest as seen by an external observer. (a) How fast, and in what direction, must the ball be thrown? (b) How can the ball's motion in the rotating frame be explained by the centrifugal and Coriolis pseudoforces?

Solution: If the pitcher throws the ball with a velocity equal but opposite to the tangential velocity of the rim as seen by the outside inertial observer, then obviously the ball will be instantaneously at rest from the outside observer's point of view. In the inertial frame no forces act on the ball after it is thrown, so if it starts at rest it will stay at rest, while the space station rotates about it.

In the rotating frame the spacestation is at rest, so an occupant sees the ball moving in a circle, skimming around at constant distance from the rim, as shown. Nothing touches the ball, so the only forces acting on it in the rotating frame are the centrifugal and Coriolis pseudoforces. The centrifugal pseudoforce $F_{\text{centrifugal}} = m\omega^2 r$ is outward, and the Coriolis pseudoforce $F_{\text{Cor}} = 2m\omega v = 2m\omega(\omega r) = 2m\omega^2 r$ is inward. Here we have used the fact that the speed of the ball in the rotating frame is $v = \omega r$. The *net* force on the ball is *inward*, equal to

$$F_{\text{net}} = F_{\text{Cor}} - F_{\text{centrifugal}} = 2m\omega^2 r - m\omega^2 r = m\omega^2 r.$$

The net force makes the ball accelerate inward, with

$$a_{\text{centripetal}} = F_{\text{net}}/m = \omega^2 r = v^2/r.$$

This is exactly the centripetal acceleration needed to make the ball circle around at radius r with speed v.

Questions and Problems from *Day Fifteen*

15Q1. A balloon is filled with helium and released in an accelerating spaceship. Does it move toward the bow or stern of the ship? Explain why it does this (a) from the standpoint of non-inertial observers inside; (b) from the standpoint of inertial observers outside.

15Q2. You are holding a cup of hot chocolate in a moving car. The driver steps on the brakes. Which way should you tip the cup so the hot chocolate won't spill out? If the car decelerates with $a = g/2$, at what angle should you tip the cup?

15Q3. A car is at rest with one of its doors open, as seen from above in the figure below. Using the principle of equivalence, describe how to close the door by accelerating the car.

15Q4. A rectangular fish tank on Earth is half full of water. One end of a rubber band is attached underwater to the middle of the tank bottom, and the other end is attached to a cork, as shown below. The rubber band is quite taut so that it almost pulls the cork under the surface. Now the tank is pushed from the left so that it accelerates from left to right with a constant acceleration $g/2$. If this keeps up indefinitely, so the water has again come to equilibrium, what will the water surface, cork and rubber band look like? Draw a picture.

15Q5. A boy places a ball inside his wagon, toward the front. When he makes the wagon accelerate quickly forward, which way does the center of the ball move (a) relative to the wagon? (b) relative to the sidewalk?

15Q6. Suggest an experiment using light rays, whose outcome you would know in a uniformly accelerating spaceship. *Hint*: Light rays travel in straight lines in inertial frames. Therefore what should happen to the rays in a ship at rest in uniform gravity?

15Q7. An onboard centrifuge was planned for the International Space Station for doing scientific and engineering experiments, but there is no artificial gravity provided for onboard astronauts. Why do you think the designers decided not to make the entire station (or a substantial portion of it) rotate to provide a comfortable (and healthy) artificial gravity for people?

15Q8. A local freely falling frame is a reference frame (a box, say, with inner coordinates) upon which the only force acting is gravitation. Show that gravity seems to have been turned off from the point of view of an observer within the box, as long as the ratio M_I/M_G of the inertial and gravitational mass is the same for all objects inside.

15Q9. A burning candle is placed upright on a circular sheet of glass. A large bell-jar is placed over the candle on top of the glass plate. The entire assembly is then dropped so it falls freely toward the floor. It is found that the candle stops burning long before the oxygen in the bell-jar is used up, and before the assembly hits the floor. Why?

15Q10. On Einstein's 76th (and last) birthday, March 14, 1955, his neighbor, Prof. Eric Rogers, gave him a toy constructed of a heavy brass ball, a spring, and other components, as shown below. The ball was attached to a string that hung outside a metal cup into which the ball could fit snugly. The string passed through a hole in the cup and down through a pipe, where it was tied to a spring. The assembly was mounted on a curtain rod so that one could easily hold on to the whole contraption. Finally, the cup and ball assembly was enclosed in a transparent glass sphere to give it a finished look.

If the spring had been strong enough it could have pulled the ball into the cup; however, it was too weak to counteract gravity, so the ball hung limply outside the cup. By shaking the curtain rod, it would be possible to pop the

ball into the cup. However, this turned out to be very difficult. The challenge was to find a way to pop the ball into the cup every time.

Einstein was delighted. He recognized immediately that the necessary trick hinged on a physical principle he himself had thought up half a century earlier. He was pleased that his friend took the trouble to remind him of what he had described as one of the happiest moments in his life. (Paraphrased from *An Old Man's Toy* by A. Zee). What was the trick?

The old man's toy. For maximum clarity, the artist shows three different views of the toy. **1.1a.** *A three-dimensional rendering.* **1.1b.** *A three-dimensional cutaway drawing.* **1.1c.** *A two-dimensional cross section.*

Section 15.1 Artificial gravity in an accelerating frame

15P1. A ball is thrown vertically "upward" with speed v_{0y} from a height y_0 above the floor by a person standing in a spaceship cabin, while the ship is accelerating upward in gravity-free space with acceleration a_{ship} relative to inertial frames. (a) Find the balls subsequent motion $y'(t)$ in the frame of the ship. (b) How long will it take the ball to return to its starting point?

15P2. A ball is thrown "horizontally" with speed $v_{0x} = 3.0$ m/s and position $x_0 = 0$ from a height $y_0 = 2$ m above the floor by a person standing in a spaceship cabin. (a) Find its subsequent horizontal and vertical motion in the frame of the ship, $x'(t)$ and $y'(t)$, if the ship is accelerating uniformly upward with $a_{\text{ship}} = 9.8$ m/s^2. (b) What is the shape $y'(x')$ of the path?

15P3. A large elevator on Earth is accelerating upward with $a = g/3$. A passenger holds a penny a distance h above the elevator floor, and drops it. Find the time t it takes the penny to hit the floor (a) working in the elevators frame of reference (b) working from the point of view of an outside inertial observer, who is instantaneously at rest relative to the elevator when the penny is dropped.

Section 15.2 Newton's law $F = ma$ in uniformly accelerating frames

15P4. A chandelier of mass m hangs on a cord from the ceiling of a spaceship cabin. The ship has uniform acceleration a_{ship} as it moves in gravity-free space. Find the tension T in the cord, working from the perspective of (a) an external inertial observer; (b) a passenger in the ship. Begin each part with an isolation diagram.

15P5. A frictionless, massless pulley is placed in an elevator on Earth; a string over the pulley is attached to objects with masses m_1 and m_2, as shown below. If the elevator accelerates upward with $a_e = g/2$, find (a) the tension T in the string; (b) the acceleration a_2 of the "m_2" object relative to the elevator; (c) the force F_0 exerted on the pulley by its axle.

15P6. A train coming into the station slows with constant deceleration $a = g/4$. A suitcase of mass $m = 12$ kg, which has a coefficient of sliding friction $\mu = 0.20$ with the floor, starts to slide along the aisle. What is the acceleration of the suitcase relative to (a) the train? (b) the ground?

15P7. During a takeoff in which a plane accelerates with $a_p = 0.20\ g$ along the runway, an unprepared flight attendant accelerates toward the rear of the plane at $a_f = 0.06\ g$ down the aisle, relative to the plane. (Here g is the gravitational field on Earth.) (a) What is the coefficient of kinetic friction μ_k between the aisle and the flight attendant? (b) Could a passenger who knew the numerical values of a_p and μ_k determine the attendant's mass m by measuring a_f?

15P8. A block of mass m on Earth is placed on the frictionless sloping side of a wedge of mass M and angle $\theta = 45^o$, as shown below. There is a real, uniform gravity g downward. The wedge sits on a horizontal frictionless table. With what force F_0 must we push the wedge from the left, so that the block will stay at the same position relative to the wedge? (a) Find F_0 working in the accelerating frame of the wedge, beginning with an appropriate isolation diagram in that frame. (b) Find F_0 working in the inertial frame of the table, beginning with an isolation diagram in that frame.

Section 15.3 Inertial and gravitational mass

15P9. It can be shown that the period of a small-amplitude simple pendulum (the time for a small object on a string to swing once back and forth, if the maximum angle relative to the vertical is small) is $2\pi\sqrt{m_G g/m_I \ell}$, where m_G and m_I are the gravitational and inertial masses of the object, and ℓ is the length of the string. In his *Dialogues Concerning Two New Sciences*, Galileo Galilei described an experiment he had performed:

... I took two balls, one of lead and one of cork, the former being more than a hundred times as heavy as the latter, and suspended them from two equal thin strings, each four or five brachia long. Pulling each ball aside from the vertical, I released them at the same instant, and they, falling along the circumferences of the circles having the strings as radii, passed through the vertical and returned along the same path. This free oscillation, repeated more than a hundred times, showed clearly that the heavy body kept time with the light body so well that neither in a hundred oscillations, nor in a thousand, will the former anticipate the latter by even an instant, so perfectly do they keep step.

Suppose that Galileo established that an upper limit to the fractional difference in the periods of the lead and cork balls was

$$\left|\frac{(T_{\text{lead}} - T_{\text{cork}})}{T_{\text{cork}}}\right| = 10^{-3}$$

(i.e., one part in a thousand.) Show that in so doing he established an upper limit to the fractional difference in the mass ratios that was

$$\left|\frac{(m_G/m_I)_{\text{lead}} - (m_G/m_I)_{\text{cork}}}{(m_G/m_I)_{\text{cork}}}\right| = 2 \times 10^{-3}.$$

15P10. Suppose that the gravitational and inertial masses of any particle on Earth are related by $m_G = \alpha m_I$, where α is a universal constant, the same for all particles. Show that if the gravitational field g on Earth is changed by an appropriate factor, then all mechanics experiments on Earth would give the same results as if $\alpha = 1$. What is the appropriate factor? (This result demonstrates that experiments cannot guarantee that the inertial and gravitational mass are equal to each other for all particles, but only that the ratio m_G/m_I is the same for all particles.)

Section 15.4 The principle of equivalence

15P11. A hypothetical material X has an inertial mass twice as large as its gravitational mass. A block of this material is placed on a frictionless plane inclined at angle θ to the horizontal, and released from rest at a vertical altitude h above the floor. (a) How long will it take the block to reach the floor? (b) What will be its speed when it arrives?

15P12. Approaching a station, the string of an unwound yo-yo in equilibrium is observed to make a steady angle of $\theta = 12^o$ with the vertical inside a uniformly decelerating train on Earth. (a) What is the magnitude a of the trains deceleration, expressed as a fraction of g? (b) During this same period of time a weighing scale is tipped so that a passenger of mass m can stand on it. What will be the weight of the passenger in terms of m and g?

15P13. An elevator on Earth rises with constant acceleration $a = g/2$. If a flea on the elevator floor leaps vertically upward with initial velocity v_0, how long will it be before it lands again on the floor?

15P14. Two small blocks rest on the top and sloped sides of a large block; the slope is at angle θ to the horizontal, as shown below. A light string connecting the small blocks hangs over a frictionless peg at the corner of the large block. There is no friction between the small blocks and large block, and there is no friction between the large block and the table on which it rests. With what force F_0 must we push on the large block from the left, so that both small blocks will remain fixed relative to the large block? (Assume that the inertial and gravitational masses are both m for each small block, and that they are both M for the large block.)

15P15. Answer the question posed in 15P14, assuming instead that the gravitational masses of both small blocks are twice as large as their inertial masses m, but that the inertial and gravitational masses of the large block are both equal to M.

15P16. A laser is aimed horizontally in a room on Earth, a distance y_0 above the floor, and a pulse of light is emitted. (a) How much will the pulse fall by the time it reaches the opposite wall, a distance L away? The gravitational field is g and the speed of light is c. *Hint*: First find how far such a pulse would fall in a similar room in a spaceship accelerating upward with an acceleration numerically equal to g. (b) What must be the length L of the room if the pulse falls by 0.1 nm? (0.1 nm is roughly the diameter of a hydrogen atom.)

Section 15.5 A rotating space station

15P17. A cylindrical space station rotates about its axis of symmetry; occupants live on the inside rim of the cylinder. (a) If the radius of the cylinder is R and it rotates once in time T, find the weight W of an occupant whose mass is m. (b) The occupant climbs a ladder a distance d toward the central axis. What is the occupant's weight W when $d = R/2$? When $d = R$? (c) What is the effective gravity g_{eff} at the rim? When $d = R/2$? When $d = R$?

15P18. Colonists live on the inside rim of a cylindrical rotating space colony of radius $R =$ 1 km. (a) What is the rotation period T of the cylinder about its symmetry axis if the weight of a colonist of mass $m = 53$ kg is $W = 520$ N? (b) What is the magnitude a of her acceleration? (c) She spots a rope stretched between the axis the a point on the rim. How far up the rope must she climb to reduce her weight to W = 260 N?

15P19. A spaceship includes living quarters in the configuration of a rotating torus, to provide the travelers with an artificial gravity. The outer rim of the torus has radius R and rotates with period T. (a) Find the effective gravity g_{eff} experienced by travelers standing on the outer torus rim, as shown below. In terms of R, T, and v, find g_{eff} in the frame of a traveler running with speed v around the rim (b) in the rotation direction (c) against the rotation direction. (d) In which direction would it be easier to run, assuming that g_{eff} in the colony is comparable to Earth's gravity?

15P20. A cylindrical space colony of radius R rotates with period T about its symmetry axis. An apple is suspended by a string from the axis, a distance $R/2$ above a short colonist standing on the rim, as shown below. The apple is at rest in the rotating frame of the colony. The string is then cut, allowing the apple to fall. How far must the colonist run along the rim to catch it?

15P21. I. Newton is seated on the inside rim of a large cylindrical space colony of radius R, rotating about its symmetry axis with period T in the counter-clockwise direction in the diagram below. An apple is suspended from the axis by a string of length $R/2$. The string makes an angle θ with respect to the radial line from the axis to Newton; it is stationary in the rotating frame of the colony. The string is cut and the apple falls, striking Newton on the head. (a) Sketch the path of the apple, as seen by an outside inertial observer. (b) Find the angle θ in radians. (c) Sketch the path of the apple as seen by Newton.

15P22. A cylindrical space colony of radius R rotates at constant angular speed ω about its symmetry axis. A colonist stationed at the rotation axis throws a ball at velocity v directly at a second colonist standing on the inside rim. How far around the rim must the second colonist run to catch the ball? (The colonists are very short compared with R.)

15P23. A colonist with a baseball stands on the inside rim of a cylindrical space colony of radius R which rotates with constant angular speed ω about its symmetry axis. The colonist wants to throw the ball so that it will reach the axis of rotation and then fall back to him. (a) Sketch the path of the ball as seen by an outside inertial observer. (b) With what speed v_{in} must the ball be moving (to the inertial observer) so the colonist can catch the ball when it returns to the rim? (c) With what speed v_{rot}, and at what angle θ to the radial line between the colonist and rotation axis, must the colonist throw the ball in the rotating frame of the colony? (d) Sketch the path of the ball as seen by the colonist.

15P24. A giant cylindrical space hotel of radius R rotates with period T about its symmetry axis in gravity-free space. Much of the rim consists of a large swimming pool, and a high-diving board is situated a distance R from the symmetry axis, as shown below. (a) In terms of R and T, what is the effective gravity g_{eff} at the diving board? (b) A guest steps off the board. How long a time does it take her to reach the water, expressed as a multiple of T? (c) If a point P at the surface of the pool was directly beneath her when she stepped off, how far (measured around the rim) from P will she be when she enters the pool, expressed as a multiple of R? (d) Sketch the guest's trajectory as seen by people at rest in the hotel.

15P25. A rotating space hotel of radius R and period T has a diving board a distance $R/2$ above the rim pool, as described in *Sample Problem 5*. A diver steps off the board. (a) Show that the speed with which the diver impacts the water (relative to the water) has magnitude $v = \pi R/T$. (b) At what angle to the pool surface does the diver enter the water? (c) Suppose the hotel has radius $R = 100$ m and rotation period $T = 20\pi$ s $= 62.8$ s. Find the effective gravity at the rim, expressed as a fraction of one Earth gee. In this hotel, (d) How long does it take the diver to reach the pool? (e) At what speed does the diver impact the water? (f) How high must a diving board on the Earth be so that a diver stepping off the board will impact the water with the same speed found in (e)?

Special Topic: The Coriolis pseudoforce

15P26. A train runs around the circumference of a rotating cylindrical space colony of radius R and angular velocity ω. How would the total effective gravity on the passengers (including both centrifugal and Coriolis effects) depend upon the train's speed v (a) if the train travels in the rotation direction of the rim? (b) if it

travels in the opposite direction?

15P27. In *Rendezvous with Rama* by Arthur C. Clarke, a waterfall spills out of one of the endcaps of a cylindrical spaceship rotating about its axis of symmetry. Suppose the water originates at a point halfway between the rotation axis and rim, and then "falls" toward the rim. (a) Explain why the water does not fall straight as seen by people within the ship. (a) Which way does the waterfall bend, if the ship is rotating clockwise as seen by a particular outside inertial observer? Sketch the waterfall's shape.

15P28. Earth dwellers live in a rotating frame. (a) Does an outside inertial observer looking down on the North Pole see Earth turning clockwise or counterclockwise? (b) A ball is dropped from height h by someone standing still on the equator. Does it fall to the *east* or *west* of a point just beneath the position from which it was dropped?

15P29. (a) Explain why the work done in a rotating frame by the Coriolis pseudoforce is *zero*. (b) If the Coriolis pseudoforce were the only force acting on a particle, what could you conclude about the particle's speed?

Day Sixteen

Angular Momentum

"Any blockhead can cite generalities, but the mastermind discerns the particular cases they represent."

- George Eliot

16.1 Introduction

Rotation is an important feature of motion, whether it is the rotation of bicycle wheels, iceskaters, merry-go-rounds, tornados, or our rotating space station. Rotation is observed in atoms and molecules, planets, solar systems, and galaxies. How do we describe rotation, and how do rotating objects behave? That will be our goal for the next several days.

Begin with the rotation of a wheel. Consider an arbitrary point P on a wheel that turns around its center. At any instant the point P is located at some particular angle θ relative to a fixed direction in space. If the wheel has turned counterclockwise we will say that the angle θ is positive, as illustrated in Fig. 1(a); if the wheel has turned clockwise θ is negative, as shown in Fig. 1(b).

Fig. 1. A point P on a wheel, with angle θ relative to some fixed direction in space, shown here as the x-axis. (a) Positive and (b) negative angles θ

The *angular velocity* ω of the wheel is the rate at which the angle θ changes with time:

$$\omega \equiv \frac{d\theta}{dt}. \tag{1}$$

When the wheel turns with constant angular velocity, then θ increases (or decreases) at a constant rate. In that case the angular speed ω (the magnitude of the angular velocity) gives the number of radians through which the wheel turns per second. There are 2π radians in a compete revolution, so ω is related to the time for one complete revolution (called the *period* T) by the equation

$$\omega \equiv \frac{2\pi}{T}. \qquad \text{(if } \omega \text{ is constant)} \tag{2a}$$

If a wheel is made to spin faster its angular speed increases and its period decreases.

Suppose the wheel turns counterclockwise. Then θ increases with time, so the angular velocity ω is positive, according to Eq. 1. If the wheel turns clockwise, θ decreases with time, so ω is negative. The two rotation directions are indicated by the sign of ω.

If a point P on the wheel is a distance r from the rotation axis, it moves at speed

$$v = \frac{2\pi r}{T} = r\omega, \tag{2b}$$

since it moves around the length of the circumference $2\pi r$ during one period T. For given angular velocity, the farther P is from the axis, the faster it is moving.

16.2 "Experiments" with wheels

Our rotating space station comes equipped with a laboratory. The laboratory contains many kinds of wheels and other pieces of apparatus we can use to figure out how rotation works. There is a light-weight circular platform, for example, that can rotate about a strong but thin vertical axle through its center. The axle is supported from below on nearly frictionless bearings. We also find several identical solid heavy metal disks with rough surfaces. Each disk has a small hole drilled through the middle so it can be threaded onto the vertical axle.

We thread a disk onto the axle, and set it down upon the platform. When we give it a spin it keeps turning, with essentially constant angular velocity. There seems to be a *rotational inertia*, analogous to the inertia of linear motion. The wheel likes to retain its rotation rate: If it starts at rest it stays at rest, but if spun up initially, it keeps spinning at the same rate, unless something touches it (like air or a frictional bearing) to slow it down or speed it up.

A. Identical disks

Disk # 1 sits on the platform. We set it rotating with constant angular velocity ω_0 about the vertical axle. (We can find ω_0 by measuring the time T_0 for a complete revolution of the disk, and then calculating $\omega = 2\pi/T_0$.)

We thread an identical disk # 2 onto the axle, and let it fall (without any spin of its own) onto the rotating disk below. The disks scrape over one another and soon start turning as a unit, both with the same angular velocity ω_f. What is this common angular velocity ω_f? We find that

$$\omega_f = \frac{\omega_0}{2}. \tag{3a}$$

The two disks rotate together with *half* the angular velocity of the lower one initially.

We next drop *two* identical nonrotating disks onto the single rotating disk below. What is the final angular velocity of all three disks, by the time they begin to rotate as a unit? It is

$$\omega_f = \frac{\omega_0}{3}. \tag{3b}$$

In fact, if we drop N_2 identical nonrotating disks onto the single turning disk below, then

$$\omega_f = \frac{\omega_0}{N_2 + 1}. \tag{4}$$

What if we drop a *single* non-rotating disk onto a *pair* of initially rotating disks? In this case

$$\omega_f = \frac{2}{3}\omega_o. \qquad (5)$$

If in general we drop N_2 nonrotating disks onto N_1 initially rotating disks, we find that

$$\omega_f = \left(\frac{N_1}{N_1 + N_2}\right)\omega_o. \qquad (6)$$

What is the meaning of this result? Rewriting it slightly,

$$(N_1 + N_2)\omega_f = N_1\omega_o. \qquad (7)$$

That is, the total number of rotating disks at the end, multiplied by their angular velocity ω_f, equals the number of rotating disks at the beginning, multiplied by *their* angular velocity ω_o. This is a *conservation law*, valid for the experiments done so far. Some quantity (the number of rotating disks times their angular velocity) stays the same before and after.

B. Disks of various masses and radii

We next select a single disk twice as thick as the others, but made of the same material. Experiments show that it acts as though it consisted of *two* of the original disks taken as a unit, so that "N" for this single thick disk is $N = 2$.

There are also disks only *half* as thick as the original disks. They behave as though N (whether N_1 or N_2 in Eq. 6) is $N = \frac{1}{2}$ rather than some integer. In fact, for various disks of different thicknesses, but all of the same radius, the value of "N" is proportional to the thickness, which of course is also proportional to the *mass* of the disk, since all are made of the same material and all have the same radius. That is, for all disks used so far, Eq. 6 can be generalized to read

$$\omega_f = \left(\frac{M_1}{M_1 + M_2}\right)\omega_o, \tag{8}$$

where M_2 is the total mass of the (initially nonrotating) dropped disks, and M_1 is the total mass of the initially rotating bottom disks. For disks of the original standard thickness, Eq. 8 is equivalent to Eq. 6.

We also own some disks whose *radii* are different than that of our standard disks. One of them has twice the radius R and only one-fourth the thickness t of the standard disks. Its volume $\pi R^2 t$ is therefore the same as the others, so its mass is also the same, since it is made of the same stuff. If Eq. 8 works for all disks, we should find

$$\omega_f = \left(\frac{1}{1 + 1}\right)\omega_o = \frac{1}{2}\omega_o \tag{9a}$$

when we drop the new disk onto one of the regular disks. Instead, we find that

$$\omega_f = \frac{\omega_o}{5} = \left(\frac{1}{1 + 4}\right)\omega_o. \tag{9b}$$

Why is the denominator equal to $1 + 4$ instead of $1 + 1$? The important difference between the new disk and the old disk is that the new one has twice the radius, while keeping the same mass. So a generalization of Eq. 8, valid for both kinds of disks, is

$$\omega_f = \left(\frac{M_1 R_1^2}{M_1 R_1^2 + M_2 R_2^2}\right)\omega_o \tag{10}$$

which works in all experiments with one disk (#1) on the bottom and one disk (#2) on the top. For several disks of various masses and radii, we have to put the *sum* of the MR^2 of all the bottom disks in the denominator (i.e., $M_1R_1{}^2 + M_2R_2{}^2 + M_3R_3{}^2$ ..., one term for each disk), and the *sum* of the MR^2 of all the top disks in the numerator. The resulting relation then works for every single experiment carried out so far.

C. Rings

Besides the disks we also have *rings*, with all their mass essentially at the same radius R. One ring has the same radius as our original disks. It also has the same mass as the disks when we weigh it, so must be made of a very dense material, since its volume is much less than than of the disks. If Eq. 10 applies to rings as well as disks, we should find that

$$\omega_f = \left(\frac{M_1 R_1^2}{M_1 R_1^2 + M_2 R_2^2} \right) \omega_o = \left(\frac{1}{1+1} \right) \omega_o = \frac{\omega_o}{2} \tag{11a}$$

when we drop the nonrotating ring on one of the original rotating disks. That's *not* what we find. Instead,

$$\omega_f = \frac{\omega_o}{3} = \left(\frac{1}{1+2} \right) \omega_o \,, \tag{11b}$$

so MR^2 can't be the whole story. There must be an additional effect due to the *shape* or *mass distribution* within the object, distinguishing between rings and disks.

Define a quantity

$$I = f MR^2 \tag{12}$$

for each object, where f is a dimensionless "shape factor", a number representing how mass is distributed within the object. The denominator of Eq, 11b shows that

$$\frac{f_{\text{ring}}}{f_{\text{disk}}} = 2. \tag{13}$$

What are f_{disk} and f_{ring} individually? Since our experiments only show the *ratio* of the two constants, we are free to define f for one type of object; then the values of f for other objects can be found by measuring the ratio and using the definition for the single type. In fact, we'll define the shape factor of a *ring* to be

$$f_{\text{ring}} = 1 \tag{14}$$

because then

$f_{\text{ring}} = 1$

$$I_{\text{ring}} = MR^2, \tag{15}$$

the simplest possible expression, especially appropriate since all of the mass of an ideal ring is located at the same distance R from the ring's center. Using this definition our experiments have shown that $f_{\text{disk}} = 1/2$, so

$$I_{\text{disk}} = \frac{1}{2}MR^2 . \tag{16}$$

$f_{\text{disk}} = \frac{1}{2}$

We call I the *moment of inertia* (or *rotational inertia*) of the object, whether a ring or a disk. For any circular object of mass M and radius R, the moment of inertia I has the form $I = f MR^2$ given by Eq. 12. Note that I has the dimensions $[\,I\,] = \text{ML}^2$, mass times length squared, and so in Standard International (SI) units, I can be expressed in the units $\text{kg} \cdot \text{m}^2$.

The moment of inertia enters our experiments through the relationship

$$\omega_{\text{f}} = \left(\frac{I_1}{I_1 + I_2}\right)\omega_{\text{o}} = \left(\frac{f_1 M_1 R_1^2}{f_1 M_1 R_1^2 + f_2 M_2 R_2^2}\right)\omega_{\text{o}} \tag{17}$$

when we drop a nonrotating object with moment of inertia I_2 onto an initially rotating object with moment of inertia I_1. We can measure the shape factor f of an object this way, by dropping it onto a rotating disk or ring whose shape factor we already know.

When we drop an assortment of nonrotating objects, with differing masses, radii, and shape factors, onto an assortment of rotating objects, also perhaps with differing M, R, and f, the generalization is

$$\omega_f = \left(\frac{\Sigma I_1}{\Sigma I_1 + \Sigma I_2}\right)\omega_o \qquad (18)$$

where the ratio inside parentheses is the sum of the moments of inertia of the upper objects divided by the sum of the moments of inertia of the lower objects. Rewriting Eq. 18, we have the *conservation law*

$$\Sigma I_1 \, \omega_o = \Sigma (I_1 + I_2) \, \omega_f \, , \qquad (19)$$

satisfied in all the experiments carried out so far. The total moment of inertia of all objects initially rotating with the common angular velocity ω_o, multiplied by ω_o, equals the total moment of inertia of all objects finally rotating with the angular velocity ω_f, multiplied by ω_f.

16.3 Conservation of angular momentum

The quantity $I\omega$, the product of the moment of inertia of a wheel and its angular velocity, appears prominently in the new conservation law. So we define the *angular momentum* L of an object about a rotation axis to be

$$L = I\omega \qquad (20)$$

where I is the moment of inertia of the object about the given axis, and ω is the object's angular velocity. The angular velocity ω can be positive or negative, depending upon whether the object rotates counterclockwise or clockwise about the axis. The angular momentum about the axis can therefore also be positive or negative, depending upon whether the object rotates CCW or CW.

Eq. 19 is equivalent to the conservation of angular momentum of a set of objects, some of them initially spinning together, and all of them spinning together as a unit when they all

come together. That is,

$$(L_{\text{TOTAL}})_i = (L_{\text{TOTAL}})_f \tag{21}$$

where the total *initial* angular momentum $(L_{\text{TOTAL}})_i = (\Sigma L)_i$ and the total *final* angular momentum $(L_{\text{TOTAL}})_f = (\Sigma L)_f$.

Conservation of angular momentum applies in a much wider set of circumstances than shown by our "experiments" so far. For example, it also applies to two disks (#1 and #2) initially spinning at different rates, which touch briefly and then separate, spinning at different rates, as shown. That is,

$$I_1(\omega_1)_o + I_2(\omega_2)_o = I_1(\omega_1)_f + I_2(\omega_2)_f . \tag{22}$$

In fact, experiments suggest that the conservation law applies to *any* isolated system of objects rotating about a fixed common axis.

Sample Problem 1. A ring of mass M and radius R spinning about an axis with angular velocity ω_o falls onto a disk of mass $2M$ and radius $2R$, spinning initially with angular velocity $2\omega_o$. The ring then bounces back up, spinning now twice as fast as before, with angular velocity $2\omega_o$. What is the angular velocity ω_f of the *disk* just after the ring bounces up?

Solution: The moment of inertia of the ring is $I_{\text{ring}} = MR^2$, and the moment of inertia of the disk is

$$I_{\text{disk}} = \frac{1}{2}M_{\text{disk}}R_{\text{disk}}^2 = \frac{1}{2}(2M)(2R)^2 = 4MR^2.$$

The total initial angular momentum is therefore

$$L_i = I_{disk}(2\omega_o) + I_{ring}\omega_o = 4MR^2(2\omega_o) + MR^2\omega_o = 9MR^2\omega_o$$

and the total final angular momentum is

$$L_f = I_{disk}\omega_f + I_{ring}(2\omega_o) = 4MR^2\omega_f + MR^2(2\omega_o)$$

Angular momentum is conserved: $L_f = L_i$, so $4MR^2\omega_f + MR^2(2\omega_o) = 9MR^2\omega_o$. Therefore

$$\omega_f = \frac{7}{4}\omega_o .$$

The disk has slowed its rotation rate from $2\omega_o$ to $\frac{7}{4}\omega_o$, to make up for the higher rotation rate of the ring. Overall angular momentum stays constant throughout the process.

The *angular momentum*

$$L = I\omega \tag{23}$$

of a rotating disk or ring has similarities with the *momentum* $p = mv$ (sometimes called the *linear* momentum) of a moving object. Each is the product of an inertial property of the system (its moment of inertia or its mass) and a property of its motion (angular velocity or linear velocity). That doesn't mean that angular momentum is just a new kind of momentum, however; L has different dimensions (and different units) than p. Angular momentum has the dimensions $[L] = [I\,\omega] = ML^2 \cdot \frac{1}{T}$ and SI units $\frac{kg \cdot m^2}{s}$, while momentum has dimensions $[p] = [m\,v] = M \cdot \frac{L}{T}$ and SI units $\frac{kg \cdot m}{s}$. Angular momentum is an *entirely new quantity*. It makes no more sense to add a particle's angular momentum to its linear momentum, for example, than to add the particle's angular momentum to its energy, or to its mass, or to its electric charge.

16.4 Summing moments of inertia

The "experiments" of Section 16.2 showed that the angular momenta $L = I\omega$ of various disks and rings add up to form a total angular momentum. Therefore if a set of objects all rotate with the same angular velocity, their moments of inertia must add up as well:

$$I_{\text{object}} = \Sigma I_{\text{constituent parts}} \cdot \tag{24}$$

For a ring, we *defined* the shape-factor $f_{\text{ring}} = 1$, so the moment of inertia is $I_{\text{ring}} = MR^2$ about an axis perpendicular to the ring and passing through its center. We then found from experiment that moment of inertia of a disk is $I_{\text{disk}} = \frac{1}{2}MR^2$ about a similar axis.

However, we can think of a disk as a nested set of rings, as shown. Can we *derive* $I_{\text{disk}} = \frac{1}{2}MR^2$ by summing up the moments of inertia of all the rings that make up the disk?

Suppose the disk has a uniform volume density ρ (ρ = mass per unit volume), so

$$M_{\text{disk}} = \rho \cdot \text{Volume} = \rho \cdot \pi R^2 t, \tag{25}$$

where t is the thickness of the disk. Then the moment of inertia of a single thin ring of radius r, width Δr, and mass Δm is

$$\begin{aligned} \Delta I_{\text{ring}} &= \Delta m\, r^2 = (\rho \cdot \text{Volume}) r^2 \\ &= \rho(2\pi r\, \Delta r\, t)r^2 = 2\pi\, \rho\, t\, r^3\, \Delta r\,, \end{aligned} \tag{26}$$

assuming very narrow rings so their radius are well characterized by a single radius r. Now we have to sum over all the rings. We can do this by letting the rings have infinitesimal width dr (rather than the small but finite width Δr) and integrating over an infinite number of them to get the entire disk:

$$I_{\text{disk}} = \int dI_{\text{ring}} = 2\,\pi\,\rho\,t \int_0^R r^3 dr = 2\,\pi\,\rho\,t \left.\frac{r^4}{4}\right|_0^R = \frac{\pi}{2}\rho\, t\, R^4. \tag{27}$$

The mass of the disk is $M_{\text{disk}} = \rho \cdot \text{Volume} = \rho \cdot \pi R^2 t$, so (eliminating ρ in the expression for I_{disk}), we find that

$$I_{\text{disk}} = \frac{1}{2} MR^2 \tag{28}$$

as in Section 16.2. We *can* find the moment of inertia of an object by summing up the moments of inertia of all the pieces that make it up.

We have split up the disk into a set of nested rings. We can also further split up each ring into a set of small slices as shown, all at the same distance from the center, at various angles around the ring. In the limit of an infinite number of slices, each becomes essentially a point particle.

What is the moment of inertia I_{particle} of a *single particle* of mass m, a distance R from the center? It must be

$$I_{\text{particle}} = mR^2 \tag{29}$$

about a rotation axis a distance R away, because only then can we sum up all of them to get the moment of inertia of the entire ring, $I_{\text{ring}} = MR^2$. That is,

$$I_{\text{ring}} = \Sigma mR^2 = (\Sigma m)R^2 = MR^2 \tag{30}$$

because each particle is the same distance R from the center.

Sample Problem 2. A disk of mass M and radius R rotates with angular velocity ω_0 on a platform.

(a) A small gob of putty of mass m drops onto the disk at a point a distance $R/2$ from the rotation axis. The putty sticks to the disk. What is the final angular velocity ω_f of the combination?

(b) A similar disk rotates with angular velocity ω_o. A small chunk of the disk rim, of mass m and located at radius R, breaks off the disk. What is the angular velocity ω_f of the remaining disk?

Solution: Angular momentum is conserved in both problems. The initial angular momentum of the disk in both cases is $L_{\text{initial}} = I_{\text{disk}}\omega_o = \frac{1}{2}MR^2\omega_o$.

(a) The putty has no initial angular momentum; the total initial angular momentum is that of the disk alone. After the putty sticks to the disk, the total moment of inertia of the combination is

$$I_{\text{total}} = I_{\text{disk}} + I_{\text{putty}} = \frac{1}{2}MR^2 + m\left(\frac{R}{2}\right)^2,$$

so the final angular momentum of disk and putty is $L_{\text{final}} = (\frac{1}{2}MR^2 + m\left(\frac{R}{2}\right)^2)\omega_f$.

Setting $L_{\text{initial}} = L_{\text{final}}$, we get $\omega_f = \left(\frac{M}{M+(m/2)}\right)\omega_o$. Note that $\omega_f < \omega_o$.

(b) As the rim chunk m breaks off, according to the law of inertia it "remembers" its former tangential velocity $v = R\omega_o$, so immediately after leaving the disk it still has tangential velocity $v = R\omega_o$. Therefore it still rotates instantaneously about the center with the same angular velocity as before. Its angular momentum is still $L_{\text{chunk}} = mR^2\omega_o$, just as it was before it broke off. Conserving total angular momentum, it follows that the angular momentum of the rest of the disk must also stay the same as before. The moment of inertia of this portion of the disk (the whole disk minus the rim chunk) hasn't changed, so its angular velocity can't change either. Its final angular velocity remains $\omega_f = \omega_o$.

Few wheels are exact disks or rings, so we need to learn how to find moments of inertia for objects of various shapes, like broomsticks or artificial satellites.

Start with the moment of inertia of a point mass m a distance r from a rotation axis,

$$I_{\text{point mass}} = mr^2. \qquad (31)$$

We can find the moment of inertia of *any* object about some axis by summing over the moments of inertia of all the masses that make it up. That is, we sum over all the masses m_i in the object, weighted by the square of their *perpendicular* distances r_i from the axis of rotation,

$$I = \sum_i m_i r_i^2 \qquad \text{(summing over discrete masses).} \qquad (32)$$

Note that the moment of inertia depends upon the mass distribution within the object, *and also upon where the axis is located.* Even the moment of inertia of a single particle depends upon the axis. The farther the rotation axis is from the particle, the larger its moment of inertia.

When the object contains a huge number of particles we can approximate it by a continuous distribution of mass. That allows us to integrate rather than sum, which is usually much easier for a large number of particles. Instead of summing over discrete masses m_i we replace m_i by the infinitesimal mass $dm = \rho\, dV$, the mass within the infinitesimal volume element dV, where ρ is the local mass density of the object. We then have to sum (i.e., integrate) over all of the infinitesimal mass elements, weighted by the square of their distances from the rotation axis. That is,

$$I = \int r^2 dm \qquad \text{(for a continuous distribution of mass).} \qquad (33)$$

Sample Problem 3. Find the moment of inertia of a uniform thin stick of mass m and length ℓ, about an axis through its center.

Solution: On dimensional grounds, the moment of inertia of the stick must be $I = k\, m\, \ell^2$, where k is some dimensionless number. The mass within a small slice Δx of the uniform stick is $\Delta m = \lambda\, \Delta x$, where λ is the mass per unit length $\lambda = m/\ell$. So the moment of inertia of the stick about an axis through its center is (since in this case $r = x$)

$$I = \int r^2 dm = \int_{-\ell/2}^{\ell/2} \lambda x^2 dx$$

$$= \lambda \frac{x^3}{3}\Big|_{-\ell/2}^{\ell/2} = \frac{\lambda}{3}\left(\frac{\ell}{2}\right)^3 \cdot 2 = \frac{1}{12}\frac{m}{\ell}\ell^3 = \frac{1}{12} m\, \ell^2.$$

In terms of m and the total length ℓ, the dimensionless shape factor is $f = \frac{1}{12}$ for a thin stick about an axis through its center.

Sample Problem 4. Find the moment of inertia of a hoop of mass M and radius R about an axis passing through a diameter of the hoop.

Solution: The diagram shows a small piece of the hoop of length Δs, at angle θ from the axis of rotation. The mass per unit length of the hoop is $\lambda = M/2\pi R$, so the mass in the small piece is $\Delta m = \lambda\, \Delta s = \lambda(R\Delta\theta)$. (We assume all the mass is essentially at the same radius.) The distance of this piece from the rotation axis is $r = R\sin\theta$, so its moment of inertia is

$$\Delta I = \Delta m\, r^2 = (\lambda R\, \Delta\theta)\, R^2 \sin^2\theta = \lambda R^3 \sin^2\theta\, \Delta\theta.$$

The moment of inertia of the entire hoop about the diameter is the sum of the moments of inertia of all these pieces,

$$I = \int dI = \lambda R^3 \int_0^{2\pi} \sin^2\theta\, d\theta = \lambda R^3 \frac{1}{2}\int_0^{2\pi}(1 - \cos 2\theta)\, d\theta$$

$$= \frac{\lambda R^3}{2}\left[2\pi - \frac{\sin 2\theta}{2}\Big|_0^{2\pi}\right] = \pi\lambda R^3 = \frac{1}{2}MR^2.$$

The moment of inertia of a hoop about a diameter is just half as large as its moment of inertia about an axis perpendicular to the hoop, through the center.

Sample Problem 5. Find the moment of inertia of a thin spherical shell of mass M and radius R about an axis through its center.

Solution: Dimensionally, the moment of inertia must have the form $I = f\,MR^2$, where f is a dimensionless shape factor. To find f, we can start by cutting the spherical shell into very thin slices perpendicular to the rotation axis. Two nearby parallel slices cut out a thin ring, as shown. If we find the moment of inertia of each ring, we can add them up to find the total moment of inertia of the shell. A typical ring is shown in the picture: It is located at angle θ from the rotation axis, and has the small angular width $\Delta\theta$. The radius of the ring (out from the axis) is $r = R\sin\theta$, and the mass of the ring is $\Delta m = \sigma\Delta A$, where σ is the mass/area of the spherical shell and ΔA is the area of the ring. This area is the ring's *circumference* $C = 2\pi r = 2\pi R\sin\theta$ times its *width* $\Delta s = R\Delta\theta$; i.e.,

$$\Delta A = 2\pi R\sin\theta\, R\Delta\theta = 2\pi\, R^2\sin\theta\,\Delta\theta.$$

The moment of inertia of the ring is

$$\Delta I = \Delta m\, r^2 = \sigma\Delta A\,(R\sin\theta)^2 = 2\pi\sigma R^4 \sin^3\theta\,\Delta\theta.$$

The total moment of inertia of the spherical shell is the sum of the ΔI's of all the rings. In the limit of an infinite number of infinitesimal rings,

$$I = \int dI = 2\pi\sigma R^4 \int_0^\pi \sin^3\theta\, d\theta\,.$$

We include every ring *once* by letting θ go from 0 to π. The integral is

$$\int_0^\pi (1-\cos^2\theta)\,\sin\theta\, d\theta = -\cos\theta\Big|_0^\pi - \left(-\frac{\cos^3\theta}{3}\right)\Bigg|_0^\pi = 2 - \frac{2}{3} = \frac{4}{3}.$$

The mass/area of the shell is $\sigma = M/(4\pi R^2)$, so finally the moment of inertia of a spherical shell of mass M and radius R is

$$I_{\text{spherical shell}} = 2\pi\left(\frac{M}{4\pi R^2}\right) R^4 \cdot \frac{4}{3} = \frac{2}{3}\,MR^2.$$

Sample Problem 6. Find the moment of inertia of a solid uniform-density sphere of mass M and radius R about an axis that passes through the sphere's center.

Solution: Dimensionally, the moment of inertia of a solid sphere must have the form $I = f\,MR^2$, where f is a dimensionless shape factor. To find the full answer we start by splitting the sphere into a collection of spherical shells of radius r and small thickness Δr. The mass of such a thin shell is $\Delta M = \rho\,\Delta V$, where ρ is the density of the material and the shell volume is $\Delta V = 4\pi r^2\,\Delta r$, the product of the shell area and the shell thickness. As shown in *Sample Problem* 5, the moment of inertia of such a shell is $\Delta I = \frac{2}{3}\Delta M\,r^2$, so $\Delta I = \frac{8\pi}{3}\rho\,r^4\Delta r$. The total moment of inertia of all the shells together, i.e. of an infinite number of shells of infinitesimal thickness, is $I = \int dI = \int_0^R \frac{8\pi}{3}\rho\,r^4 dr = \frac{8\pi}{3}\rho\,\frac{R^5}{5}$, where R is the radius of the sphere. The density ρ of the sphere is the total mass divided by its total volume, $\rho = \mathrm{M} / \frac{4}{3}\pi R^3$, so the moment of inertia of the sphere can be written

$$I_{\text{sphere}} = \frac{2}{5}MR^2.$$

16.5 The parallel axis theorem

Most of the moments of inertia we have calculated so far have been about an axis through the *center of mass* of the object. Recall that the center of mass (CM) of an object is the distance $\mathbf{r}_{\text{CM}} = (X_{\text{CM}}, Y_{\text{CM}}, Z_{\text{CM}})$ from the origin of coordinates, where $X_{\text{CM}} = \frac{\Sigma_i\, m_i x_i}{\Sigma_i\, m_i}$ (with similar formulas for Y_{CM} and Z_{CM}). Quite often we need the moment of inertia of an object about some other axis instead, one that does not pass through the CM of object. For example, we might want to know the moment of inertia of a stick about one end of the stick, or the moment of inertia of a ring about an axis perpendicular to the ring, passing not through the center of the ring but through a point on the ring itself, as shown. How can we find the moment of inertia about such an axis, *without* starting each time with the definition $I = \sum_i m_i r_i^2$?

The *parallel-axis theorem* allows us to find the moment of inertia I of an object about any axis at all, given the object's moment of inertia I' about a *parallel* axis that passes through the CM.

The picture at the right shows an object and two coordinate systems: Here x', y', and z' are coordinates whose origin is at the CM of the object, and x, y, z are coordinates whose origin is at an arbitrary point. The two sets of axes are parallel to one another, and the z and z' axes are a distance $d = (X, Y)$ apart, as shown. We are going to compare the object's moment of inertia about an *arbitrary* axis, the z axis, with its moment of inertia about a *parallel* axis, the z' axis, which passes through the CM of the object.

The moment of inertia of the object about the z axis is

$$I = \Sigma m_i(x_i^2 + y_i^2) = \sum m_i[(X + x_i')^2 + (Y + y_i')^2]$$

$$= \sum m_i[X^2 + Y^2 + x_i'^2 + y_i'^2 + 2Xx_i' + 2Yy_i']$$

$$= \sum m_i d^2 + \sum m_i(x_i'^2 + y_i'^2) + 2X\sum m_i x_i' + 2Y\sum m_i y_i' \qquad (34)$$

since $d^2 = X^2 + Y^2$. Now recall that both $\sum m_i x_i' = 0$ and $\sum m_i y_i' = 0$, since x' and y' are coordinates relative to the center of mass. (For example, if the origin of the primed coordinates is located at the CM, then the x-distance of the CM from the origin is $X_{CM} = \frac{\Sigma_i m_i x_i'}{\Sigma_i m_i} = 0$.) Therefore we get the simple result

$$I = I' + Md^2 \qquad (35)$$

where $I' = \sum m_i(x_i'^2 + y_i'^2)$ is the moment of inertia about an axis through the CM.

In summary, we have found that the moment of inertia of an object about an *arbitrary* axis is equal to the moment of inertia of the object about an axis through the CM which is *parallel* to the arbitrary axis, *plus* the mass of the object times the square of the distance between the two axes. That is the meaning of the *parallel axis theorem* $I = I' + Md^2$.

Sample Problem 7. Find the moment of inertia of a ring of mass M and radius R about an axis perpendicular to the ring, passing through a point on the rim, as shown.

Solution: The moment of inertia of the ring about an axis through its center is $I' = MR^2$. So according to the parallel axis theorem, the moment of inertia about a parallel axis through a point on the rim is

$$I = I' + Md^2 = MR^2 + MR^2 = 2MR^2$$

since the distance d between the two axes is $d = R$. The moment of inertia of a ring of mass M and radius R about an axis perpendicular to the ring, passing through a point on the rim, is $I = 2MR^2$.

Sample Problem 8. Model a shovel by a uniform rectangular blade of mass M and length d attached to one end of a uniform handle of mass m and length ℓ, as shown. Find the moment of inertia of the shovel about the axis A-A that passes along the edge of the blade that is attached to the handle.

Solution: The moment of inertia of the shovel is the sum of the moments of inertia of the handle and the blade. The moment of inertia of the *handle* about its *center* is $\frac{1}{12}m\ell^2$ (See *Sample Problem* 3), so the moment of inertia of the handle about axis A-A, a distance $\frac{\ell}{2}$ from the axis through the handle's center, is $I_{\text{handle}} = \frac{1}{12}m\ell^2 + m\left(\frac{\ell}{2}\right)^2 = \frac{1}{3}m\ell^2$, using the parallel axis theorem. The moment of inertia of the *blade* about an axis passing through its center which is parallel to axis A-A is $\frac{1}{12}Md^2$, since we can think of the blade as a number N of thin sticks of length d and mass M/N, all lying parallel to one another. Each stick has moment of inertia $\frac{1}{12}(M/N)d^2$ about an axis through its center, so all the sticks taken together have $\frac{1}{12}Md^2$ about that axis. By the parallel axis theorem the moment of inertia of the blade about axis A-A is $\frac{1}{12}Md^2 + M\left(\frac{d}{2}\right)^2 = \frac{1}{3}Md^2$. The moment of inertia of the entire shovel about axis A-A is therefore the sum, $I_{\text{shovel}} = I_{\text{handle}} + I_{\text{blade}} = \frac{1}{3}\left(m\ell^2 + Md^2\right)$.

Summary of some moments of inertia for objects of mass M and radius R or length ℓ:

(Note that each result is the product of the mass of the object, some distance (radius or length) squared, and some dimensionless shape factor.)

$I = MR^2$ for a ring (or hoop) about its axis of symmetry

$I = \frac{1}{2}MR^2$ for a disk (or solid cylinder) about its axis of symmetry

$I = \frac{2}{3}MR^2$ for a spherical shell about any axis through the center

$I = \frac{2}{5}MR^2$ for a uniform solid sphere about any axis through the center

$I = \frac{1}{3}M\ell^2$ for a uniform stick about a perpendicular axis through one end of the stick

$I = \frac{1}{12}M\ell^2$ for a uniform stick about a perpendicular axis through the center of the stick

Figure skaters change their rotational angular velocity by changing their moments of inertia. A skater undertaking a "sit spin" begins in a crouch, with one of her legs and both arms extended, spinning slowly about a vertical axis through her center of mass, as Tara Lipinski demonstrated in the 1998 United States Championships, at the left in Fig. 2. Her angular momentum is nearly conserved as she spins on the nearly frictionless ice. When she rises up and pulls her arms and legs in (and magically turns into Michelle Kwan) as shown at the right, her moment of inertia is substantially reduced, so she spins much faster. Even though only a fairly small fraction of her total mass is initially extended, it makes a lot of difference, because the moment of inertia of a mass (like a forearm or foot) is proportional to its distance *squared* from the rotation axis. Even if the mass is pulled in only halfway to the rotation axis, its moment of inertia is reduced by a factor of four.

Angular momentum is conserved for a figure skater about a vertical axis through her center of mass. It is also conserved in the collision of wheels on a common axle, as we have shown. But in a collision the angular momentum of each individual wheel changes. And the figure skater somehow had to start spinning in the first place. What causes such *changes* in angular momentum? That is the topic we will take up tomorrow.

Fig. 2. Iceskater spinning up by decreasing her moment of inertia

(Photos from "The Official Book of Figure Skating" Simon and Schuster Editions 1998)

State of the Search

1. There is a rotational inertia analogous to the inertia of translational motion. A wheel tends to retain its rotation rate: If it starts at rest it stays at rest, but if spun up initially, it keeps spinning at the same rate, unless something causes it to slow down or speed up, like a hand tugging on it.

2. The total angular momentum of a system about a central axis is conserved for a system of objects free to rotate without friction around the axis. The system may consist of a number of objects. The angular momentum L of each such object is

$$L = I\omega,$$

where I is the moment of inertia of the object and ω is its angular velocity around the axis. The angular velocity can be positive (if the object rotates counterclockwise) or negative (if the object rotates clockwise.) Note that the formula $L = I\omega$ for angular momentum is *analogous* to the formula $p = mv$ for linear momentum. That is, angular velocity ω is analogous to linear velocity v, moment of inertia I is analogous to mass m (both representing a kind of inertia), and angular momentum L is analogous to linear momentum p. These are only analogies, however; the analogous quantities are definitely not the same as one another. They have different dimensions, for example, and different units.

3. The moment of inertia (or rotational inertia) of an object of mass M and radius R is

$$I = fMR^2$$

where the "shape factor" f depends upon the mass distribution within the object. For a ring, for example, all the mass resides at the same radius R; its moment of inertia is $I_{\text{ring}} = MR^2$, corresponding to a shape factor $f_{\text{ring}} = 1$. The moment of inertia of a solid disk is $I_{\text{disk}} = (1/2)\, MR^2$, corresponding to $f_{\text{disk}} = 1/2$, and the moment of inertia of a solid sphere about an axis through its center is $I_{\text{sphere}} = 2/5\, MR^2$, corresponding to $f_{\text{sphere}} = 2/5$.

4. The moment of inertia of a point mass m is

$$I_{\text{point mass}} = mR^2,$$

if it is a distance R from the axis of rotation. The moment of inertia of any object is the sum of the moments of inertia of the point masses from which it is composed. Also solid spheres can be composed from disks, and disks from rings.

5. The moment of inertia of an arbitrary object about a given axis is

$$I = \sum_i m_i r_i^2$$

for a discrete set of particles composing the object, where m_i is the mass of the ith particle and r_i is the distance of that particle from the rotation axis. For a continuous mass distribution, the moment of inertia is the integral

$$I = \int r^2 dm \ .$$

6. The *parallel axis theorem* states that the rotational inertia I of an object about an arbitrary axis is equal to the rotational inertia I' of the same object about a parallel axis passing through its center of mass, plus the mass M of the object times the square of the distance d between the two axes. That is,

$$I = I' + Md^2 \ .$$

The parallel axis theorem shows that moments of inertia depend not only upon the object, but upon the axis of rotation. The same is true for angular momentum; the angular moment of a wheel depends not only on the mass distribution within the wheel, but also upon where the rotation axis is located.

7. Previously we have learned about the conservation of momentum and the conservation of energy. The *momentum* of a system of particles is conserved if there is no net external force on the system. We can't get momentum from nowhere; if we could, a car might accelerate without any applied force. The *mechanical energy* (the sum of kinetic and potential energies) of a system is conserved if there are no frictional heating losses and if no outside work is done on the system. We also can't get energy out of nothing; if we could, homes might be heated without burning fuel or tapping other energy sources.

Now we have met a third conservation law in mechanics; the conservation of angular momentum. Is angular momentum always conserved? If not, how can it be changed? That is the topic of *Day Seventeen*.

PICTURE QUIZ 16

1. In each of the following comparisons (a)-(c), which example (A or B) leads to a greater final common angular speed when the two objects meet and stick together? Explain why.

2. A nonrotating ring of radius $R/2$ is dropped symmetrically onto a disk of mass M and radius R, freely rotating with angular velocity ω_0. If they reach a common final angular velocity $\omega_f = \omega_0/3$, what is the mass of the ring?

(A) M (B) 2M (C) 3M (D) 4M (E) 5M

3.* The stick-figure shown is performing a simple trampoline stunt known as a "back-drop." In this stunt, he bounces, tucks, and lands on his back. If the trampolinist has zero angular momentum at "B" is it possible for him to land on his back as in "D"?

(A) Yes, a trained trampolinist could successfully complete the stunt as shown.

(B) No, even a well-trained person could not do that stunt if at "B" he had no rotating motion.

4.* One standard but difficult dive from a three-meter springboard is a forward one and one half somersault with three twists (the "twist" is about an axis from the divers' head to her feet.) If she has initiated only somersaulting motion on the board, is it possible for her to do multiple twist dives like the "triple twister" without beginning the twist until after leaving the board? Could a diver learn to twist in either direction, without beginning the twist on the board?

(A) Yes, a diver really can do a multiple twist dive, and can choose the twist direction after leaving the board.

(B) Yes, a diver can do multiple twist dives after her feet leave the board. However, no matter how hard she trains the direction of the twist is already determined before her feet leave the board.

(C) No, even a well-trained diver could only do a multiple twist dive if she initiated the twisting motion while her feet still touched the board.

5. Two rotating space stations a and b are each cylindrical shells of the same radius and total mass; they also rotate about their symmetry axes with the same angular velocity. Station a has length D and is formed from a uniform-density material that includes two endcaps. Station b has length $2D$ and is formed from a uniform-density material with *no* circular endcaps. About their symmetry axes,

(A) Station a has more angular momentum than b.

(B) Station b has more angular momentum than a.

(C) Stations a and b have the same angular momentum.

*Questions 3 and 4 are from an article by Cliff Frohlich, Amer. J. Physics 47, 7 (1979). Of 59 physics graduate students, postdocs, and faculty at a prestigious university, 66% got question 3 right, but only 44% got 4 right!

PROBLEM QUIZ 16

1. A disk of mass M and radius R rotates with angular velocity ω_0 on a frictionless horizontal shaft. A second disk, of mass $2M$ and radius $R/2$, rotates with angular velocity $\omega = -2\omega_0$ on the same shaft. The two disks collide, and the first disk has angular velocity $\omega_0/2$ after the collision. What is the final angular velocity of the second disk?

2. A washer is made from a piece of sheet metal. The washer has mass M, inner radius R_1, and outer radius R_2, as shown. Find the moment of inertia of the washer about an axis through its center and perpendicular to its plane.

3. Four small objects of equal mass are connected by spokes of negligible mass and length d to a central axis, as shown. (a) If the moment of inertia of the system about the central axis is I, find the mass m of each object in terms of I and d. (b) What is the angular momentum L of the system if each object is moving counterclockwise with speed v in a circle about the axis? Express L in two ways: (i) in terms of I, d, and v (ii) in terms of m, d, and v.

4. Which of the following is the moment of inertia of a thin rod of mass M and length D about an axis through its center at $x = 0$, if the mass density (the mass per unit length) of the stick is $\lambda(x) = \lambda_0(1 - 2|x|/D)$ for $|x| \leq D/2$?

(A) $\frac{1}{3}MD^2$ (B) $\frac{1}{6}MD^2$ (C) $\frac{1}{12}MD^2$ (D) $\frac{1}{24}MD^2$

5. Find the moment of inertia of a solid cylinder of mass M, radius R, and length L, about an axis drilled lengthwise along the cylinder, a distance $R/2$ from the symmetry axis, as shown.

Special Topic: Core Collapse of Stars during Supernova Explosions

Every hundred years or so some heavy star in our galaxy blows up, becoming what is called a Type II Supernova. The supernova may be visible during the daytime for a few weeks, and at night for much longer. The remnants of a supernova seen by Chinese and Japanese observers in the year 1054 A.D. is no longer visible to the naked eye, but can be seen through telescopes as an object called the Crab Nebula. A relatively nearby supernova was observed in 1987 in the southern hemisphere; it was not in our own Milky Way galaxy but in the Large Magellanic Cloud, a small galaxy orbiting our own. Supernovae are often detected in distant galaxies, and are so bright they outshine all the other stars in their galaxy put together, at least for a few days.

Heavy stars "burn" their nuclear fuel quickly, in so-called nuclear fusion reactions, in a few million years rather than the four or five billion years our sun has been shining and will continue to shine. Hydrogen is converted into helium, helium into carbon, oxygen, neon, and so on. This burning provides the energy which supports the star against gravitational collapse. The stars are entirely gaseous, so without the enormous heat energy provided by nuclear fusion they would be crushed by gravity. Eventually the core of the star, the hottest portion where the heavier elements are produced, becomes largely iron and elements neighboring iron in the periodic table. The nuclei of these elements are especially stable and will not burn into still heavier nuclei in such a way as to release energy. No more heat energy is produced to prevent collapse. Most of the star undergoes an almost free infall, collapsing in approximately one minute.

The outer parts of the star, which were never as hot as the core, contain unburned hydrogen, which suddenly ignites due to compressional heating as the star collapses. Gravitational potential energy is converted into heat energy, so the ignition temperature for thermonuclear fusion reactions is suddenly reached for great quantities of material. The star explodes, spreading material outward with enormous energy. The core of the star, however, continues to implode. If the core is not too heavy (not greater than about two or three times the mass of our sun) it will stop, forming a neutron star. The neutron star, formed primarily of neutrons pushed together about as close as they can get, has a density comparable to that of an atomic nucleus, and a radius

of about 10 km. We observe such neutron stars as what are called *pulsars*; they rotate, radiating like a lighthouse, beaming energy once per revolution toward the Earth.

Before it became a supernova, the original heavy star certainly rotated (all stars rotate, including the Sun.) As the star explodes to become a supernova, the angular momentum contained in the outer layers is dispersed throughout a large volume of space. The core continues to shrink, so as its moment of inertia I gets smaller, its angular speed ω must increase proportionately to conserve angular momentum $L = I\omega$. What effect does the collapse have on the period of rotation?

Suppose for example the core of the original heavy star has a radius $R = 10^5$ km and rotates once every 30 days. (Our Sun has a radius of $7 \cdot 10^5$ km and rotates with period of about 27 days.) We take the core of the heavy star to be smaller than the Sun, even though its mass is larger, because it is under great pressure from the rest of the star. We assume the core is approximately spherical even though it is rotating; that is a good approximation for estimating the core spinup. The moment of inertia of the core is then $I = fMR^2$, where the shape-factor f depends on the density distribution: $f = 2/5$ if the density is uniform throughout.

As the core collapses from $R_i = 10^5$ km to $R_f = 10$ km, its moment of inertia falls by a factor of about $(R_i/R_f)^2 = 10^8$. Therefore the core's angular velocity must increase by a factor of 10^8 to conserve angular momentum $L = I\omega$. The angular velocity $\omega = 2\pi/T$ is inversely proportional to the rotation period T, so T will fall by a factor of 10^8, going from 30 days ($= 2.5 \cdot 10^6$ seconds) to a fraction of a second (.025 seconds, taking the initial conditions as stated.) The entire core, now a neutron star, with a mass exceeding that of our Sun and a radius of 10 km, rotates some forty times per second!

We do in fact observe pulsars that rotate with periods from a millisecond or so up to several seconds. Many neutron stars probably spin with periods of a few milliseconds when they are first formed, and then spin down (increase their periods) as they lose energy and angular momentum to escaping particles and radiation, which we will discuss during Day 19.

Questions and Problems from *Day 16*

16Q1. In recent times ice sheets have been melting in polar regions. What likely effect has that had on the length of Earth's day?

16Q2. The Mississippi River flows from north to south, carrying sediment with it. What effect does this tend to have on Earth's rotational velocity?

16Q3. An acrobat is standing on a rotating turntable with both arms extended. (a) As he turns, he grabs a dumbbell in each hand, held out to him by two people standing on the floor. Does his angular velocity decrease, stay the same, or increase? Explain. (b) Holding the two dumbbells with arms extended, continuing to turn, he drops both dumbbells. Does his angular velocity decrease, stay the same, or increase? Explain.

16Q4. Standing at rest on a frictionless sheet of ice wearing a spacesuit in a perfect vacuum, without throwing anything away, one's angular momentum about a vertical axis remains always zero no matter what contortions one goes through. Is it therefore impossible to turn completely around, alternately facing east, north, west, south, and finally east again?

16Q5. A cat is held upside down at rest. If the cat is dropped, it will turn around and land on its feet. How does it do this, keeping its angular momentum zero at all times?

16Q6. In a simple springboard dive, the diver enters the water head-first, face down. Is it possible for a diver to enter the water head first, face up (back to the water) instead? If so, how could the diver do it?

16Q7. Two three-dimensional objects have exactly the same shape and the same mass density, but the second has twice the linear size as the first. What is the ratio of their moments of inertia about similar (parallel) axes through their centers of mass?

16Q8. Must an object be rotating to have a moment of inertia? Why or why not?

16Q9. Consider two solid wooden spheres, where sphere 2 has twice the radius of sphere 1. By what factor does the moment of inertia through the center of sphere 2 exceed the moment of inertia through the center of sphere 1?

16Q10. Is the angular momentum of disk a, shown below, larger than, smaller than, or equal to the angular momentum of disk b?

16Q11. Four $T's$ are made from two identical rods of equal mass and length, as shown below. Rank in order, from largest to smallest, the moments of inertia I_a to I_d for rotation about the dashed line.

Section 16.2 "Experiments" with disks

16P1. A wheel of radius $r = 12$ cm turns through a 75° angle in 5.0 s, at constant angular velocity. (a) What is the angular velocity, in rad/s? (b) What is the rotational period, in seconds? (c) What is the speed of a point on the rim?

16P2. A car is traveling at speed $v = 55$ km/hr. The tires, which are not slipping relative to the road, have a radius $R = 32$ cm. (a) What is the angular velocity of the wheel, in rad/s? (b) What is the wheels period of rotation? (c) Through how many complete rotations will the wheel have turned in one hour?

16P3. (a) Find the angular speed ω of Earth's rotation about its axis, in rad/s. (b) Earth's radius is $R = 6400$ km. What is the speed v of a point on Earth's surface at (i) the equator? (ii) 45° north latitude?

16P4. A phonograph record of radius $r = 15$ cm rotates at 33 rpm (revolutions per minute). Find (a) the angular speed ω of the record, in rad/s; (b) the rotation period T; (c) the speed v of a point on the edge.

16P5. (a) A nonrotating ring of mass M and radius R is dropped concentrically onto a disk of mass $2M$ and radius $2R$, rotating initially with angular velocity ω_0. What is the final angular velocity ω_f after they begin turning together? (b) The two are now reversed, with the same ring (of mass M and radius R) at the bottom, initially rotating with ω_0 , and the nonrotating disk dropped upon it. What now is the final angular velocity ω_f?

16P6.(a) A nonrotating disk of thickness t and radius R is dropped upon a disk of thickness $2t$ and radius $R/2$, made of the same material, rotating initially with angular velocity ω_0, as shown below. What is the final angular velocity ω_f of the combination? (b) Suppose everything is the same as in part (a), except that the bottom disk is made of a material whose density (mass/volume) is twice as large as that of the top disk. What now is the final angular velocity of the combination?

16P7. (a) A nonrotating disk of mass M and radius R is dropped upon a ring of mass $2M$ and radius $R/2$, rotating initially with angular velocity ω_0. What is the final angular velocity ω_f after they begin turning as a unit? (b) The objects are reversed, with the disk initially rotating with angular velocity ω_0 at the bottom. After the initially nonrotating ring is dropped, what is their final mutual angular velocity ω_f?

16P8. A disk of mass M and radius R rotates on the platform with angular velocity ω_0. Three other nonrotating disks, each of radius R, but with masses $M, 2M$, and $3M$, respectively, are suspended above it. (a) If all three upper disks are dropped together onto the rotating disk, what is the final angular velocity ω_f of all four together? (b) If the three upper disks are dropped instead one at a time, each time allowing all bottom disks to come to a common angular velocity before the next disk is dropped, what now is the final common angular velocity of all four disks?

Section 16.3 Conservation of Angular Momentum

16P9. A disk of mass M and radius R, rotating with angular velocity ω_{10} , falls onto a second disk, of mass $3M$ and radius $R/2$, rotating with angular velocity $\omega_{20} = -\omega_{10}$. Their axes coincide. The first disk bounces upward after the collision, rotating with angular velocity $\omega_{1f} = -\omega_{10}/2$. What then is the angular velocity ω_{2f} of the second disk, as a multiple of ω_{10} ? Which way is it rotating?

16P10. Two disks are free to move along the thin horizontal axle of a truck transmission. The first disk has mass M and radius R , and is turning with angular velocity ω_0 . It runs into a second disk, also of mass M and radius $2R$,

initially *non*turning. As they bounce apart, the first disk has angular velocity $\omega_0/4$. What then is the angular velocity of the second disk?

16P11. A circular sander is a power tool in which a disk of mass m and radius R (to which sandpaper can be attached) is caused to rotate about a narrow axle by a motor contained in the main body of the tool. Model the main body as a solid cylinder of mass M , radius R , and moment of inertia $(1/2)MR^2$, as shown below. The tool is set at rest in empty space, and the (internally battery powered) motor turned on to cause the disk to rotate at angular velocity ω_0 relative to the tool body. Find then the angular velocity of (a) the disk, and (b) the tool body, in an inertial frame of reference.

16P12. A chief executive officer of mass M is seated on a disklike piano stool of mass $M/20$ and radius R, which can turn almost frictionlessly. Model the CEO as a vertical, solid cylinder, also of radius R , equivalent to a stack of disks. The CEO, who is initially stationary, is handed a bicycle wheel of mass $m = M/30$ and radius $r = 2R$, rotating CCW with angular speed ω_0 about a vertical axle as seen from above. (The CEO holds both ends of the axle.) The CEO then turns the wheel upside down, so the wheel now turns CW as seen from above, rotating with the same angular speed ω_0 it had before. Estimate the angular speed of the CEO.

Section 16.4 Summing Moments of Inertia

16P13. A wheel consists of a uniform disk of mass M and radius R , with a thin tire of mass m and radius R around the circumference of the disk, as shown below. Find the moment of inertia of the wheel about an axis through its center and perpendicular to its plane.

16P14.(a) Explain why the moment of inertia of a thin-walled hollow cylinder of mass M and radius R, about its axis of symmetry, is $I = MR^2$. (b) Find the moment of inertia of a solid cylinder of mass M and radius R. about its axis of symmetry.

16P15. A solid cone has mass M and height h; the radius at the top is R, as shown below. Show that the moment of inertia about its symmetry axis is $I = MR^2$. *Hint*: Slice the cone into horizontal disks, and integrate over them.

16P16. A hollow cone has mass M and height h. At the top (which is open; there is no end cap) its radius is R, as shown below. Find the moment of inertia about its axis of symmetry, in terms of any or all of M, R, and h.

16P17. An empty tin can (with both ends in place) is made of uniform-density metal. It has radius R, length ℓ, and mass M. Find the moment of inertia of the can about its symmetry axis.

16P18. A yo-yo consists of three disks, all made of the same kind of wood. Each of the two outer disks has radius R and width w, and the inner disk has radius $R/3$ and width $w/2$, as shown below. Find the moment of inertia of the yo-yo about its symmetry axis, in terms of R and its total mass M.

16P19. The radius of gyration ρ of a body about a particular axis is the distance from that axis to a point at which the bodys entire mass can be considered to be concentrated, $I = M\rho^2$. Find the radius of gyration of (a) a ring (b) a disk (c) a solid sphere.

16P20. Find the moment of inertia of a propeller modeled by four uniform sticks, each of mass m and length L, with their centers placed together at the rotation axis, as shown below.

16P21. Find the moment of inertia of a solid sphere of mass M and radius R, not by adding up spherical shells (as is done in *Sample Problem 6*), but by adding up an infinite stack of infinitesimally-thick flat parallel disks.

16P22. A sphere of radius $R/2$ is removed from the center of a uniform-density sphere of radius R . What is the moment of inertia of the resulting hollow sphere about an axis through its center, in terms of R and its remaining mass M?

16P23. The mass of a disk of radius R and thickness t is not uniformly distributed; it has mass density $\rho(r) = \rho_0/[1 + r^2/R^2]$, where ρ_0 is the density at the center and r is the distance from the axis of symmetry. Find (a) the total mass M of the disk, and (b) the moment of inertia around an axis perpendicular to the disk and passing through its center. (c) Express the moment of inertia in terms of M and R , and compare the result with the rotational inertia of a disk with the same M and R , but with uniform mass density. Comment on the difference.

16P24. "Plane lamina" are objects confined to a plane. Thin disks or hoops are examples, but any thin flat object is included. Let I_x, I_y, and I_z be the moments of inertia of such an object about its x -axis, y -axis, and z -axis, respectively, where the z -axis is perpendicular to the plane, and the x and y axes lie in the plane of the object, as shown below. Prove that $I_z = I_x + I_y$.

16P25. Use the result of 16P24 to find the following moments of inertia. (a) The moment of inertia of a uniform thin disk of mass M and radius R about an axis through its CM and perpendicular to the disk is $I_z = (1/2)MR^2$. Find I_x , the moment of inertia about an axis in the plane of the disk, passing through its CM. (b) A non-uniform thin disk of mass M and radius R has a density distribution which depends only on r , the distance from the center, such that its moment of inertia about an axis in the plane of the disk which passes through its CM is $(1/3)MR^2$. What is the moment of inertia of the disk about an axis through its center which is perpendicular to the disk?

16P26. (a) Find the moment of inertia of a thin uniform square of mass M and side L about an axis through the CM, which is perpendicular to the square. (b) Then find the moment of inertia of the thin square about any axis in the plane of the square passing through the CM, such as the axis shown below. (*Hint*: Review the result of 16P24.)

16P27. The three masses shown below are connected by massless, rigid rods. (a) Find the coordinates of the center of mass. (b) Find the moment of inertia about an axis that passes through mass A and is perpendicular to the page. (c) Find the moment of inertia about an axis that passes through both B and C.

16P28. Find the moment of inertia about the axis of the object shown below.

Section 16.5 The Parallel Axis Theorem

16P29. A sphere of radius $R/2$ is removed from a uniform-density sphere of mass M and radius R. The center of the removed sphere is a distance $R/2$ from the center of the larger sphere. What is the moment of inertia of the resulting hollow sphere about the axis shown below?

16P30. Find the moment of inertia of a uniform stick of mass M and length ℓ about an axis through one end of the stick (a) by integrating (b) by using the parallel axis theorem and the moment of inertia of the stick about an axis through its center of mass, as found in *Sample Problem 3*

16P31. Find the moment of inertia of an object consisting of a uniform stick of mass M and length ℓ with a small mass m at each end of the stick, if the axis is perpendicular to the stick and passes through (a) the center of the stick; (b) one end of the stick.

16P32. A uniform wire of mass M and length 4ℓ is bent to form the sides of a square, as shown below. Find the moment of inertia of the square about an axis through the squares center of mass, which is (a) perpendicular to the square; (b) in the plane of the square, parallel to two opposing sides.

16P33. Find the moments of inertia of (a) a straight wire of mass M and length ℓ about an axis through one end of the wire. (b) a wire of mass M and length ℓ which has been bent to form an equilateral triangle, about an axis perpendicular to the plane of the triangle, which passes through one vertex.

16P34. A thin uniform rectangular board of mass M has width a and length b. (a) Find the moment of inertia of the board around the symmetry axis running lengthwise down the board. (b) Find the moment of inertia of the board about an axis passing through its center which is perpendicular to the board.

16P35. Find the moments of inertia of (a) a thin uniform square of cardboard of mass m and side ℓ, about an axis that is perpendicular to the square, and that passes through its center. (b) a uniform cube of mass M and edge-length ℓ, about an axis which passes through the center of the cube, and is perpendicular to two of its faces.

16P36. A bicycle wheel consists of a rim of mass M and radius R, and 60 uniform spokes, each of mass m, extending from the center of the wheel to its rim. (a) Find the moment of inertia I of the wheel about an axis through its

center, perpendicular to the plane of the wheel. (b) Find I numerically if $M = 1.15$ kg, $R = 0.35$ m, and $m = 0.012$ kg. (c) What fraction of the total moment of inertia is due to the spokes?

16P37. Starting with a uniform-density metallic disk of mass M and radius R, a machinist cuts out a disk of radius $R/2$ extending from the center to the edge of the original disk. What is the moment of inertia of the remaining object about an axis through the center and perpendicular to the plane of the object, in terms of R and the initial mass M?

More conservation of angular momentum problems

16P38. A solid sphere of mass M and radius R is sliced in two, and one of the hemispheres is placed on a frictionless vertical axle so the flat side is on top, as shown below; the hemisphere is made to rotate at angular velocity ω_0. A nonrotating ring of mass $M/2$ and radius $R/2$ is dropped concentrically onto the hemisphere. (a) If they come to equilibrium, find their common final angular velocity ω_f. (b) Suppose instead the ring bounces up, rotating with angular velocity $\omega = \omega_0/2$. What then is the angular velocity of the hemisphere?

16P39. A disk of mass M and radius R rotates on a platform with angular velocity ω_0. A nonrotating ring of mass M and radius $R/2$ (and with glue on its lower surface) is dropped down onto the disk so that the center of the ring is a distance $R/2$ from the center of the disk, as shown below. What is their final mutual angular velocity ω_f? Assume the axle is frictionless.

16P40. If all seven billion people on Earth simultaneously embarked on polar expeditions, estimate the change in the length of the day by the time they arrive.

16P41. A child of mass m is standing at the center of a disk-like merry-go-round of mass M and radius R, which is rotating on frictionless bearings with angular velocity ω_0. There is a straight, light horizontal rail extending from the center of the merry-go-round to its rim, as shown below. The child walks beside the rail outward toward the rim. (a) Find the angular velocity ω of the merry-go-round as a function of the distance r of the child from the center. Model the child as a point mass. (b) Upon reaching the rim, the child falls off. What then is the merry-go-round's angular velocity?

16P42. A daredevil physics instructor of typical weight and size is seated on a piano stool which can rotate with little friction about the vertical axis. She holds a 3-kg dumbbell in each hand, with her arms outstretched, and is then given a small angular velocity, as shown. By roughly what factor will her angular velocity increase if she pulls the dumbbells in close to her body?

16P43. A disk of mass M and radius R is rotating freely about its symmetry axis with angular velocity ω_0. A small bug of mass m is standing on the rim of the disk. The bug then crawls inward to the center of the disk. Find the angular speed ω_f of the disk when the bug has completed its journey.

16P44. A space colony is a thin hollow cylinder of mass M and radius R, with two endcaps, each of mass $M/4$ and radius R. It rotates about its symmetry axis with angular velocity ω_0, as shown below. (a) What is the moment of inertia I of the colony about its symmetry axis? (b) Two cables of negligible mass are stretched from the axis to the rim, as shown below. Two elevators, each of mass m, "descend" from the axis to opposite points on the rim. When they arrive, what is the angular velocity ω of the colony, in terms of M, m, and R?

16P45. Five hundred people, each of mass m, live on the inside rim of a rotating space colony of radius R and moment of inertia I, not counting the colonists. The effective gravity at the rim is g. If half the people travel inwards to the colony hub, what is the new effective gravity to those still living at the rim, in terms of m, R, I, and g?

16P46. The rotational period $P = 0.089$ s of the Vela Pulsar has on several occasions been observed to suddenly decrease (on a time-scale of minutes) by about 200 ns (1 ns $= 10^{-9}$ s). This sudden decrease is called a *glitch*. One suggested explanation is that the surface crust of the star near the equator suddenly cracks and settles, reducing the rotational inertia of the star, causing it to spin faster to conserve angular momentum. (a) Find the fractional change in the star's moment of inertia $\Delta I/I$ required for an observed glitch. (b) In a crude model in which the pulsar has the mass of our Sun, a radius of 10 km, and uniform density throughout, how far (in centimeters) would the entire star have to shrink to produce the observed decrease in period?

16P47. Planet Earth has moment of inertia $I = 0.33MR^2$ about its north-south pole axis of rotation, where M is its mass and R is its radius. (a) Does this mean that its density is more concentrated in the center or in the outer portions of the planet? Explain. (b) There is a ring of communication satellites in the equatorial plane of Earth, at an altitude such that they rotate around once/day. (In that way each satellite remains above the same location on Earth's equator.) Suppose the total mass of these satellites is $m = 1.5 \times 10^6$ kg. If they all crash to the Earth's surface, will the length of the day increase or decrease? By how much?

16P48. A particular merry-go-round has the form of a disk of mass M and radius R, supported frictionlessly at its center. A person of mass m is standing at the rim, and the entire system is rotating at angular velocity ω_0. If the person starts to walk around the rim at speed v_0 relative to the merry-go-round rim, in the same direction in which the rim is rotating relative to the ground, what is the new angular velocity ω_f of the merry-go-round?

Special Topic: Core Collapse during Supernova Explosions

16P49. The core of a heavy star rotates with period $T = 15$ days. The star explodes, so the core shrinks from a radius $R_0 = 17{,}000$ km to final neutron-star radius $R_f = 11$ km. (a) How many times/second does the resulting neutron star rotate? (b) What is its initial angular mo-

mentum L_0 and its final angular momentum L_f assuming the core has uniform density and that its mass is twice that of our Sun? (c) To answer parts (a) and (b) you had to make some simplifying physical assumptions. List two or three of them.

16P50. Immediately after forming, a neutron star of radius $R = 12$ km is rotating with period $T = 0.025$ s. (a) If the core radius of the original star was $R_0 = 15{,}000$ km, what was the rotation period T_0 of the original star? (b) What are the core's initial angular momentum L_0 and final angular momentum L_f, assuming that it has uniform density and that its mass is three times that of our Sun? (c) To answer parts (a) and (b) you had to make some simplifying physical assumptions. List two or three of them.

Day Seventeen

Torque

"If you are out to describe the truth, leave elegance to the tailor." – Albert Einstein

Yesterday we found that a quantity called *angular momentum* is conserved when wheels collide. The total angular momentum $L = I_1\omega_1 + I_2\omega_2$ stays the same, although the angular momentum of each individual wheel changes. Why does the angular momentum of each wheel change? What do the wheels do to one another to change their individual angular momenta?

Similarly, why does a bicycle wheel start to spin and what makes it slow down? What exactly causes it to change its angular momentum?

17.1 Experiments

A wheel is mounted on an axle projecting from one wall of the lab. It is a single casting of aluminum, with three coaxial "subwheels". Each subwheel is channeled like a pulley, so cord can be wound around it. The largest subwheel has twice the radius of the middle one, and the middle one has twice the radius of the smallest.

Two views of a wheel

We can use the wheel to carry out some experiments. We start by winding a cord around the smallest pulley, and pull on it to spin up the wheel. We have to pull quite hard to get the wheel turning. When we wind a cord around the largest pulley instead, the wheel spins up more easily. Finally, we wind one cord clockwise (CW) around the largest pulley, and another cord counterclockwise (CCW) around the smallest, and pull both cords with equal force, as shown. Which way will the wheel start to turn?

The wheel turns CW. To keep the wheel from turning, the cord around the smallest pulley must be pulled harder than the cord around the largest pulley.

Can we measure the two forces *quantitatively*? We attach a spring to each cord, as shown in Fig. 1. To keep the wheel from turning, we have to stretch the inner spring four times as much as the outer one. Four times as much force is being exerted by the inner spring. It is interesting that the ratio of the two pulley radii is *also* 4:1 ; that is,

$$\frac{F_1}{F_2} = 4 = \frac{r_2}{r_1}, \tag{1}$$

so $r_1 F_1 = r_2 F_2$: The product of the force and the distance from the axis at which the force acts is the same for both. Is this just a coincidence?

Fig. 1. The wheel stays in equilibrium when the inner spring stretches four times as much as the outer spring.

We try different combinations. For the second and third pulleys, the required forces have a 2:1 ratio, and so do the pulley radii. The same holds true for the first and second pulleys. In this way we confirm that equilibrium is maintained if

$$r_a F_a = r_b F_b \tag{2}$$

for any of the pulley pairs.

We will call the product $r F$ the *torque* τ exerted by the force F on the wheel, acting tangentially (i.e., tangent to the wheel) at a radius r from the axis of rotation: That is,

$$\tau = r F . \tag{3}$$

The torque can be doubled by doubling the force or by doubling the radius at which the force acts.

The wheel stays in equilibrium when the clockwise torque $\tau_{CW} = r_2 F_2$ is balanced by the counterclockwise torque $\tau_{CCW} = r_1 F_1$, as shown in Fig. 2. The figure shows two ways for the inner cord to exert the same CCW torque. In either case the wheel stays in equilibrium.

Fig. 2. Two ways for the inner cord to exert the same CCW torque on the wheel, to balance the CW torque due to the outer cord.

We can use the same sign convention for torque that we used during Day 16 for angular momentum. We take a counterclockwise torque to be *positive* and a clockwise torque to be *negative*. Then in Fig. 2 the counterclockwise torque is positive, $\tau_1 = r_1 F_1$, and the clockwise torque is negative, $\tau_2 = -r_2 F_2$. The wheel then remains in equilibrium if there is no *net* torque:

$$\tau_{net} = \tau_1 + \tau_2 = 0. \tag{4}$$

What if we pull on all *three* pulleys? We wind one cord CW around the largest pulley, and the other two CCW around the other pulleys. By measuring the stretch in each spring we find that the wheel stays at rest if the total torque is zero:

$$\tau_{net} = \tau_1 + \tau_2 + \tau_3 = r_1 F_1 + r_2 F_2 - r_3 F_3 = 0, \tag{5}$$

where we have again taken the CCW torques to be positive and the CW torque to be negative. By trying various combinations we confirm that the wheel's rotational equilibrium is always maintained if

$$\tau_{net} = 0. \tag{6}$$

There is an alternative way to attach cords to the wheel. There are small pins that fit into holes in the wheel at various points, so part of the pin sticks out a ways, perpendicular to the plane of the wheel, as shown at the right. We tie a small loop of cord and slip it over a pin, and then pull on the wheel at various angles, as illustrated in Fig. 3. Depending on how the wheel is oriented, we can pull in any of the ways shown, all with the same pin in the same hole.

Fig. 3. We can pull on the wheel in various directions (a) **F** parallel to **r** (b) and (c) **F** perpendicular to **r** (d) **F** neither parallel nor perpendicular to **r**

When attached as shown in Fig. 3(a), the wheel doesn't turn at all. When attached as in (b), the wheel turns CW. When attached as in (c) or (d), the wheel turns CCW. It is harder to make the wheel turn when in position (d) than when in position (c). It seems reasonable that the wheel should not turn in picture (a); however, the product rF is the same in each picture, so it can't be simply rF that causes turning. There must be something more to it.

How can we tell the difference between the four pictures? If we think of $\boldsymbol{r}$ as the vector pointing from the axis of rotation to the point where the vector force $\boldsymbol{F}$ is applied, it is clear what the difference is. In picture (a) the two vectors are parallel, in pictures (b) and (c) the vectors are perpendicular, and in picture (d) they are neither parallel nor perpendicular. It looks as though the only part of the force $\boldsymbol{F}$ that causes rotation is the part $F_\perp$ *perpendicular* to $\boldsymbol{r}$. We can do the usual spring experiments to test the idea, and find that it works. The parallel part of $\boldsymbol{F}$ doesn't matter.

We have to refine the definition of torque, keeping the name "torque" as the quantity that causes the wheel to start rotating. The magnitude of the torque is

$$\tau = r F_{\perp} = r F \sin\phi \tag{7}$$

(remembering that CCW torques are given a positive sign and CW torques a negative sign). Here ϕ is the smaller angle between the two vectors $\boldsymbol{r}$ and $\boldsymbol{F}$. We find ϕ by moving $\boldsymbol{r}$ so the tails of the two vectors $\boldsymbol{r}$ and $\boldsymbol{F}$ coincide; then ϕ is the smaller of the two angles

Fig. 4. (a) The angle ϕ between **r** and **F**, and the meaning of $F_{\perp}$. (b) The angle ϕ between **r** and **F**, and the meaning of $r_{\perp}$.

between them, as illustrated in Fig. 4(a). The angle ϕ is therefore always less than or equal to 180°, so that $\sin\phi$ will be nonnegative. Clearly the refined definition of torque reduces to $\tau = r F$ if $\boldsymbol{r}$ and $\boldsymbol{F}$ are perpendicular to one another, since then $\phi = 90^{\circ}$, making $\sin\phi = 1$.

There is still *another* way to express torque: It is also true that $\tau = r_{\perp} F$, the magnitude of F times the component of r perpendicular to F, since $r_{\perp} = r \sin\phi$. Here $r_{\perp}$ is sometimes called the *lever arm* or the *moment arm*; $r_{\perp}$ is illlustrated in Fig. 4(b). So the torque can be expressed in three equivalent ways:

$$\tau = r F \sin\phi = r F_{\perp} = r_{\perp} F, \tag{8}$$

where in each case we take the sign to be positive for CCW torques and negative for CW torques.*

* There is yet one *more* way to express torque, as we will discuss fully during *Day Twenty*. Torque can be expressed in terms of the so-called *cross product* of the **r** and **F** vectors, $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. In this notation the torque becomes a vector, perpendicular to the plane defined by the two vectors **r** and **F**. The magnitude of the torque vector is $r F \sin\phi$, the same value given in Eq. 8.

Sample Problem 1. Forces F_0 and $2F_0$ are exerted tangentially on the inner two pulleys, as shown. In terms of F_0, what force F_1 must be exerted on the outermost pulley, oriented at 60° to the radius vector, so there is no net torque on the wheel?

Solution: The clockwise torque on the innermost pulley is $-RF_0$, where R is the radius of that pulley, and the counterclockwise torque on the middle pulley is $(2R)(2F_0) = 4RF_0$; the sum of these is $3RF_0$, a net counterclockwise torque. (Recall that CCW torques are positive and CW torques are negative.) The clockwise torque on the outermost pulley is $-(4R)F_1 \sin 60^\circ$, which must equal $-3RF_0$, to make the total torque equal to zero. Therefore

$$F_1 = \frac{3RF_0}{4R\sin 60^0} = \frac{\sqrt{3}}{2}F_0 .$$

17.2 Gravitational torques

The wheels we have used so far are all circular. We can also try *non*circular wheels, such as a stick mounted on an axle through its center, as shown. Forces can be applied at various points along the stick and conditions found for equilibrium. The results are the same as before: The wheel can only be in equilibrium if there is no net torque on it.

What happens if we mount a wheel, circular or not, on an axle that does *not* pass through the wheel's center of mass? For example, suppose we mount a stick of mass M and length L on an axle that passes through the left end of the stick, as shown in Fig. 5(a). If we hold the stick in a horizontal position and then let go of it, the stick starts to rotate clockwise. There must be a net clockwise torque on the stick about the rotation axis. The only force that could exert this torque is the gravitational force pulling downward on every atom in the stick. How large is the resulting gravitational torque?

Fig. 5. (a) A stick with rotation axis at the left end. (b) Slicing the stick into small sections. (c) The radius x and force ΔF exerted on a typical section.

Gravity acts on every part of the stick. Imagine slicing the stick into small sections, as in Fig. 5(b). Let the ith section, which is a distance x_i from the rotation axis, have mass Δm_i. The gravitational force on that section is $\Delta F_i = \Delta m_i\, g$, as shown in Fig. 5(c), so the torque on it is

$$\Delta\tau_i = r_\perp \Delta F_i = x_i \Delta F_i = g\, \Delta m_i\, x_i. \tag{9}$$

The total torque is the sum of the torques $\Delta\tau$ over all the sections in the stick,

$$\tau = \sum_i \Delta\tau_i = g \sum_i \Delta m_i\, x_i\,. \tag{10}$$

The sum on the right is related to the stick's center of mass, defined by $X_{\text{CM}} \equiv \dfrac{\sum_i \Delta m_i\, x_i}{M}$, where $M = \sum_i \Delta m_i$ is the total mass of the stick.* The total gravitational torque on the stick can therefore be written

$$\tau = Mg\, X_{\text{CM}}. \tag{11}$$

The gravitational torque on the stick is equal to the total gravitational force Mg on the stick, multiplied by the distance X_{CM} of the stick's center of mass from the rotation axis. In other words, *the gravitational torque on the stick is as though the entire mass of the stick were concentrated at the stick's center of mass.* If the stick is uniform, for example, the gravitational torque is as though all the stick's mass were at its center.

*See Day Four.

This result is easy to generalize. The stick doesn't have to be horizontal, for example. There is also nothing special about a one-dimensional stick. It is straightforward to generalize the derivation to show that the total torque exerted by uniform gravity on any three-dimensional body is as though the mass of the body were entirely concentrated at its center of mass. Gravity alone cannot start an object rotating about its center of mass. If we drop an extended object with no initial rotation, its center of mass will fall straight and gravity will not cause any rotation to begin.

Sample Problem 2. A stick of length ℓ has a mass per unit length λ that increases linearly with distance from the left-hand end of the stick. That is, $\lambda(x) = kx$, where k is a constant. Beneath what point on the horizontal stick should you place your finger so the stick will be balanced in Earth's gravitational field?

Solution: Place your finger directly beneath the center of mass of the stick, since gravity would then exert no net torque on the stick. (The gravitational torque is as though all the mass of the stick were placed at the stick's CM, so if your finger is directly beneath that point no net gravitational torque is exerted.) The CM of the stick is located at

$$X_{\mathrm{CM}} \equiv \frac{\sum_i \Delta m_i\, x_i}{M} = \frac{\int_0^\ell dm\, x}{\int_0^\ell dm} = \frac{\int_0^\ell dx\, \lambda(x)\, x}{\int_0^\ell dx\, \lambda(x)} = \frac{k\int_0^\ell dx\; x^2}{k\int_0^\ell dx\, x} = \frac{\ell^3/3}{\ell^2/2} = \frac{2}{3}\ell$$

17.3 Static Equilibrium

For a wheel of any shape to stay in equilibrium, it is *necessary* that the net torque be zero. Is that also a *sufficient* condition? That is, if the net torque is zero, does that guarantee that the wheel will stay in equilibrium?

The answer is definitely *NO*. The *net force* has to be zero as well. Balancing torques doesn't guarantee that the forces are balanced, or *vice versa*. Take a long thin stick, for example, with forces of equal magnitude acting at either end. If the forces are in opposite directions, as in Fig. 6(a), the net force is $F = F_1 + F_2 = F_1 + (-F_1) = 0$, so the center of mass (CM) of the stick won't accelerate. If the CM starts at rest, it stays at rest. But the stick is obviously not in equilibrium; it starts to rotate, because there is a net CCW torque on it. *Both* forces exert CCW torques about the center of the stick.

Suppose instead that the two forces are in the *same* direction, as shown in Fig. 6(b). There is then no net torque, so the stick will not start to twist; one force exerts a CW torque

Fig. 6(a) There is a net torque on the stick, but no net force. (b) There is a net force on the stick, but no net torque.

and the other causes a compensating CCW torque about the stick's center. However, now there *is* a net force on the stick, so the CM of the stick (and the entire stick for that matter) will accelerate in the direction of the net force. The stick is not in equilibrium.

The moral is: *For an object to stay in equilibrium, both*

$$F_{\text{net}} = 0 \qquad \textit{and} \qquad \tau_{\text{net}} = 0 . \tag{12}$$

For example, the center of mass of our aluminum wheel hasn't moved once since we began our experiments, which implies the net force on it must have been zero throughout. The axle must exert a force that cancels out the sum of the forces due to the cords.

Sample Problem 3. A bead of mass m is threaded onto a wire of length ℓ and negligible mass. The wire is supported horizontally by strings at both ends, as shown. If the bead is a distance $\ell/3$ from the left-hand string, what is the tension in each string?

Solution: The net torque about any point must be zero. If we choose the bead as the origin, the clockwise torque is $(\ell/3)\,T_1$ due to the left-hand string and the counterclockwise torque is $(2\ell/3)\,T_2$ due to the right-hand string. Therefore to balance torques we must have $T_1 = 2T_2$. The total force on the system wire + bead must also be zero, so the sum of the upward forces must equal the downward force, $T_1 + T_2 = mg$. Combining results, it follows that

$$T_1 = 2mg/3 \qquad \text{and} \qquad T_2 = mg/3.$$

Sample Problem 4. Rounding a curve while driving in the mountains, we spot an automobile instantaneously at rest with its rear wheels at the edge of the cliff and its front wheels over the edge, as shown. Will the car fall? Why or why not?

Solution: The forces on the car include gravity mg acting at the car's center of mass and a normal force N of the ground upward on the rear wheels, as shown in the isolation diagram.

Using $\mathbf{F} = m\mathbf{a}$ *alone* we can't tell whether the car will fall or not! (If $N = mg$, for example, then there is no net force on the car, so its CM cannot accelerate, meaning the car will not fall.) What about torques? Placing the origin at the rear wheels (for example), the normal force exerts no torque and gravity exerts a clockwise torque on the car. The net torque is therefore clockwise, so the car will start to rotate clockwise. This means, of course, that it will fall. In falling its CM accelerates downward, showing (from $\mathbf{F} = m\mathbf{a}$) that the normal force $N < mg$.

Sample Problem 5. One end of a horizontal uniform stick of mass m and length d is placed on the edge of a tabletop, and a vertical string tied to the ceiling is attached a distance ℓ from the same end of the stick, as shown. (a) Find the tension in the string if the stick is in equilibrium. (b) Find the force exerted by the tabletop.

Solution: (a) We can think of the stick as a (noncircular) wheel, with various forces and torques exerted on it. What point of the stick should we choose as the rotation axis? The answer is that it doesn't matter! *If the stick is in equilibrium about one point (neither accelerating or rotating about it) it doesn't accelerate or rotate (so is in equilibrium) about any other point also.** For example, choose the right-hand end of the stick as the rotation axis. Then the table exerts no torque on the stick, since the normal force N it exerts is located *at* the rotation axis, so $\tau = r \cdot N = 0 \cdot N = 0$. The torque due to the string is $\tau_{\text{string}} = -\ell T$, negative because it is clockwise, where T is the unknown tension. There are also counterclockwise gravitational torques acting on every piece of the stick. As shown in Section 17.2 these are equivalent to a single gravitational force mg acting at the center of mass of the stick, a distance $d/2$ from the end of the stick.

The net torque on the stick must be zero if it is in equilibrium:

$$\tau_{\text{net}} = -\ell T + mg\frac{d}{2} = 0,$$

so the tension is

$$T = mg\frac{d}{2\ell}.$$

(b) Balancing forces, $T + N = mg$, so

$$N = mg - T = mg\left(1 - \frac{d}{2\ell}\right).$$

Notice that if ℓ is increased, N must also increase, which makes sense.

*We prove this in general during *Day Twenty*.

Sample Problem 6. *A self-locking post.*

A post of mass M and length ℓ stands vertically on the ground. A guy wire making an angle ϕ with the vertical is attached from the ground to the top of the post, as shown. A horse pulls horizontally on the post with a force F_O by means of a rope attached to the center of the post. What is the minimum coefficient of friction μ_S between the ground and the post, so the post will not slip no matter how large the force exerted by the horse?

Solution: The isolation diagram of the post is as shown. T is the tension in the guy wire, N is the normal force due to the ground, and f is the force of friction. For the post to remain in equilibrium, the total force must be zero in both the x and y directions:

F_X: $$F_O - T\sin\phi - f = 0$$

F_Y: $$N - Mg - T\cos\phi = 0.$$

The total torque must also be zero about any point. If we choose the origin at the bottom of the post, three of the torques (those due to the weight, the normal force, and friction) are *zero*, because in each case the lever arm $r_\perp = 0$ in $\tau = r_\perp F$, as shown in the diagram. The guy wire exerts a (positive) CCW torque about the chosen origin, given by $\ell T\sin\phi$, and F_O exerts a (negative) CW torque, given by $(\ell/2)F_O$. The net torque is therefore

τ: $$\tau_{net} = \ell T\sin\phi - (\ell/2)F_O = 0.$$

In addition to the F_X, F_Y, and τ equations there is the relationship $f \le \mu_s N$ between the static friction force f and the normal force N on the stick; the coefficient of static friction therefore obeys

$$\mu_s \ge \frac{f}{N}.$$

We can find both f and N in terms of F_O, and so find a lower limit for μ_s in terms of F_O. The τ equation gives $T = \dfrac{F_O}{2\sin\phi}$, so the F_X equation gives

$$f = F_O - T\sin\phi = F_O - \frac{F_O}{2} = \frac{F_O}{2}.$$

Similarly, the F_Y equation gives

$$N = Mg + T\cos\phi$$

$$= Mg + \frac{F_O}{2\sin\phi}\cos\phi = Mg + \frac{F_O}{2}\cot\phi.$$

Therefore

$$\mu_s \geq \frac{f}{N} = \frac{\frac{F_O}{2}}{Mg + \frac{F_O}{2}\cot\phi}$$

if the post remains in equilibrium. It follows that if F_O is very small compared with Mg, we can neglect the second term in the denominator, so $\mu_s \geq \dfrac{F_O}{2Mg}$; in that case, even a very small coefficient μ_s is sufficient to keep the post from slipping. As the horse pulls harder, a larger μ_s is required. In the limit $F_O \to \infty$ (where $\dfrac{F_O}{2}\cot\phi >> Mg$), the first term in the denominator can be neglected, so equilibrium is ensured only if $\mu_s \geq \dfrac{1}{\cot\phi} = \tan\phi$. The minimum coefficient of friction is therefore

$$(\mu_s)_{\min} = \tan\phi$$

to make sure the post won't slip for an arbitrarily large force F_O.

Note that as the force F_O exerted by the horse increases, so does the tension in the guy wire, so the post is pushed more firmly against the ground by the guy wire, increasing the normal force and therefore the friction as well. That is why the post doesn't slip, even if the horse pulls with unlimited force. Note also that the minimum coefficient of friction is larger for large ϕ than for small ϕ. Is that what you would expect using physical intuition?

State of the Search

1. The net external *torque* on a wheel causes it to start rotating. The torque τ exerted on a wheel by a force $\mathbf{F}$ acting a distance $\mathbf{r}$ from the center of the wheel has the magnitude

$$\tau = r F \sin\phi = r F_{\perp} = r_{\perp} F$$

   where ϕ is the angle between the vectors $\mathbf{r}$ and $\mathbf{F}$. Here $F_{\perp}$ is the component of the force $\mathbf{F}$ perpendicular to $\mathbf{r}$, and $r_{\perp}$ is the component of the radius vector $\mathbf{r}$ perpendicular to $\mathbf{F}$. A torque is positive if it tends to make a wheel turn counterclockwise, and negative if it tends to make it turn clockwise.

2. Uniform gravity $\mathbf{g}$ acting on an object exerts a torque as though the entire mass of the object were located at its center of mass. Therefore gravity won't cause the object to rotate if the object is supported beneath its center of mass. This provides a quick way to find the center of mass of an object.

3. An object is in equilibrium if and only if both the net torque and the net force on it are zero. If there is a net torque the object starts to rotate; if there is a net force the center of mass of the objects starts to move.

During *Day Sixteen* we explored the conservation of angular momentum for rotating wheels. During *Day Seventeen* we found that the net external torque on a wheel is what causes it to start rotating. Is there a relationship between angular momentum and torque? How fast does a wheel spin up when a net torque is exerted on it? Our next goal is to pursue these questions.

PICTURE QUIZ 17

1. Each of the following wheels is instantaneously at rest. Which of them remain at rest about their central rotation axis, with applied forces as shown?

(a)

(b)

(c)

(d)

2. A weight of mass 1.0 kg is hung on the left end of a horizontal, uniform stick. It is found that the system remains in equilibrium if the stick is supported at a point 1/3 of the way from the left end. The mass of the stick is

A. 2.0 kg

B. 1.5 kg

C. 1.0 kg

D. 0.75 kg

E. 0.50 kg

3. What must be the mass M of a weight hung on the right side of a pulley of radius R, to balance a weight of mass 10 kg rigidly attached to the pulley by a very light rod of length $2R$ tilted at angle 30° to the horizontal, as shown?

A. 5.0 kg (B) $(10/\sqrt{3})$ kg (C) 10 kg (D) $10\sqrt{3}$ kg (E) 20 kg

PROBLEM QUIZ 17

1. A weight of mass M is hung from one end of a cable of negligible mass. The cable is fixed to a uniform strut of mass m, and the cable is then run horizontally to a wall, as shown. The strut is oriented at angle θ ($0 < \theta < 90°$) to the horizontal. A hinge attaches the left end of the strut to the wall. Find the tension in the cable at point (a) and at point (b), and find the magnitude and direction of the force exerted by the hinge on the strut, assuming the system is in equilibrium.

2. A wheel of mass M and radius R sits in the street against a curb of height h, as shown. A force F_0 is exerted horizontally at the wheel's axle. What is the maximum force $(F_0)_{max}$ such that the wheel will remain in equilibrium in the position shown? What happens if $F_0 > (F_0)_{max}$?

3. A uniform ladder leans against a wall at angle 60° to the horizontal, as shown. If there is no friction between the ladder and wall, what is the minimum coefficient of static friction required between the ladder and floor?

Questions and Problems from *Day Seventeen*

17Q1. Why is less torque required to unscrew the lid of a peanut-butter jar if the lid is first immersed in hot water?

17Q2. Why is a car with a low center of mass less likely to roll than one with a high center of mass?

17Q3. A rock climber is supported by two ropes, each having the same tension. In which of the configurations shown below would the ropes be under less tension? What can be learned by balancing torques on the system (climber + ropes), about the support point of the left-hand rope?

17Q4. How could you construct a circular wheel which would sit on a sloped surface without rolling down it? No glue allowed!

17Q5. Suppose that the sum of the forces and the sum of the torques on an object are both zero. Does this imply that the object is neither translating nor rotating?

17Q6. The dimensions of torque (in terms of mass M, length L, and time T) are the same as which of the following? (a) angular momentum (b) linear momentum (c) energy

17Q7. In changing a tire, the lug nuts clamping tire to wheel must be removed. Is it the force applied or the torque applied that is most important in loosening the nuts?

17Q8. A resident of Maine had trouble driving in the ice and snow, so put a log in the trunk of her car to increase traction. The trunk was in the rear, and the car had front-wheel drive. How might the log affect the cars driveability, for the better or for the worse? Consider the cases where the log is directly above the rear wheel, and where it is behind the rear wheel. (Question posed on "Car Talk" on *National Public Radio.*)

17Q9. Alice (50 kg) and Bob (90 kg) are standing on a 100 kg rigid plank resting on the supports shown in the figure below. If Alice stands on the left end, can Bob walk all the way to the right end without the plank tipping over? If not, how far can he get past the support on the right?

17Q10. A person holds a meter stick straight out with one or more masses dangling from it, as shown below. Rank in order, from most difficult to least difficult, how hard it will be for the person to keep the meter stick from rotating.

Section 17.1 Experiments

17P1. A force of magnitude F_1 is directed tangent to a wheel at its rim, as shown below. What is the minimum additional force F_2 required, acting anywhere on the wheel, to keep the wheel in rotational equilibrium?

17P2. A stick of length ℓ can turn about a vertical axis which is a distance $\ell/3$ from the left end of the stick, as shown below in a view from above. A spring of force-constant k is attached to the left end, oriented perpendicular to the stick, and stretched a distance Δx_0. Another spring, of force-constant $2k$, is attached to the right end of the stick. When oriented at angle ϕ to the stick, it must be stretched a distance Δx so the stick will be in equilibrium. What must be the value of ϕ to minimize this stretch Δx, and what is the value of Δx (in terms of Δx_0) in that case?

17P3. A force $\mathbf{F}$ acts at the rim of a wheel of radius $\mathbf{r}$, at angle $\phi = 60^o$ to the radial line, as shown below. Find (a) $F_\perp$, the component of $\mathbf{F}$ perpendicular to $\mathbf{r}$. (b) $r_\perp$, the component of $\mathbf{r}$ perpendicular to $\mathbf{F}$. (c) Then evaluate $F_\perp r$ and $r_\perp F$, and compare each with $rF\sin\theta$.

17P4. A spring of force-constant k is attached to a wheel at a point a distance $R/2$ from the center of the wheel. The spring is then stretched a distance Δx, with the spring oriented at angle 45° to a radial line, as shown below. Find, relative to the center of the wheel, (a) $F_\perp$, the component of $\mathbf{F}$ perpendicular to $\mathbf{r}$. (b) $r_\perp$, the component of $\mathbf{r}$ perpendicular to $mathbfF$. (c) Then evaluate $F_\perp r$ and $r_\perp F$, and compare each with $rF\sin\phi$.

17P5. A force of magnitude F_1 acts on a wheel of radius R at a point a distance $R/3$ from the center, and directed at 45° to the radial line at that point, as shown below. What is the minimum force F_2 needed at the wheel rim to keep the wheel in equilibrium?

17P6. Two forces act upon a wheel in equilibrium, in addition to the force holding the axle in place. The first has magnitude $F_1 = 2.3$ N and acts tangentially at radius $r_1 = 0.50$ m, as shown below. The second force has magnitude $F_2 = 1.5$ N and acts at radius $r_2 = 1.00$ m. What angle ϕ must F_2 have to the tangential direction?

17P7. A force with magnitude F_1= 1.7 N acts on a wheel at radius $r_1 = 4.6$ cm from the wheels fixed axis of rotation. A second force with magnitude $F_2 = 3.2$ N acts at the axis itself. If the net torque on the wheel is τ= 0.050 N m, find the two possible angles ϕ between the two vectors representing F_1 and r_1.

17P8. Two strings are attached to a wheel. The first is attached to a point a distance $R/2$ from the axis and is oriented at angle 30° to the horizontal, as shown below. The second string is attached to a point on the rim, and oriented at angle ϕ to the vertical, as shown. Find ϕ if the tensions in the two strings are related by $T_2 = T_1/(2\sqrt{3})$ and the wheel is in equilibrium.

17P9. Four forces act on a wheel in equilibrium. F_1, of magnitude 2.0 N, acts at radius $r_1 = 2.0$ m; F_2, of magnitude 3.5 N, acts at radius $r_2 = 1.5$ m; F_3, of magnitude 4.0 N, acts at radius $r_3 = 1.0$ m. These forces all act tangentially, in the directions shown below. A fourth force, of magnitude 15 N, acts at radius $r_4 = 0.5$ m. Find the angle this force makes with the outward radius vector. Does this force exert a CW or a CCW torque?

17P10. Two strings pull on a wheel of radius R, free to rotate about its central axis. String 1 is wound around a subwheel of radius $R/2$, and pulls to the right. One end of string 2 is attached to the wheel at a point D above the axis, and the other end is attached to a point $2R$ to the left of the axis, as shown below. If the wheel is in equilibrium, find the tension in string 2 in terms of R, D, and F.

Section 17.2 Gravitational torques

17P11. A light, rigid rod of length $2R$ extends radially outward from the rim of a pulley of radius R, and a weight of mass 1.0 kg is attached to the far end of the rod. A weight of mass 2.0 kg is also hung from the pulley, as shown below. At what angle θ to the horizontal should the rod be oriented so the system is in equilibrium?

17P12. A uniform rod of length ℓ can turn about an axis a distance $\ell/6$ from one end. A weight of mass m is hung from that end, as shown below. If the system is in equilibrium, what is the mass of the rod?

17P13. A thin stick of length has mass per unit length $\lambda = kx^2$, where k is a constant and x is the distance from the left end of the stick. If you want to balance the horizontal stick on your finger, how far from the left end of the stick should you place your finger?

17P14. Model a tennis racket as a uniform ring of mass M and radius R attached to a uniform handle of mass m and length ℓ. If you can balance the racket in a horizontal position by placing your finger under the point where the handle and ring meet, find m/M in terms of ℓ/R.

17P15. A piece of sheet metal is cut to form an isoceles triangle whose smallest angle is 30° and whose long sides have length ℓ. We hang the triangle in a horizontal orientation by means of a string from the ceiling attached to point P on the triangle. How far is P from vertex A as shown?

17P16. A uniform brick of length ℓ is placed on a table. As much as half the brick (a distance $\ell/2$) can jut out beyond the tables edge without the brick falling off. (a) If two identical bricks are placed on the table, one above the other, what is the maximum distance the upper brick can extend beyond the tables edge without either brick falling off? (b) If three identical bricks are placed on the table, one above the other, what now is the maximum distance the uppermost brick can extend beyond the tables edge? (c) Repeat part (b) for n bricks. (d) Is it possible to stack enough bricks so that the

top brick will be a distance beyond the tables edge?

Section 17.3 Static equilibrium

17P17. A uniform rectangular sign of width w , height h, and mass M is supported by a wire making a 30° angle with the sign, and by a hinge at point A, as shown below. (a) Find the tension T in the wire, (b) Find the vertical and horizontal components of the force exerted on the sign by the hinge.

17P18. A weight of mass M is supported by a cable and strut system, as shown below. The strut, of mass $M/2$, is attached to the wall by a pivot, and sits at an angle of 60° to the horizontal. Find the tension in (a) cable 1 (b) cable 2. Also (c) find the horizontal and vertical components of the force exerted on the strut by the pivot.

17P19. A pair of forces with equal magnitudes but acting in opposite directions is called a *couple.* A couple acts on a stick as shown below, with the two forces F acting at points a distance d apart. (a) Find the total torque on the stick due to the couple, about an axis which is a distance ℓ to the left of the upward force. (b) Now find instead the total torque on the stick about an axis a distance h below the axis in part (a). Can you draw any general conclusions about the torque due to a couple?

17P20. A weight of mass M is supported by two ropes, as shown below. Rope 2 is horizontal, and rope 1 is inclined at angle ϕ to the horizontal. Find the tension in (a) rope 1 (b) rope 2, in terms of M and ϕ.

17P21. A carpenter's square is made of a uniform material, and consists of a 9 inch arm and a 12 inch arm forming a right angle. Suppose for simplicity that each arm is very thin. The square is hung over a nail as shown below, and is in equilibrium when the 12 inch arm makes an angle ϕ with the vertical. Calculate ϕ.

17P22. A nonuniform, horizontal stick of mass m and length ℓ is held in equilibrium by two strings, making angles θ and ϕ with the vertical, as shown below. (a) Find the distance x from the left-hand end of the stick to its center of mass. (b) Find the tension in each cord.

17P23. A uniform ladder of length and mass m leans against a frictionless wall. It is found that the largest distance the base of the ladder can have from the wall without slipping is d. (a) Draw an isolation diagram of the ladder, showing all forces acting upon it. (b) Then find the horizontal and vertical components of the force exerted on the ladder at A and at B, in terms

of gravity g and given parameters. (c) Find the coefficient of static friction between the ladder and floor, in terms of given parameters.

17P24. A uniform ladder of length $\ell = 3.20$ m and mass $M = 15.4$ kg leans against a wall. If the ladder has no friction with the wall, and has coefficient of static friction $\mu_s = 0.52$ with the ground, what is the maximum distance d the base of the ladder can have from the wall so that it will not slip?

17P25. A uniform ladder of length ℓ leans against a wall. If d_0 is the farthest the base of the ladder can be from the wall such that the ladder does not slip, what is the coefficient of static friction μ_s between the ladder and floor and between the ladder and wall, assuming that these friction coefficients are equal to one another?

17P26. A worker of weight $W_w = 150$ lb begins to climb a uniform ladder of weight $W = 25$ lb and length $= 15$ ft. The ladder rests against a frictionless wall (which means that the wall can only exert a normal force on the ladder), with the bottom of the ladder a distance $d = 9$ ft from the wall. If the coefficient of static friction between the floor and ladder is $\mu_s = 0.40$, how far up the ladder can the worker safely climb?

17P27. A uniform chain of mass m is hung between two points at the same height, as shown below; each end makes an angle ϕ with the vertical. (a) Balancing torques about the left end of the chain, find the magnitude of the force exerted on the chain at its right endpoint. (b) Verify the result by balancing the net vertical force on the chain.

17P28. Two uniform boards of equal mass lean against one another, as shown below. Each is tilted at angle ϕ to the vertical. What is the minimum coefficient of static friction between each board and the floor, so the system will stay in equilibrium?

17P29. A horse pulls horizontally with force F_0 on a self-locking post of mass M and length ℓ (as in Sample Problem 6), by means of a rope attached to the post at distance $\alpha\ell$ from the ground, where $\alpha < 1$. (a) Find the tension in the guy wire in terms of α, F_0, and ϕ . (b) What is the minimum coefficient of static friction between the ground and the post, so that the post will not slip no matter how large the force exerted by the horse?

17P30. A uniform strut of mass m is hinged to a wall, and extends at angle $\phi = 30°$ to the horizontal, as shown below. A cord, attached

to the wall and fixed to one end of the strut, supports a weight of mass M. Above the strut the cord makes a 30° angle with the horizontal. Find (a) the horizontal (b) the vertical component of the force exerted by the hinge on the strut if the system is in equilibrium.

17P31, A triangular block of Swiss cheese sits on a cheese board, as shown below. As the board is slowly tilted more and more by increasing the angle θ, will the cheese start to slide before it topples over, or will it topple before starting to slide? The coefficient of static friction between the wood and cheese is 0.90. (From *Physics* by Randall D. Knight)

17P32. A 5.0 kg cat and a 2.0 kg bowl of tuna fish are at opposite ends of a 4.0 m long seesaw, as shown below. How far to the left of the pivot must a 4.0 kg cat stand to keep the seesaw balanced? (From *Physics* by Randall D. Knight)

Day Eighteen

Rotation of a Wheel

"Don't hesitate to be as revolutionary as science." - Calvin Coolidge

What causes a wheel to start turning? Yesterday we learned that the cause is the *net torque* on the wheel. If the torques on the wheel sum to zero (i.e., the clockwise torques exactly balance the counterclockwise torques), a wheel initially at rest remains at rest. Now we need to figure out what happens when the net torque is *not* zero. How fast does the wheel spin up? What is the change in angular momentum $L = I\omega$ of the wheel?

18.1 Torque and angular momentum

When a force $\mathbf{F}$ acts on a wheel at a point some distance r from the rotation axis, it produces a torque

$$\tau = rF_{\perp} \qquad (1)$$

where $F_{\perp}$ is the component of $\mathbf{F}$ perpendicular to the radius vector $\mathbf{r}$ from the axis to the point where $\mathbf{F}$ is applied.* What does this torque τ do to the wheel?

Start with a very simple "wheel", a single particle of mass m held at constant distance r from the rotation axis by a spoke of negligible mass. A force on the particle produces a torque

$$\tau = rF_{\perp} = rma_{\perp} \qquad (2)$$

where we have used Newton's second law $\mathbf{F} = m\mathbf{a}$ for the perpendicular (i.e., tangential) components of $\mathbf{F}$ and $\mathbf{a}$.**

* The parallel (i.e., radial) component of the force exerts no torque.

**The spoke exerts only a *radial* force on the particle, so it doesn't influence $F_{\perp} = ma_{\perp}$.

An acceleration $a_\perp$ tangent to the circular path in the direction of motion means that the particle moves faster and faster around the rotation axis. The tangential speed is $v_\perp = r\,\omega$ where ω is the angular velocity of the particle, so

$$a_\perp = \frac{dv_\perp}{dt} = r\frac{d\omega}{dt}, \tag{3}$$

showing that the angular velocity also increases as time goes on. The torque on the particle can then be written

$$\tau = rF_\perp = r\,m\,a_\perp = mr^2\,\frac{d\omega}{dt}. \tag{4}$$

How does this torque affect the particle's angular momentum

$$L = I\omega = mr^2\,\omega\,? \tag{5}$$

As ω increases, the angular momentum increases at the rate

$$\frac{dL}{dt} = mr^2\,\frac{d\omega}{dt} \tag{6}$$

since r is constant. Comparing Eqs. 4 and 6, it is clear that for a single-particle wheel, the torque and angular momentum are related by

$$\tau = \frac{dL}{dt}. \tag{7}$$

The torque on a single particle moving in a circle causes the angular momentum of the particle to change according to $\tau = \dfrac{dL}{dt}$.

Sample Problem 1. A ball of mass m is whirled around at speed v_0 in a horizontal circular orbit on the end of a string of length r. Starting at $t = 0$ a constant frictional drag force F_{drag} acts upon the ball, opposite to its direction of motion. Find the ball's angular momentum as a function of time.

Solution: The ball's initial angular momentum is $L = I\omega = mr^2\omega = mr(r\omega) = mrv_0$. The torque on the ball is $\tau = -rF_{drag}$, minus because the torque acts to decrease the angular momentum. Integrating Eq. 7 over time gives $\int_{L_0}^{L} dL = L - L_0 = \int_0^t \tau\, dt = -r F_{drag}\, t$, since r and F_{drag} are constant. Solving for L,

$$L = L_0 - r F_{drag}\, t = mrv_0 - r F_{drag}\, t.$$

Three comments:

(1) The equation $\tau = \dfrac{dL}{dt}$ is the rotational analog of Newton's second law $F = \dfrac{dp}{dt}$ for one-dimensional motion, relating the change in momentum of a particle to the net force exerted upon it. Torque is the rotational analog of force, and angular momentum is the rotational analog of momentum.

(2) If the torque on the particle is kept constant, the particle's angular momentum increases at a constant rate.

(3) For a *system* of particles in one-dimensional motion, Newton's second law is still given by $F = \dfrac{dp}{dt}$, where now F is the net external force acting on the system and p is the total momentum of the system. (The internal forces between different particles within the system all cancel out because of Newton's third law: The force of particle 1 on particle 2 is equal but opposite to the force of particle 2 on particle 1.) Can Eq. 7 similarly be extended to a system of particles such as the particles in a solid wheel?

We can think of an arbitrary wheel as consisting of a huge number of particles. Each particle feels many forces. Some of the forces (called *internal* forces) on the ith particle are due to neighboring particles. After all, nearby particles are attached to the ith particle, and will generally exert forces on it; otherwise the wheel would not turn rigidly. Other forces (called *external* forces) may include gravity,

string tension, etc. Given an axis of rotation, these internal and external forces generally cause torques on the ith particle, both internal torques and external torques.

The *net* torque τ_i on the ith particle is

$$\tau_i = \tau_i\,(\text{ext}) + \tau_i\,(\text{int})\,, \tag{8}$$

the sum of all the external and internal torques. We already know from Eq. 7 that the torque on the ith particle changes its angular momentum according to

$$\tau_i = \frac{dL_i}{dt}\,, \tag{9}$$

so summing Eq. 9 over all the particles,

$$\begin{aligned}\tau_{\text{net}} &= \textstyle\sum_i \tau_i \equiv \tau_1 + \tau_2 + \ldots \tau_N = \sum_i \tau_i(\text{ext}) + \sum_i \tau_i(\text{int}) \\ &= \frac{d}{dt}\left(L_1 + L_1 + \ldots L_N\right) = \frac{dL}{dt}\,.\end{aligned} \tag{10}$$

Here $L = L_1 + L_1 + \ldots L_N$ is the total angular momentum of the wheel, the sum of the angular momenta of all the particles in the wheel. Therefore for the entire wheel $\tau_{\text{net}} = \frac{dL}{dt}$, just as it was for a single-particle wheel. However, the left-hand side of Eq. 10 is complicated, since we have to sum over all the internal as well as external torques acting on all the particles within the wheel. Fortunately, the sum of all the internal torques is *zero**, as shown in the Appendix to this chapter. Therefore a general rigid wheel obeys

$$\boxed{\tau = \frac{dL}{dt}} \tag{11}$$

where the torque is the net *external* torque on the wheel. The external torques are often simple and few in number; they might include torques due to string tensions or friction or even gravity if the rotation axis does not coincide with the center of mass of the wheel.

* The result is a consequence of Newton's third law.

Sample Problem 2. A rotating disk-like wheel of mass M and radius R is used to sharpen knives. A motor turns the wheel with angular velocity ω_o, and a knife edge is pressed against the outer surface of the wheel which is covered with a fine, gritty material. Suppose there is a coefficient of kinetic friction μ_k between a knife and the surface, and that the knife is pressed against the surface with force N. If the motor is disconnected and the wheel rotates freely about its axis, how long does it take for the wheel to come to rest?

Solution: The force N, which we will assume acts directly toward the axle, exerts no torque on the wheel, while the frictional force exerts torque $-RF_f = -R\mu_k N$ (minus because the torque acts to slow down the wheel.). Then $\int_{L_o}^{L} dL = L - L_o = \int_0^t \tau\, dt = -R\mu_k N\, t$, using $\tau = \frac{dL}{dt}$, so $L = L_o - R\mu_k N\, t = \frac{1}{2}MR^2\omega_o - R\mu_k N\, t$. The wheel stops when $L = 0$, at time

$$t = \frac{\frac{1}{2}MR^2\omega_o}{R\mu_k N} = \frac{MR\omega_o}{2\mu_k N}.$$

18.2 Torque and angular acceleration

A rigid wheel is a wheel in which every particle has the same angular velocity about the rotation axis. In that case the wheel's angular momentum is

$$L = I\omega, \qquad (12)$$

so the time rate of change of angular momentum is

$$\frac{dL}{dt} = I\frac{d\omega}{dt} = I\alpha, \qquad (13)$$

where $\alpha \equiv \frac{d\omega}{dt}$ is the angular acceleration of the wheel, the rate of change of the angular velocity. Therefore for a rigid wheel (for which I is constant), Eq. 11 takes the useful form

$$\boxed{\tau = I\alpha} \tag{14}$$

which is the rotational analog of $F = ma$. Torque is the rotational analog of force, the moment of inertia is the rotational analog of mass, and angular acceleration is the rotational analog of linear acceleration.

If no net torque acts on a wheel, then $\alpha = 0$, so the angular velocity $\omega = \dfrac{d\theta}{dt}$ = constant. As a shorthand, we will often use the *dot notation* $\omega = \dfrac{d\theta}{dt} \equiv \dot{\theta}$ to represent the time derivative.* Then the *angular acceleration* of the wheel is

$$\alpha \equiv \ddot{\theta} \equiv \frac{d^2\theta}{dt^2}, \tag{15}$$

with two dots over the θ to represent the second time derivative. Then Eq. 14 can also be written

$$\tau = I\ddot{\theta}. \tag{16}$$

A constant net torque causes a constant angular acceleration of the wheel.

Writing Eq. 16 in the form

$$\ddot{\theta} = \frac{\tau_{\text{net}}}{I} \tag{17}$$

emphasizes the fact that the angular acceleration $\ddot{\theta}$ is caused by the net torque and that it is inversely proportional to the moment of inertia, which is the "sluggishness" of the wheel, its reluctance to change its rotation rate. This is the rotational analog of Newton's second law

$$\ddot{x} = \frac{F_{\text{net}}}{m}$$

for the motion of a particle, showing that the acceleration $\ddot{x}$ is caused by the net force and that it is inversely proportional to the mass, which is the "sluggishness" of the particle, its reluctance to change its velocity.

The dot notation for time derivatives was invented by Isaac Newton. He of course was one of the inventors of calculus itself.

Sample Problem 3. A cord is wound around a disk-like pulley of mass M and radius R, and a weight of mass m is hung on the cord, as shown. Find the angular acceleration of the pulley, in terms of M, m, R, and g.

Solution: Suppose the tension in the cord is T, so the external torque on the pulley about its rotation axis is $\tau = RT$. An isolation diagram of the pulley is shown at the right. (Note that the support force F_O exerts no torque, because it acts at the rotation axis.) We can then find the angular acceleration using $\tau = I\ddot{\theta} = \frac{1}{2}MR^2\ddot{\theta}$. Solving for $\ddot{\theta}$, the pulley's angular acceleration is

$$\ddot{\theta} = \frac{RT}{\frac{1}{2}MR^2} = \frac{2T}{MR}.$$

Now we need to find the tension T. We draw an isolation diagram for the weight, as shown. The net force downward on the weight is $F_{\text{net}} = mg - T$, so from $F_{\text{net}} = ma$ we have $ma = F_{\text{net}} = mg - T$. The tension in the cord is therefore $T = m(g - a)$, *less* than mg.* The cord doesn't slip on the wheel, so if the weight drops a certain distance, the wheel rim must turn by this same distance. The same goes for the acceleration of the weight and the tangential acceleration of the wheel rim. Therefore $a = R\ddot{\theta}$, so

$$\ddot{\theta} = \frac{2T}{MR} = \frac{2}{MR}m(g-a) = \frac{2m}{MR}(g - R\ddot{\theta}) = \frac{2mg}{MR} - \frac{2m}{M}\ddot{\theta}.$$

Solving for $\ddot{\theta}$, we find that the angular acceleration of the wheel is

$$\ddot{\theta} = \frac{2m}{(2m+M)}\frac{g}{R}.$$

Note that the angular acceleration $\ddot{\theta}$ is *constant*, and that the acceleration $a = R\ddot{\theta}$ of the falling weight is also *constant*. If the weight's mass m is zero, there is no acceleration at all; if the disk's mass M is negligible, the acceleration of the weight is $a = R\ddot{\theta} = g$, as expected. According to the equations, what would happen if all the masses were doubled? Does the answer make sense physically?

*People often think that the cord tension is *always* mg if there is a weight mg hanging on it. If that were true here, the net force on the weight would be *zero*, so it would not start falling.

18.3 A special case: constant torque

Sometimes the torque τ acting on a wheel is independent of time. Then the angular acceleration $\ddot{\theta} = \frac{\tau}{I}$ of the wheel will also be constant, so by straightforward integration we can find the wheel's angular velocity $\dot{\theta}$ and orientation angle θ.

We can first find the angular velocity $\dot{\theta} \equiv \omega$. Calling the angular acceleration α and integrating over time,

$$\int_{\omega_0}^{\omega} d\omega = \int_0^t \alpha \, dt \tag{18}$$

where ω_0 is the initial angular velocity. Therefore

$$\omega(t) = \omega_0 + \int_0^t \alpha \, dt . \tag{19}$$

If the torque is constant, the angular acceleration $\alpha_0 = \frac{\tau}{I}$ is also constant, so can be brought outside the integral. Then

$$\omega = \omega_0 + \alpha_0 t . \tag{20}$$

The change $\omega - \omega_0$ in the angular velocity of the wheel is equal to the angular acceleration multiplied by the time.

Then we can integrate $\omega = d\theta/dt$ with respect to time,

$$\theta = \theta_0 + \int_0^t \omega(t) \, dt . \tag{21}$$

Using Eq. 20 for $\omega(t)$, the result is

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha_0 t^2 . \tag{22}$$

The net rotation $\theta - \theta_0$ of the wheel depends upon time and also upon the wheel's initial angular velocity ω_0 and the constant angular acceleration α_0.

Two additional formulas are useful in constant angular-acceleration motion. Eliminating α_0 between Eqs. 20 and 22, we get

$$\theta = \theta_0 + \frac{\omega + \omega_0}{2} t, \tag{23}$$

relating the turn angle $\theta - \theta_0$ to the initial and final angular velocity and time; the turn angle is equal to the average angular velocity times the time. If instead we eliminate the time t between Eqs. 20 and 22, we get an equation without time,

$$\omega^2 = \omega_0^{\ 2} + 2\alpha_0(\theta - \theta_0), \tag{24}$$

relating the angular velocity to the angular acceleration and the turn angle.

Sample Problem 4. A helicopter rotor consists of four blades, each of length $\ell = 6.5$ m and mass $m = 270$ kg. At time $t = 0$, when the blades are at rest, the helicopter engine begins to exert a constant torque $\tau = 16{,}700$ N · m on the rotor. At a later time $t = 0.50$ min, find (a) the angular velocity of the rotor; (b) the speed of the blade tips; (c) the number of revolutions of the wheel. Model the blades as thin rods of uniform mass density, and pretend there is no air resistance.

Solution: (a) The moment of inertia of each blade is $I_{\text{blade}} = \frac{1}{3}m\ell^2$, so the rotor has moment of inertia $I_{\text{rotor}} = \frac{4}{3}m\ell^2$, neglecting the moment of inertia of the central hub, which is much smaller because its radius is small. The angular acceleration of the rotor is therefore

$$\ddot{\theta} = \frac{\tau}{I} = \frac{\tau}{\frac{4}{3}m\ell^2} = \frac{16{,}700 \text{ N m}}{\frac{4}{3}(270 \text{ kg})(6.5 \text{ m})^2} = 1.1 \text{ s}^{-2}$$

which is constant, so we can use Eq. 21 to get $\omega = \omega_0 + \alpha_0 t = 0 + 1.1 \text{ s}^{-2}(30 \text{ s}) = 33 \text{ s}^{-1}$.

(b) The speed of the blade tips is $v = r\omega = (6.5 \text{ m}) \cdot 33 \text{ s}^{-1} = 210$ m/s.

(c) The angle through which the wheel turns in 30 seconds is

$$\theta - \theta_0 = (1/2)\alpha_0 t^2 = \frac{1}{2}\left(1.1 \text{ s}^{-2}\right)(30 \text{ s})^2 = 495 \text{ radians},$$

so the number of revolutions of the wheel is $\frac{\theta - \theta_0}{2\pi} = \frac{495}{6.28} = 79$, to two significant figures.

Summarizing the preceding formulas, all valid *only for constant angular acceleration* α_0,

$$\omega = \omega_0 + \alpha_0 t \tag{25a}$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha_0 t^2 \tag{25b}$$

$$\theta = \theta_0 + \frac{\omega + \omega_0}{2} t \tag{25c}$$

$$\omega^2 = \omega_0^{\ 2} + 2\alpha_0(\theta - \theta_0). \tag{25d}$$

These formulas are analogous to those for the one-dimensional motion of a particle undergoing constant linear acceleration a_0. Here ω is analogous to v, θ is analogous to x, and α_0 is analogous to a_0.

$$v = v_0 + a_0 t \tag{26a}$$

$$x = x_0 + v_0 t + \frac{1}{2} a_0 t^2 \tag{26b}$$

$$x = x_0 + \frac{v + v_0}{2} t \tag{26c}$$

$$v^2 = v_0^{\ 2} + 2a_0(x - x_0). \tag{26d}$$

Sample Problem 5. A light string is wound around the largest subwheel of the aluminum wheel of Section 17.1, at a radius R from the axis of rotation. A weight of mass m is attached to the other end of the string, as shown, and released from rest. It then falls a distance d in time t_f.
(a) What is the angular velocity of the wheel at time $t = t_f$?
(b) What is the rotational inertia of the wheel?

Solution: (a) Assuming the string doesn't slip on the wheel, the wheel rim turns a distance $\theta = \theta_0 + \frac{\omega + \omega_0}{2} t$ with $\theta_0 = 0$ and $\omega_0 = 0$, giving at time $t = t_f$,

$$\omega_f = \frac{2\theta_f}{t_f} = \frac{2d}{R\,t_f}.$$

We can check this result from the fall of the weight m. The weight's acceleration is constant, so the distance it has fallen is

$$d = d_0 + \frac{v + v_0}{2} t.$$

The weight starts at rest at $d_0 = 0$, so its velocity at time t_f is $v = \frac{2d}{t_f}$. The string does not slip on the wheel, so $v = R\omega$; therefore $\omega = \frac{2d}{R\,t_f}$ as we found before.

(b) We can't use the formula $I = f MR^2$ for the moment of inertia of the pulley because we know neither its mass M nor its shape-factor f. However, we can find I from $\tau = I\alpha$ relating torque and angular acceleration. The torque on the wheel is $\tau = RT$, the product of the wheel radius and string tension. The tension is $T = m(g - a)$, as shown in *Sample Problem* 3. Therefore the torque is $\tau = Rm(g - a) = Rm(g - \alpha R)$ and the wheel's (constant) angular acceleration is

$$\alpha = \alpha_o = \frac{\omega}{t} = \frac{2\,d}{R\,t_f^2},$$

from Eq. 25(a). The rotational inertia of the wheel is

$$I = \frac{\tau}{\alpha_o} = \frac{Rm(g - \alpha_o R)}{\alpha_o} = \frac{Rm\,[g - 2d/t_f^2]}{2d/Rt_f^2} = mR^2\left[\frac{gt_f^2}{2d} - 1\right].$$

Suppose for example the weight drops a distance $d = 2.8$ m in the time $t_f = 1.0$ s. The rotational inertia of the wheel is then found by experiment to be

$$I = mR^2\left[\frac{9.8\ \text{m/s}^2\ (1.0\ \text{s}^2)}{2(2.8\ \text{m})} - 1\right] = [1.75 - 1]mR^2 = 0.75\,mR^2$$

in terms of its radius R and the mass m of the hanging weight.

Sample Problem 6. *Atwood's machine.* A rope is hung over a pulley of radius R and moment of inertia I; the rope does not slip on the pulley. Weights with masses m_1 and m_2 (where $m_1 > m_2$) are attached to the ends of the rope, as shown.. Released from rest, when m_1 has fallen a distance h, how fast is it moving?

Solution: Isolation diagrams of the two weights and of the pulley are as shown.

The tensions T_1 and T_2 on the left and right sides of the rope are different; if they were the same, there would be no net torque τ on the pulley, so it would not start to turn.* The upward force F_A on the pulley is caused by the supporting axle.

Choose the positive rotation direction of the pulley to be counterclockwise, because that is the direction in which it starts to turn. For consistency, the positive direction of motion of the weight m_1 is downwards, so $F = ma$ becomes

$$(1) \quad m_1 g - T_1 = m_1 a$$

for m_1. The positive direction of motion of m_2 is upwards, so

$$(2) \quad T_2 - m_2 g = m_2 a.$$

The acceleration of each weight has the same magnitude, since we assume the rope doesn't stretch. That is because if m_1 falls a distance $s(t)$, m_2 must rise by this same distance $s(t)$, so by differentiation the speed must be the same for each weight at all times. A second derivative shows that the magnitude of the acceleration must also be the same at all times.

*Mass m_1 accelerates downward, so the tension T_1 on the left-hand side must be *less* than $m_1 g$. Mass m_2 accelerates upward, so the tension T_2 on the right-hand side must be *greater* than $m_2 g$. You can feel these effects by suspending a weight on a string, holding the upper end of the string. When you start to raise the string the weight feels heavier (the string tension has increased); when you start to lower the string it feels lighter (the tension has decreased). (Note that in Motion I we assumed that the tension in a rope over a pulley is the same on both sides of the pulley. That assumption is only valid for *massless* pulleys, however, because only pulleys with negligible mass can be made to rotate with a negligible net torque.)

The rope doesn't slip on the pulley, so the angle θ through which the pulley turns must be related to s by $s = R\theta$, as shown. Differentiating, the speed of any point on the rope is $v = R\,d\theta/dt = R\omega$, where ω is the angular velocity of the wheel. Differentiating again, the acceleration of any point on the rope is $a = R\alpha$, where α is the angular acceleration of the wheel. The equation $a = R\alpha$ (i.e., $\ddot{y} = R\ddot{\theta}$, where y is the vertical altitude of mass m_2) is sometimes called an *equation of constraint*, because it is a mathematical consequence of the fact that the rope is constrained not to slip relative to the pulley.

Applying $\tau = I\alpha$ to the wheel, and noting that the two torques oppose one another, we have $(T_1 - T_2)R = I\alpha = Ia/R$, so

$$(3) \quad (T_1 - T_2) = Ia/R^2.$$

We can solve for all three unknowns T_1, T_2, and a using equations (1), (2), and (3). The tensions T_1 and T_2 cancel when the equations are added together, so $(m_1 - m_2)g = (m_1 + m_2)a + Ia/R^2$, giving

$$a = \frac{(m_1 - m_2)g}{\left(m_1 + m_2 + I/R^2\right)}.$$

The acceleration is constant (because g, I, m_1, m_2, and R are all constants), so the speed of m_1, after falling a distance h, is

$$v = \sqrt{2ah} = \sqrt{\frac{2(m_1 - m_2)gh}{m_2 + m_1 + I/r^2}},$$

using the constant-acceleration formula $v^2 = v_0^2 + 2ah$.

Having found the acceleration of the weights, we can also find the string tensions and the angular acceleration of the pulley; see problem 18P28.

This device, consisting of the pulley, cord, and two hanging weights, is called an *Atwood machine*. It can be used to slow down the gravitational acceleration of weights. That is, the acceleration a can be made arbitrarily small by choosing the masses m_1 and m_2 to be nearly equal to one another, so the weights rise and fall slowly enough that the properties of constant-accelerated motion can be easily observed. [*Note*: We can also apply $F = ma$ to the pulley. Since the center of the pulley is not accelerating, this tells us that $F_A = T_1 + T_2$, which can be used to find F_A but is useless in finding the motion of the weights.]

18.4 Time-dependent torques

A child pushes on a playground merry-go-round to get it spinning. Given the force exerted by the child, how does the merry-go-round respond?

Suppose that the merry-go-round has radius R and the child pushes with constant force F tangent to the rim. Then the torque $\tau = R\,F$ is also constant, so we can use the formulas generated in the preceding section to find the angular velocity and turn-angle of the merry-go-round as functions of time.

If instead the child pushes with a force that depends upon time, then the torque $\tau(t) = R\,F(t)$ is also time-dependent so we have to start from scratch. The angular acceleration of the merry-go-round is

$$\ddot{\theta}(t) = \frac{\tau(t)}{I},$$

where I is the merry-go-round's moment of inertia. Knowing $\ddot{\theta}(t) \equiv \dfrac{d\omega(t)}{dt}$, we can integrate to find the angular velocity $\omega(t)$: That is,

$$\int_0^t d\omega(t) = \int_0^t \ddot{\theta}(t)\,dt,$$

so

$$\omega(t) = \omega_0 + \int_0^t \ddot{\theta}(t)\,dt. \tag{27}$$

For given $\tau(t)$ and I, we can in principle do the integrations. Having found $\omega(t)$, we can integrate once more to obtain $\theta(t)$,

$$\theta = \theta_0 + \int_0^t \omega(t)\,dt = \theta_0 + \omega_0 t + \int_0^t \left(\int_0^{t'} \frac{\tau(t'')}{I}\,dt'' \right) dt'. \tag{28}$$

In summary, given the applied torque $\tau(t)$ and the moment of inertia I of the merry-go-round, we can in principle find the rotation rate $\omega(t)$ and the orientation angle $\theta(t)$.

Sample Problem 7. A child standing on the ground exerts an increasing force $F = At$ tangentially at the rim of a disk-like merry-go-round of mass M and radius R that is initially at rest, where A is a constant and t is the time. (a) Find the angular velocity of the merry-go-round as a function of time. (b) How long does it take the merry-go-round to turn once around?

Solution: (a) The angular *acceleration* of the merry-go-round is

$$\alpha(t) = \frac{\tau(t)}{I} = \frac{R\,F(t)}{\frac{1}{2}MR^2} = \frac{2F(t)}{MR} = \frac{2A}{MR}\,t\,,$$

so its angular velocity is

$$\omega(t) = \omega_0 + \int_0^t \alpha\,dt = 0 + \frac{2A}{MR}\int_0^t t\,dt = \frac{A}{MR}t^2,$$

increasing quadratically with time.

(b) Setting $\theta(0) = \theta_0 = 0$, the orientation angle of the wheel as a function of time is

$$\theta(t) = \int_0^t \omega(t)dt = \frac{A}{MR}\int_0^t t^2 dt = \frac{A\,t^3}{3MR}\,.$$

The wheel turns once around when $\theta = 2\pi$, which requires the time $t = \left(\frac{6\pi MR}{A}\right)^{1/3}$.

The next cycle (assuming the child can run fast enough to keep up with the rotation) won't take this long, because at the beginning of the second cycle the merry-go-round will have an initial angular velocity, and also the angular acceleration keeps increasing all the time.

Appendix: Is the net *external* torque on an object equal to the sum of the torques on all the particles in the object?

Is it true that $\tau_{\text{ext}} = \sum \tau_i$? Is the torque we exert on the wheel by pulling on it with cords, for example, equal to the sum of the torques on all the individual particles in the wheel? The torque τ_i on the ith particle is the sum of a large number of "internal" torques due to other particles in the wheel (especially its nearest neighbors), and also any external torques due to causes external to the wheel, like attached cords. The internal torques due to other particles in the wheel have got to be there; these torques ensure that the wheel turns together as a unit.

Altogether, the torque on the ith particle is the sum of these internal and external torques, $\tau_i = \tau_{i\,\text{int}} + \tau_{i\,\text{ext}}$. Summing over all particles,

$$\sum \tau_i = \sum \tau_{i\,\text{int}} + \sum \tau_{i\,\text{ext}} .$$

The second term on the right is the total torque due to external causes, which is the torque exerted by the cords in our experiments. So to agree with our experiments, where we only took account of external torques, we have to show that the remaining part, the sum over internal torques $\sum \tau_{i\,\text{int}}$, is *zero*.

Consider two nearby particles a and b within the wheel. The force-counterforce law (Newton's third law) states that if a exerts a force $\boldsymbol{F}_0$ on b, then b exerts an equal but opposite force $-\boldsymbol{F}_0$ on a, as illustrated below. If r is the common distance of a and b from the rotation axis, then notice that the torque exerted by a on b is also equal but opposite to the torque b exerts on a about the rotation axis, because the components of $\boldsymbol{F}_0$ which are perpendicular to $\boldsymbol{r}$ (those are the components used in calculating the torque) are also equal but opposite. In the illustration, the torque on a due to b is counterclockwise and the torque on b due to a is clockwise. So these two internal torques cancel out. If we extend the argument to internal forces between any other pair of neighboring particles, we see that the total sum $\sum \tau_{i\,\text{int}} = 0$, as we needed to show.

State of the Search

1. The net torque on a wheel is responsible for changes in its angular momentum $L = I\omega$ according to

$$\tau = \frac{dL}{dt}.$$

2. The angular momentum of the wheel is conserved if and only if there is no net torque on it. The internal torques between particles within the wheel all cancel out, leaving the external torque alone as the cause of any change in angular momentum. Note that if the wheel is in rotational equilibrium its angular momentum remains zero, so the net torque on must also be zero.

3. For a wheel of fixed moment of inertia, the derivative $\frac{dL}{dt} = \frac{d(I\omega)}{dt} = I\frac{d\omega}{dt}$, so

$$\tau = I\alpha$$

where $\alpha = \frac{d\omega}{dt}$ is the angular acceleration of the wheel. The dot notation for time derivatives is often useful, so that $\omega \equiv \dot{\theta}$ and $\alpha \equiv \ddot{\theta}$. Therefore we can also write

$$\tau = I\ddot{\theta}.$$

4. The angle θ through which a wheel of radius R turns is related to the distance s through which its rim turns by $s = R\theta$. The tangential speed v_t and acceleration a_t of the rim are similarly related to the wheel's angular speed $\omega \equiv \dot{\theta}$ and angular acceleration $\alpha \equiv \ddot{\theta}$ by the relations $v_t = R\omega$ and $a_t = R\alpha$.

5. If the torque is known as a function of time, the angular acceleration $\ddot{\theta}(t) = \tau(t)/I$ is known as well. The angular velocity $\omega(t) = \omega_o + \int_0^t \ddot{\theta}(t)\,dt$ can then be found by integration, and the orientation angle can be found by integrating a second time, $\theta = \theta_0 + \int_0^t \omega(t)\,dt$.

6. There is a striking similarity between the rotational quantities we have used and analogous translational quantities. Rotational variables include the orientation angle θ, the angular velocity ω, the angular acceleration α, the moment of inertia I, the angular momentum L, and the torque τ. Analogous translational variables are the position x, the velocity v, the acceleration a, the mass m, the momentum p, and the force F. A table of analogous variables is as follows.

translational variable	*rotational variable*
x	θ
v	ω
a	α
p	L
m	I
F	τ

Relationships among these variables are also analogous.

translational relationships	*rotational relationships*
$v = \dfrac{dx}{dt}$	$\omega = \dfrac{d\theta}{dt}$
$a = \dfrac{dv}{dt}$	$\alpha = \dfrac{d\omega}{dt}$
$p = mv$	$L = I\omega$
$F = ma$	$\tau = I\alpha$
$F = \dfrac{dp}{dt}$	$\tau = \dfrac{dL}{dt}$

These analogs can be helpful in remembering relationships among rotational quantities.

PICTURE QUIZ 18

1. A weight of mass M is hung on a pulley, also of mass M, by a cord wrapped around the pulley. When the weight is released from rest it accelerates downward and the pulley turns faster and faster. If the quantity M were doubled keeping all else the same, the acceleration of the falling weight would

A. double

B. increase by less than a factor of two

C. remain the same

D. decrease by less than a factor of two

E. be cut in half

2. A weight of mass M is hung on a pulley of mass m, by a cord wrapped around the pulley. When the weight is released from rest it accelerates downward and the pulley turns faster and faster. If the quantity M were doubled keeping all else (including m) the same, the acceleration of the falling weight would

A. double

B. increase by less than a factor of two

C. remain the same

D. decrease by less than a factor of two

E. be cut in half

3. Strings are wound around two bicycle wheels at rest, one with twice the mass and twice the radius of the other. If the strings are pulled for five seconds, each string with the same tension, which wheel has the greater angular momentum?

(A) The smaller wheel

(B) The larger wheel

(C) Their angular momenta will be the same.

PROBLEM QUIZ 18

1. A string is wound around a bicycle wheel of mass M and radius R, and a weight of mass m is attached to the other end of the string, as shown. Essentially all the wheel's mass is located at R, and the wheel is supported on a frictionless axle. If the weight is released when the system is at rest, find (a) the acceleration a of the weight (b) the angular acceleration $\ddot{\theta}$ of the wheel (c) the momentum p of the weight at time t (d) the angular momentum L of the wheel at time t.

2. A neutron star is rotating with angular velocity ω_0 about an axis through its center; its moment of inertia about this axis is I. Beginning at time $t = 0$, a constant external torque τ_0 acts upon it, in a direction tending to slow it down. Assume that I remains constant. (a) How long does it take the neutron star to come instantaneously to rest? (b) Through how many revolutions has it turned by the time it comes to rest?

3. An alien race spins up its flying saucer (of radius R and moment of inertia I about its center) with the help of two rocket engines of small mass, attached to opposite sides of the saucer, as shown. If the saucer is initially nonspinning and each rocket supplies force F_0 to the saucer rim, (a) what will be the angular velocity ω_0 of the saucer by time t_0? (b) Now the rocket engines suddenly shut off. What happens to the saucer?

4. A wheel of moment of inertia I has two subwheels, as shown. The outer subwheel has radius R and inner subwheel has radius $R/2$. A lightweight cord is wound around the inner subwheel, and a weight of mass M is attached to the free end, and then released. Find the tension in the cord at time t ($t > 0$).

5. Beginning at time $t = 0$, the engine of a truck exerts the time-dependent torque $\tau = A\sin\beta t$ on a flywheel of moment of inertia I, where A and β are constants. Find the wheel's (a) angular acceleration $\ddot{\theta}(t)$ (b) angular velocity $\dot{\theta}(t)$ (c) rotation angle $\theta(t)$, assuming the wheel starts at $\theta = 0$ with angular velocity $\dot{\theta} = \omega_0$.

Special Topic: Trebuchets

In the year 1304, King Edward I of England wanted to expand his realm by subduing the Scots. This required the subjugation of Stirling Castle, a formidable defensive structure perched on the edge of a precipice in southern Scotland. Ordinary archers or knights on horseback had no hope of overcoming the defenders, who were protected by five-foot thick castle walls and who could launch arrows through narrow slits. So the king's army laid seige to the castle, bringing to bear a huge *seige engine* they named *Warwolf.*

Warwolf was very likely a type of *trebuchet.* Trebuchets could hurl massive stones more than 200 meters, so those operating the engine would be relatively safe from defender's arrows. Trebuchets were built as follows: A long pole made from a tree trunk was supported by a structure that allowed the pole to pivot on an axle placed perhaps one-third of the way from the pole's thicker end. The axle passed through a hole drilled through the pole, and the ends of the axle were supported by the wooden support structures on either side, as illustrated.

In this configuration the larger gravitational torque on the longer end of the pole would cause that end to sink to the ground while the shorter end rose into the air, as shown above.

However, the builders attached a heavy counterweight to the short end of the pole. The counterweight could be lead weights strapped to the end, or it could be a kind of swinging bucket, holding tons of sand or rocks. With sufficient added counterweight the gravitational torque on the short end could now exceed that on the long end, causing the pole to twist in the opposite direction, with the long end in the air and the short end at the ground, as shown below.

Now ropes attached to the long end of the pole could be used (with the help of a block and tackle) to pull the long end of the pole down to the ground. That is, the added torque caused by the tension in the rope could exceed the net gravitational torque tending to make the short end sink. Should the ropes be released, the pole would turn rapidly clockwise with the short end descending and the long end rising.

Now one end of a strong rope was attached to the long end of the pole. The rope was laid out along the ground underneath the pole, and a rope basket was attached to the far end. A payload (possibly a spherical stone to punch a hole; possibly a dead animal to spread disease within the castle) was placed in the basket.

Finally, the ropes holding down the long end of the pole were suddenly released, so the pole could freely swing about the axle. As the long end began to rise, the rope with basket started to drag the stone along the ground. As the long end continued to rise the payload was pulled off the ground, and then, in a whiplike action, the payload was made to rise very quickly in a circular arc, and then leave the basket to enter a projectile-motion trajectory. If by trial and error the engine was aligned properly and if the moment of the payload's released was properly adjusted, a heavy stone could be given sufficient momentum to undertake a long free-fall trajectory ending with a strike to the castle wall. Such a heavy stone could punch a hole clear through a five-foot thick castle wall.

Given plenty of time, attackers could reduce a castle wall to rubble. However, the Stirling Castle defenders, seeing the "handwriting" on the wall, surrendered soon after the stones started flying, realizing that resistance was futile.

As seige engines, trebuchets reigned supreme in Europe for two hundred years. They were made obsolete with the advent of gunpowder and cannons in the late fifteenth century.

The Medieval Centre in Denmark is an experimental institution presenting a late 14th Century environment to the public. Its exhibits include siege engines (in action), medieval ships and a functioning shipyard. There are also blacksmiths and other craftsmen to be seen - plus performances of knightly tournaments.

Questions and Problems from *Day Eighteen*

18Q1. When pushing a door open, will the door swing open more easily if you push on the door near the hinges or near the doorknob? Why?

18Q2. A weight is attached to a long light rod in one of two positions, as shown below. If the weight is attached near the top of the rod, the CM of the system is high; if the weight is attached near the bottom, the CM of the system is low. In which configuration is it easiest to balance the system by supporting it from below on one finger? Why?

18Q3. Two rods of identical length and mass have their masses distributed differently. The first rod has most of its mass in the middle, while the second rod has most of its mass at the ends. If each is grabbed in the middle, which is easier to twist back and forth?

18Q4. How might you measure the angular acceleration of the pulley in an Atwoods machine?

18Q5. A yo-yo falls as the cord unwinds and rises as the cord rolls back up. Does the yo-yo change its rotation direction at the bottom? Taking the yo-yos center of mass as origin, what force produces the torque that makes it spin faster as it falls and slower as it rises? What is happening when a yo-yo "sleeps" at the bottom? (A sleeping yo-yo spins rapidly without winding or unwinding the cord.)

18Q6. Two unicyclists start from rest side-by-side. The riders, unicycles, and wheels have the same mass. If one unicycle wheel has twice the radius of the other and each rider exerts the same torque, which unicycle will go farther in the same length of time?

18Q7. In slalom ski races frequent turns are made, while in downhill races the skier tries to move nearly straight down the slope. How should slalom and downhill skis differ?

18Q8. The solid cylinder and cylindrical shell shown below have the same mass, same radius, and turn on frictionless, horizontal axes. (The cylindrical shell has light-weight spokes connecting the shell to the axle.) A rope is wrapped around each cylinder and tied to a block. The blocks have the same mass and are released simultaneously. Which hits the ground first? Or is it a tie? Explain. (From *Physics* by Randall D. Knight.)

18Q9. Rank in order, from largest to smallest, the angular accelerations α_a to α_d in the situations below. (From *Physics* by Randall D. Knight.)

Section 18.1 Torque and angular momentum

18P1. The angular momentum of a bowling ball about an axis passing through its center varies with time according to $L = b + ct - dt^2$, where b, c, and d are constants. (a) In terms of the fundamental dimensions M, L, and T, what are the dimensions of b, of c, and of d? Then as functions of time, find (b) the net torque on the ball and (c) the ball's angular speed ω in terms of b, c, d and the balls mass m and radius r.

18P2. A bicycle wheel, at rest at time $t = 0$, has mass M, radius R, and shape-factor $f = 0.9$. The torque on the wheel about its axle varies with time according to $\tau = b + ct + dt^2$, where b, c, and d are constants. As functions of time, find the wheels (a) angular momentum (b) angular velocity (c) total number of revolutions.

18P3. Two disk-like wheels have the same mass density and the same width. Disk 1 has twice the radius of disk 2, and they are connected by a belt as shown. (a) If disk 1 turns with angular velocity ω_1, what is the angular velocity ω_2 of disk 2? (b) What is the ratio of their angular momenta, L_1/L_2? (c) What ratio of constant net torques τ_1/τ_2 was required to spin them up?

18P4. A force acts tangentially at the rim of a disk-like wheel of mass $M = 2.0$ kg and radius $R = 0.30$ m. As a function of time, the orientation angle θ of the wheel is $\theta = 1.0 + 2.0t - 1.0t^2$ in radians, for t measured in seconds. (a) Sketch a graph of $\theta(t)$ for the time interval $0 \leq t \leq 3.0$ s. Find (b) the wheels angular momentum $L(t)$ (c) the tangential force $F(t)$.

18P5. A ringlike wheel, with essentially all of its mass M located at its outer radius R, is turning so that the orientation angle θ of the wheel (measured from the horizontal) is given by $\theta(t) = \beta t^3 - \gamma \sin(\delta t)$ as a function of time, where β, γ, and δ are constants. As functions of time, find (a) the angular momentum of the wheel (b) the net torque acting on the wheel.

18P6. A yo-yo is constructed from two disks of radius R and a central shaft of radius $R/2$. The yo-yo is set upon a table with which there is a frictional force. A string around the shaft emerges at angle θ to the vertical, and is gently pulled, as shown below. If $\theta =$ (a) 45° (b) 30° (c) 15°, does the yo-yo start rolling to the left, to the right, or does it not roll at all along the table? *Hint*: Make an interesting choice of origin, to minimize the number of torques that must be considered.

18P7. A light cord is wound around a disk-like pulley of mass M and radius R, so the cord wont slip. The other end of the cord is attached to a weight of mass m, as shown below. If the weight is dropped when the system is at rest, find (a) the momentum p of the weight, and (b) the angular momentum L of the wheel, both as functions of time.

18P8. A long thin rod of mass M and length D is rigidly attached to a ring of mass M and radius $D/4$, as shown below. The system is free to rotate about an axle perpendicular to the page, passing through the center of the rod and ring. A light cord is wound around the ring, and a weight, also of mass M, is attached to the free end of the cord. The weight is dropped from rest, spinning up the wheel. As functions of time, find (a) the momentum p of the weight (b) the angular momentum L of the ring-plus-rod system about the axle.

Section 18.2 Torque and angular acceleration

18P9. The orientation angle of a disk-like wheel of mass M and radius R is given by $\theta(t) =$

$\beta t^2 + \gamma e^{\delta t}$ as a function of time, where β, γ, and δ are constants. As functions of time, find (a) the angular momentum of the wheel (b) the net torque τ acting on the wheel, assuming the rotation axis passes through the center of the disk.

18P10. An automobile wheel has mass M, radius R, and shape-factor 4/5. If the torque acting on the wheel as a function of time is given by $\tau = \tau_0 + \beta t$, where τ_0 and β are constants, find the angular velocity ω of the wheel as a function of time if $\omega = 0$ at $t = 0$.

18P11. A pulley with moment of inertia I consists of two coaxial wheels, one of radius R and the other of radius $R/2$. Strings are wound around each wheel; a weight of mass m_1 is hung from the inner string and a weight of mass m_2 is hung from the outer string, as shown below. When the system is allowed to rotate freely, find (a) the angular acceleration of the pulley (b) the tension in the outer string (c) the tension in the inner string.

18P12. A cord is hung over two disk-like pulleys, each of mass M_0 and radius R, as shown below; each pulley is supported at its center by a frictionless axle. The cord at the left is attached to weight A of mass M_A and the cord at the right is attached to weight B of mass M_B ($M_B > M_A$); the weights are then allowed to move. Find (a) the acceleration of weight A; (b) the acceleration of weight B; and the tensions (c) in the cord at the left (d) in the cord at the right; (e) in the portion of the cord between the two pulleys.

18P13. Three identical pulleys, each with radius R and moment of inertia I, are supported on frictionless axles in the locations shown below. A light cord is threaded around the pulleys, and a weight of mass M_A is hung on the left end of the cord and a weight of mass M_B (with $M_B > M_A$) is hung on the right end. When the system is released, find the acceleration a of each of the weights, in terms of given quantities and the gravitational field g.

18P14. The axle of a disk-like pulley P_1 of mass M and radius R is attached to the ceiling. A light cord is also attached to the ceiling, at a point to the left of the pulley. The cord passes beneath a second, identical pulley P_2, and is then draped over P_1. A weight of mass M is hung from the right-hand end of the string, and a weight of mass $2M$ is hung from the axle of P_2, as shown below. When the system is released, find the acceleration of weight M in terms of any or all of R, M, and the gravitational field g. *Hint*: Begin by showing that if weight M rises a distance h, weight $2M$ falls a distance $h/2$, and that P_2 turns through angle $\theta = h/2R$.

18P15. Two weights A and B are supported by cables wrapped around the compound cable drum shown below. The drum has moment of inertia I and the weights have masses M and m, respectively. The radii of the outer and inner

portions of the drum are R_A and R_B, respectively. When the weights are allowed to move, find (a) the angular acceleration of the cable drum. (b) the tensions T_A and T_B in the two cables.

18P16. Two weights of equal mass M are attached to the two ends of a light cord. The cord is draped over two disk-like pulleys, each of mass M and radius R, whose fixed axles are a distance $4R$ apart and at the same elevation. The center of the cord is pulled down between the pulleys, so a disk-like wheel, also of mass M and radius R, can be placed over the cord so that the cord supports it, as shown below. The wheel is held momentarily in place, and then released at time $t = 0$. (a) In terms of any or all of M, R, and g, what is the magnitude and direction of the acceleration of the left-hand weight? (b) What is the angular velocity of the right-hand pulley as a function of time?

Section 18.3 A special case: constant torque

18P17. A wheel with essentially all its mass M at the same radius R (i.e., a ring-wheel) is pulled tangentially at R by a constant force F, starting at time $t = 0$ when the wheel is at rest. At some later time t, (a) what is the wheels angular velocity? (b) Through what total angle has the wheel turned?

18P18. The motor in a phonograph exerts a constant torque on a disk-like turntable to spin it up to 33 revolutions per minute (rpm). If a phonograph record of the same radius and 1/4 the mass is placed on the turntable, (a) how much longer will it take to spin up the turntable to 33 rpm, expressed as the ratio

$$t_{\text{turntable+record}}/t_{\text{turntable alone}}?$$

(b) How many more turntable revolutions will it require to reach 33 rpm, expressed as the ratio

$$\frac{\text{No. of Rev}_{\text{turntable+record}}}{\text{No. of Rev}_{\text{turntable alone}}}?$$

18P19. A pulley consists of a uniform disk of radius R and an unknown mass. A constant force F is applied tangentially at the rim, starting with the pulley at rest, and it is found that it takes a time T for the pulley to turn through 12 complete revolutions. (a) Using dimensional reasoning, find how the mass M of the pulley must depend upon the given parameters. (b) Then determine the dimensionless constant in the formula found in part (a).

18P20. A wheel is a ring of mass M and unknown radius. A constant force F is applied tangentially at the rim, starting when the wheel is at rest, and it is found that after five complete revolutions the angular velocity of the wheel is equal to ω_f. (a) Using dimensional reasoning, find how the radius of the wheel must depend upon the given parameters. (b) Then determine the dimensionless constant that appears in the formula found in part (a).

18P21. A pulley consists of a uniform disk of mass M and radius R. A string is wound around the rim so it wont slip, and a weight, also of mass M, is hung on the string as shown below. The weight is released from rest at time $t = 0$. At some later time t, in terms of M, R, and g, (a) what is the acceleration of the weight? (b) what is the torque on the wheel about its center? (c) how fast is the weight moving? (d) through how many revolutions has the pulley

turned?

18P22. An aluminum wheel of moment of inertia $I = 0.55$ kg m^2 consists of two subwheels; the inner one has radius $r_1 = 16$ cm, and the outer one has radius $r_2 = 32$ cm. A light string is wound around the outer subwheel, and a weight of mass $m_2 = 100$ g is attached to it; another string is wound around the inner subwheel in the opposite direction, and a weight of mass $m_2 = 50$ g is attached to it, as shown below. If the weights are released from rest, at the end of time t= 5 s, (a) what is the net torque on the wheel? (b) through how many radians has the wheel turned?

18P23. A helicopter rotor consists of three blades, each of length $\ell = 7.0$ m and mass $m = 125$ kg. A length of time t= 45 s is required for the blade tips to reach the speed $v = 220$ m/s, starting at rest, when a constant torque is applied to the rotor. (a) What torque τ does the engine apply to the rotor? (pretend there is no friction or air resistance) (b) What is the maximum forward velocity (in km/hour) the helicopter can have such that no rotor tip will exceed the speed of sound in air (330 m/s)?

18P24. We want to measure the moment of inertia I of a bicycle wheel of radius R about its supported axle, by attaching one end of a light cord to a point on the rim, winding the cord around the rim, tying a weight of mass M to the free end of the cord, and measuring the time t_f it takes the wheel to turn once around, starting from rest. Derive an equation for I (in terms of the measured parameters M, R, g, and t_f) into which the experimental data can be inserted to calculate I.

18P25. Two identical uniform rectangular doors have different rotation axes. Door 1 rotates about a vertical axis through its center, while door 2 is hinged at one side. Someone pushes with constant force F_0 on the end of each door as shown below, where F_0 remains perpendicular to the door. If it takes time t_0 for door 1 to swing through 90°, starting at rest, how much time does it take for door 2 to swing through the same angle?

18P26. A weight of mass M is attached to one end of a light cord, and the cord is draped over a pulley of radius R and moment of inertia I. The other end of the cord is wound around a wheel of radius R_0 and moment of inertia I_0, which is attached to the rotor of a motor, as shown below. If τ_0 is the maximum torque which the motor can exert to lift the weight, through how many revolutions has the pulley turned in time

t if the motor is turned on at $t = 0$?

18P27. A yo-yo consists of two solid plastic disks, each of mass m, radius R, and width w, joined by a solid plastic shaft of radius $R/10$ and width $w/2$. (a) Show that the moment of inertia of the shaft is less than 1% that of the disks, taken about the symmetry axis of the yo-yo. (b) A light string of length $20R$ is wound around the shaft, all at essentially the same radius from the symmetry axis. The yo-yo is dropped from rest, with one end of the string held fixed. If the string does not slip around the shaft, find the speed of the yo-yos CM just before the string is all unwound, in terms of any or all of m, R, w, and g, valid to within 99% accuracy.

18P28. Find the tensions T_1 and T_2 and the angular acceleration α of the pulley in the Atwoods machine described in Sample Problem 6, in terms of m_1, m_2, g, I, and R.

18P29. A cylindrical space station is fashioned in the form of a huge empty tin can. Its radius is R, its length is $L = 2R$, and its total mass is M. The mass/area is the same for the end caps as for the cylindrical surface. (a) Find the moment of inertia of the station about its symmetry axis, in terms of M and R. (b) Two external rockets are attached to opposite sides of the cylindrical rim, pointed in opposing directions, as shown below. Each rocket exerts a constant force F_0 on the station tangential to the rim, to spin up the station about its symmetry axis. In terms of M, R, g_{eff}, and F_0 , how long will it take the station to achieve an artificial gravity equal to g_{eff} for people living on the inside rim of the cylinder? Neglect the mass of rockets and fuel, and assume the station is initially nonrotating.

18P30. The symmetry axes of two nonrotating cylindrical space colonies are parallel to one another, held a constant distance D apart by struts between them, as shown. Each colony has radius R and moment of inertia I about its symmetry axis. To spin up the colonies, a cable is wound around the cylindrical surface of one colony, carried to the other colony and wound around it as well, as shown below. A motor attached to the cable halfway between the colonies begins to draw the two cable halves together, establishing a constant tension T in the cable. If the colonists want to establish an effective gravity g due to rotation for people living on the inside rim of each colony, how long must the motor maintain the constant tension, in terms of given parameters?

18P31. A uniform hockey puck, a disk of mass M and radius R sitting flat on the ice, is set spinning with angular speed ω_0 about its center. The coefficient of kinetic friction between the puck and ice is μ_k. (a) Find the torque on the puck due to the ice, about the symmetry axis of the puck. (b) How long will it take the disk to stop spinning? Neglect air resistance. *Hint*: Consider the puck to be composed of a

large number of concentric rings.

Section 18.4 *Time-dependent torques*

18P32. A force $F = At^2$ is applied tangentially to the rim of a ring-like wheel of radius R and unknown mass. It takes time T for the ring to turn through one complete revolution, starting from rest. What is the mass of the ring, in terms of A, R, and T? First find how the mass depends on these parameters using dimensional reasoning, and then solve the entire problem to find the dimensionless constant.

18P33. In Sample Problem 7, a child pushes tangentially with force $F = At$ on the rim of a disk-like merry-go-round of mass M and radius R, where A is a constant and the merry-go-round starts at rest. By the time the merry-go-round has turned around once, how fast is the child running? Express the answer in terms of A, M, and R.

18P34. The angle through which a wheel with moment of inertia I turns during time t is $\theta = -At^2 + 3Bt^3$, where A and B are constants. As functions of time, find (a) the wheels angular velocity; (b) its angular acceleration; and (c) the net torque exerted on the wheel.

18P35. A time-dependent torque $\tau = At$ is applied to a disk-like wheel of mass M and radius R, starting at time $t = 0$ (A is a constant; the wheel starts at rest.) By the time it has turned through two complete revolutions, what is the wheels angular velocity?

18P36. Children perform two experiments with a playground merry-go-round, by holding onto the outer rim and running around on the ground beside it. (i) They exert a constant tangential force F_0 on the merry-go-round, which started at rest. They find that it takes time T for the merry-go-round to turn around once. (ii) They stop the merry-go-round, and then exert the time-dependent force $F = At - Bt^2$ tangentially, starting at time $t = 0$. In terms of F_0, T, A, and B, find in experiment (ii) (a) the angular velocity ω as a function of time; (b) the number of revolutions of the merry-go-round by the time it has again come instantaneously to rest.

Special Topic: Trebuchets

18P37. A trebuchet is made with a uniform swinging pole of mass M and length D. The pole is suspended one-third of the way from one end, as described in the special topic. (a) What is the minimum counterweight that must be bolted to the short end of the pole so that the pole can be in equilibrium without touching the ground? (b) If the counterweight is 50% heavier than the value found in part (a), what is the angular acceleration of the pole-counterweight system when the pole is horizontal? (Suppose no payload has been attached to the pole.)

18P38. Trebuchets can be put on wheels, as illustrated on page 18-24. Describe how the wheels are likely to move (forward or backward) during each stage of the payload launch. Here forward is the ultimate launch direction of the payload. (The addition of wheels produces a longer payload throw.)

Day Nineteen

Rotational Energy

"They say hard work never hurt anybody, but I figure why take the chance?" - Ronald Reagan

The rotation of a wheel has similarities with the motion of a particle. One is caused by a net torque, the other by a net force. The wheel responds with an angular acceleration α; the particle responds with a linear acceleration a. The moment of inertia I characterizes the inertia, or "sluggishness" of a wheel, while the mass m characterizes the inertia of a particle. The equation $\tau = I\alpha$ for a rotating wheel is therefore a close analog to Newton's law $F = ma$ for a particle. The angular momentum $L = I\omega$ of a spinning wheel is also a rotational analog to the linear momentum $p = mv$ of a moving particle. Angular momentum is conserved if there is no net torque; linear momentum is conserved if there is no net force.

What about *energy*? What is the rotational analog of the kinetic energy $K = \frac{1}{2}mv^2$ of a moving particle? We expect that a rotating wheel also has kinetic energy; we can see the wheel spinning, and that energy of motion should be convertible into other kinds of energy, like raising a weight or heating water.

19.1 Kinetic Energy of a Rotating Wheel

How can we find the kinetic energy of a rotating wheel?

We can simply sum over the kinetic energies of all the individual particles that make up the wheel. All the particles move around the rotation axis in circular paths, so (except for the point-particle in the middle) all have kinetic energy. Figure 1 shows an object rotating about a central axis. A particle of mass m_i at distance r_i from the axis moves with speed $v_i = r_i\omega$, where ω is the wheel's angular velocity. The total kinetic energy is therefore

Fig. 1. Motion of particles in a wheel. The two particles shown, with masses m_a and m_b, move with velocities v_a and v_b, perpendicular to their radius vectors from the rotation axis.

$$K = \sum_i \tfrac{1}{2} m_i v_i^2 = \sum_i \tfrac{1}{2} m_i (r_i \omega)^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right) \omega^2 = \frac{1}{2} I \omega^2, \qquad (1)$$

because the moment of inertia I is defined to be $I = \sum_i m_i r_i^2$.

This is a nice result, because it extends our analogy between translational and rotational motion. The rotational inertia I is the analog of the inertial mass m, and the angular velocity ω is the analog of the translational velocity v, so the kinetic energy $\frac{1}{2} I \omega^2$ of a rotating wheel is the analog of the kinetic energy $\frac{1}{2} m v^2$ of a moving particle. From the derivation, however, it is clear that $\frac{1}{2} I \omega^2$ really *is* the kinetic energy of the rotating wheel, and not just an *analog* of kinetic energy.

Sample Problem 1. *The rotational energy of Earth.*

Earth has mass $M = 5.98 \cdot 10^{24}$ kg and an average radius of $R = 6{,}370$ km. Estimate its rotational energy.

Solution: If its density were uniform throughout, and it were a sphere with no equatorial bulge, Earth's moment of inertia would be

$$I = \tfrac{2}{5} M R^2 = (0.4)\,(5.98 \cdot 10^{24}\text{kg})\,(6.37 \cdot 10^6\ \text{m})^2 = 9.71 \cdot 10^{37}\ \text{kg m}^2.$$

Its angular velocity of rotation is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{24\ \text{hrs}}\left(\frac{1\ \text{hr}}{3600\ \text{s}}\right) = 7.27 \cdot 10^{-5}\ \text{rad/s},$$

so in the constant-density, spherical model,

$$K = \frac{1}{2} I \omega^2 = (0.5)\,(9.71 \cdot 10^{37}\ \text{kg m}^2)\,(7.27 \cdot 10^{-5}\ \text{rad/s})^2 = 2.57 \cdot 10^{29}\ \text{J}.$$

We expect that this is an *overestimate* of K, since Earth has higher density in the middle than it has near the surface, so the moment of inertia should be less than the value calculated above. In fact, Earth's moment of inertia is known to be only about $0.33 MR^2$, instead of $0.40\, MR^2$. Even so, its rotational kinetic energy is enormous; it is comparable to the energy humans would use over billions of years at present rates of consumption.

Sample Problem 2. A rope is hung over a pulley of radius R and moment of inertia I. The rope does not slip on the pulley. Weights of mass m_1 and m_2 are attached to the ends of the rope, as shown. Released from rest, m_1 falls and m_2 rises. When m_1 has fallen a distance h, how fast is it moving?

Solution: The sums of the kinetic and potential energies of the system before and after are the same. There is no kinetic energy before, and if we define the zero of potential energy of each weight to be at its starting point, there is also no potential energy beforehand.*

When m_1 has fallen a distance h, its potential energy becomes $U_1 = -m_1gh$; then m_2 has risen this same distance h, so its potential energy is now $U_2 = +m_2gh$. The *total* potential energy after m_1 has fallen a distance h is therefore

$$U_T = -(m_1 - m_2)gh.$$

The two weights are now both moving; their kinetic energies are

$$K = \tfrac{1}{2}m_1v^2 + \tfrac{1}{2}m_2v^2 = \frac{1}{2}(m_1 + m_2)v^2,$$

and the rotational kinetic energy of the pulley is $\frac{1}{2}I\omega^2$. The total kinetic energy of the system is therefore

$$K_T = \frac{1}{2}(m_1 + m_2)v^2 + \tfrac{1}{2}I\omega^2 = \frac{1}{2}(m_1 + m_2)v^2 + \tfrac{1}{2}I(v/R)^2,$$

using the fact that $\omega = v/R$. Both the initial potential and kinetic energies are zero, so $E = 0 = \frac{1}{2}(m_1 + m_2 + I/R^2)v^2 - (m_1 - m_2)gh$, which we can solve for v to give

$$v = \sqrt{\frac{2(m_1 - m_2)gh}{m_2 + m_1 + I/r^2}}.$$

(*Note:* This problem was solved as *Sample Problem 6* during *Day Eighteen*, using $F = ma$ and $\tau = I\alpha$. Energy conservation is an easier approach in this case.)

*As always for uniform gravity, the zero for potential energy is arbitrary, and can be chosen for convenience.

Sample Problem 3: *Spindown of the Crab Pulsar*

The Crab Nebula is the remnant of a supernova explosion recorded by Chinese and Japanese observers in 1054 A.D. We now know that a massive star blew up, leaving behind an expanding, diffuse gas with glowing, reddish arms appearing vaguely crablike, suggesting its name to modern astronomers. The nebula radiates electromagnetic energy (over a broad range of wavelengths including visible light and radio waves) at the rate $dE/dt \approx 10^{31}$ Watts, over four orders of magnitude greater than the power radiated by our sun. What is the source of this energy? What powers the Crab Nebula?

In 1968 astronomers discovered that a rather inconspicuous star in the nebula is a *pulsar*, a rapidly rotating neutron star formed from the core of the original exploding star. The pulsar blinks on and off with period $T = .033$ s, the period of rotation of the star. That is, the neutron star rotates thirty times each second! The blinking is too fast for the human eye to detect. It was also discovered that this rotation period is very gradually lengthening, such that $dT/dt = 4.2 \cdot 10^{-13}$ s/s, meaning that in one second its rotation period increases by $4.2 \cdot 10^{-13}$ seconds. This slowing of the rotation rate suggests that the decreasing rotational kinetic energy of the pulsar could be the source of energy powering the nebula surrounding the pulsar. Said another way, the observed radiation might in some way be responsible for the decreasing kinetic energy of the rotating neutron star.

Question: Assuming a typical neutron star mass of $M = 2M_{\text{sun}} = 4 \cdot 10^{30}$ kg and radius $R = 10$ km, is it possible that the decreasing rotational kinetic energy could be the source of energy powering the Crab Nebula's radiation?

Solution: We will model the star as spherically symmetric, and of uniform density. The star's moment of inertia is then $I = \frac{2}{5}MR^2$, so its rotational kinetic energy is

$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{2}{5}MR^2\right)\left(\frac{2\pi}{T}\right)^2 = \frac{4\pi^2 MR^2}{5\,T^2}.$$

The time rate of change of this kinetic energy is the power

$$\frac{dK}{dt} = -\frac{8\pi^2}{5}\frac{MR^2}{T^3}\frac{dT}{dt} = -7.5 \cdot 10^{31} \text{ Watts}$$

The model neutron star therefore loses kinetic energy even *faster* than is needed to explain the radiated power. This means we are free to decrease the radius of the neutron star, or to use a more realistic model in which the mass density is larger at the center of the star than at the outer layers, which would make $I < \frac{2}{5}MR^2$. We also expect that fast particles will take away some of the star's energy, competing with the energy loss in radiation. (Note that we have not specified any particular *mechanism* for converting rotational energy into radiation; all we have shown is there is enough energy to do it. The actual mechanism probably involves the interaction between the neutron star's magnetic field and the ionized gas surrounding the star.)

19.2 A work-energy theorem for pure rotation

During *Day Eleven* we derived the work-energy theorem

$$W_T = K - K_O = \frac{1}{2}mv^2 - \frac{1}{2}mv_O^2, \tag{2}$$

showing that the total work done on a particle is responsible for the change in its kinetic energy. If the particle moves in one dimension, for example, the work done on it by a constant force is $W = F\,d$, the product of the force and the displacement of the particle. If F is not constant the work is $W = \int F\,dx$, integrated along the path of the particle. If a particle speeds up, the net work done on it by all forces must have been positive; if the particle slows down, the net work done must have been negative. For example, if we drop a ball of mass m the work done on it by gravity is $Fd = mgd$ as it falls a distance d, positive because the force and displacement are in the same direction. It then follows from the work-energy theorem (neglecting any work due to air resistance) that the kinetic energy of the ball is $\frac{1}{2}mv^2 = mgd$.

Is there a similar work-energy theorem for the rotational energy of a wheel? Is there a way to find a rotational work responsible for changes in the wheel's rotational kinetic energy $\frac{1}{2}I\omega^2$?

The rotational analog of force is torque and the analog of the one-dimensional change in position dx is the change in the angle of rotation $d\theta$. So for rotation it makes sense to inspect the integral $\int \tau\,d\theta$ instead of $\int F\,dx$.* If τ_T is the *total* torque acting on the wheel, then we can use $\tau_T = I\alpha$ to write

$$\int \tau_T\,d\theta = \int_{t_O}^{t} I\alpha\,d\theta = \int_{t_O}^{t} I\frac{d\omega}{dt}\,\omega\,dt = \int_{\omega_O}^{\omega} I\,\omega\,d\omega$$

$$= \frac{1}{2}I\,\omega^2 - \frac{1}{2}I\,\omega_O^2, \tag{3}$$

*Not only are these integrals analogous, but they have the same dimensions, $[W] = [E] = ML^2/T^2$.

using the identity $\alpha = \dfrac{d\omega}{dt}$ and the relationships $d\theta = \omega\, dt$ and $\dfrac{d\omega}{dt}\, dt = d\omega$, both valid for infinitesimal time intervals. We have assumed that the angular velocity ω is the same for all particles, which is appropriate for a rigid wheel, where also the moment of inertia I is constant.

Equation 3 is a *work-energy theorem for rotation.* The integral

$$(W_T)_{rot} \equiv \int \tau_T \, d\theta \tag{4}$$

is the total rotational work done by all torques acting on the wheel, as the wheel rotates through some angle θ. The theorem shows that $(W_T)_{rot}$ is responsible for any change in a wheel's rotational kinetic energy.

Sample Problem 4: *An airplane propeller*

A single-engine airplane propeller has moment of inertia I about its rotation axis. If we want it to achieve n revolutions per minute after N revolutions of the propeller, starting from rest, what (assumed constant) torque would have to be generated by the engine? Neglect friction and air resistance.

Solution: The torque is constant, so the rotational work done by the engine is $W = \tau\,\theta$ as the propeller turns through angle θ. At the end of this period the propeller has kinetic energy $\frac{1}{2} I\omega^2 = \tau\,\theta$, if we neglect any work done by friction or air resistance; the torque is therefore

$$\tau = \frac{I\omega^2}{2\theta}.$$

The angle $\theta = 2\pi N$ and ω (in radians/second) is $\omega = n \dfrac{2\pi \text{ rad}}{\text{rev}} \dfrac{1 \text{ min}}{60 \text{ s}} = \dfrac{\pi n}{30}$, so in SI units kg m^2/s^2, the torque is

$$\tau = \frac{\pi}{3600} \frac{I n^2}{N}$$

if I is given in units kg m^2.

19.3 Combined Rotational and Translational Kinetic Energy

So far we have dealt only with the kinetic energy of wheels that rotate about a *fixed* axis, like a stationary pulley rotating about its axle. There are, however, many interesting situations involving translation as well as rotation, such as the motion of the wheels of a moving bicycle, a ball rolling along a pool table, or a yo-yo falling on its string. What is the kinetic energy in situations like these?

A. First Step: Translational and Internal Kinetic Energies

Consider an arbitrary system of particles, such as pool balls moving on a pool table. Suppose we know the total kinetic energy of the system in its center-of-mass frame, the frame of reference in which the center of mass of the system is at rest. (Recall that the location of the center of mass in the x direction is a sum over the masses and their distances from an origin,

$$X_{CM} = \frac{\sum_i m_i x_i}{\sum_i m_i}, \tag{4}$$

with similar formulas for the y and z directions.) How can we find the total kinetic energy of the system in a frame in which the CM is *moving*?

Example: The front wheel of a bicycle has a certain kinetic energy in its CM frame, the frame of the bicycle rider. This kinetic energy is $\frac{1}{2}I\omega^2$, where I is the moment of inertia of the wheel. What is the kinetic energy of the wheel from the point of view of a person on the sidewalk watching the bicycle go by?

Example: Suppose we know the total kinetic energy of stars in a distant galaxy, in the frame of reference in which the center of the galaxy is at rest. What is the total kinetic energy in our frame of reference, in which the galaxy is speeding away?

Let the system consist of N particles with masses m_i (i = 1 to N), as shown in Fig. 2. Choose some origin of coordinates in an inertial frame, our own rest frame, and let $\mathbf{R}_{CM}$ be the vector from the origin to the center of mass of the system of particles, as shown.

Fig. 2. A system of particles, where R_{CM} is the vector from the origin O to the CM of the system.

Also let $\mathbf{r}_i$ be the vector from the origin to the ith particle, and $\mathbf{r}_i'$ be the vector from the CM to the ith particle, as shown, so that by vector addition

$$\mathbf{r}_i = \mathbf{R}_{CM} + \mathbf{r}_i' . \tag{5}$$

Differentiating with respect to time, the velocity $\mathbf{v}_i$ of the ith particle in our rest frame is

$$\mathbf{v}_i = \mathbf{V}_{CM} + \mathbf{v}_i' , \tag{6}$$

the velocity of the CM plus the velocity of the particle in the CM frame.

Now we can calculate the total kinetic energy of the system of particles in our rest frame. This is the sum of the kinetic energies of all of the particles; i.e.,

$$\text{K.E.} = \sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i \left|\mathbf{V}_{CM} + \mathbf{v}_i'\right|^2 . \tag{7}$$

The quantity $\left|\mathbf{V}_{CM} + \mathbf{v}_i'\right|^2$ is the same as the dot product of the vector $\mathbf{V}_{CM} + \mathbf{v}_i'$ with itself. (The vector is parallel with itself, so the square of the vector is equal to the square of its magnitude.) Therefore the kinetic energy is

$$\text{K.E.} = \sum \frac{1}{2} m_i \left[\mathbf{V}_{CM} + \mathbf{v}_i'\right] \cdot \left[\mathbf{V}_{CM} + \mathbf{v}_i'\right] = \sum \frac{1}{2} m_i \left[V_{CM}^2 + v_i'^2 + 2\mathbf{V}_{CM} \cdot \mathbf{v}_i'\right] , \tag{8}$$

dotting all the vectors with one another.

There are three terms on the right-hand side of Eq. 8. The first is

$$\text{K.E.}_{\text{CM}} = \frac{1}{2}\left(\sum m_i\right)V_{\text{CM}}^2 = \frac{1}{2}MV_{\text{CM}}^2, \tag{9}$$

where M is the total mass. This term is the kinetic energy the particles would have if each of them moved at the same speed, the speed of the center of mass. The second term is

$$\text{K.E.}_{\text{int}} = \sum \frac{1}{2} m_i v_i'^2, \tag{10}$$

the "internal" kinetic energy of the particles, their kinetic energy in the center-of-mass frame. The third term is $\mathbf{V}_{\text{CM}} \cdot \sum m_i \mathbf{v}_i'$, which is *zero*, because $\sum m_i \mathbf{v}_i'$ is the total momentum of the particles in the CM frame, zero by definition.

Altogether, the total kinetic energy of the system of particles is

$$\text{K.E.} = \frac{1}{2}MV_{\text{CM}}^2 + \sum \frac{1}{2} m_i v_i'^2, \tag{11}$$

the kinetic energy the system would have if all particles were moving at V_{CM}, plus the internal kinetic energy of the particles in the CM frame. This is a *surprisingly simple* result! The kinetic energy of a bicycle wheel from the point of view of somebody standing in the street is just the sum of the kinetic energy as seen by the rider, and the kinetic energy the wheel would have if it were all moving with the velocity of the center of mass. Note that the separation of the kinetic energy into two parts is just a useful bookkeeping device. The kinetic energy in Eq. 11 is still the sum of the kinetic energies $\frac{1}{2}mv^2$ of all the particles in our frame of reference.

The internal kinetic energy (the second term in Eq. 11) may be organized or disorganized, or some of each. If the kinetic energy is completely disorganized, as for example in the random motions of the molecules in a gas, the kinetic energy is often called *heat energy*, and might be measured by a thermometer. The kinetic energy of organized motion might come from an organized oscillation, for example, where the entire collection oscillates in and out together as though it were breathing.

Sample Problem 5. There are N helium atoms in a helium-filled balloon, all at temperature T. The total mass of the helium is M. What is the total kinetic energy of the atoms relative to an observer on the ground, if the balloon is rising with velocity v?

Solution: In the CM frame of the balloon, the average energy of helium atoms is $(3/2)kT$, where k is called *Boltzmann's constant.* So in the balloon's frame, the total kinetic energy of the atoms is $(3/2)NkT$. According to Eq. 11, the kinetic energy of the helium in the observer's frame is simply $(3/2)NkT + (1/2)\,Mv^2$.

B. Internal Energy as Rotational Energy

An important special case of organized kinetic energy is *rigid-body rotation*, where all the particles move with the same angular velocity ω around an axis through the center of mass. Wheels made of metal or wood are normally quite rigid, while a wheel made of rubber is less rigid. The particle speeds relative to the CM in a rigid body are $v_i' = r_i'\omega$, so the internal energy in this case is

$$\sum \frac{1}{2} m_i v_i'^2 = \frac{1}{2}\left(\sum m_i r_i'^2\right)\omega^2 \quad , \tag{12}$$

where I' is the rotational inertia of the rigid body about the axis through its center of mass. For a rigid body that rotates about its CM and translates as well, the total kinetic energy is the sum

$$K = K_{\text{translation}} + K_{\text{rotation about the CM}}$$

$$= \frac{1}{2} M V_{\text{CM}}^2 + \frac{1}{2} I'\omega^2 \ . \tag{13}$$

The rotational kinetic energy *must* be taken around an axis through the center of mass.

Sample Problem 6. Find the translational and rotational kinetic energies of the planet Earth in the inertial frame in which the center of the Sun is at rest.

Solution: In the Sun's frame, Earth is both translating and rotating, so its total kinetic energy is given by Eq. 13, with $M = 6.0 \cdot 10^{24}$ kg, $V_{CM} = 30$ km/s (which is Earth's orbital velocity around the Sun, approximating the orbit as circular), $I' = 0.33\ MR^2$, where Earth's radius is $R = 6{,}400$ km, and $\omega = 2\pi/T = 2\pi/24$ hrs $= 7.3 \cdot 10^{-5}\ \mathrm{s}^{-1}$. The total kinetic energy in the Sun's frame is therefore

$$K = \frac{1}{2}MV_{CM}^2 + \frac{1}{2}I'\omega^2 = 2.7 \cdot 10^{33}\ \mathrm{J} + 2.2 \cdot 10^{29}\ \mathrm{J}\ .$$

Earth's translational kinetic energy as it orbits around the Sun is four orders of magnitude larger than its rotational kinetic energy.

Sample Problem 7. A yo-yo of mass M is constructed from two disks, joined by a shaft of radius r as shown below. A string of negligible mass is wound around the shaft, and the yo-yo is dropped from rest. What is the speed of the yo-yo after it has fallen a distance h, in terms of the yo-yo's mass and its moment of inertia about its axis of symmetry ?

Solution: As the yo-yo falls, it also spins up. Its gravitational potential energy decreases, while both its translational and rotational kinetic energies increase. If we measure gravitational potential energy from the yo-yo's initial position, the potential energy is $-Mgh$ when it has fallen a distance h. Conserving energy, with no initial kinetic or potential energy,

$$0 = -Mgh + \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2\ ,$$

where v is the speed of the center of the yo-yo. Assuming all of the string is wound at the same radius r, and that the string does not slip on the shaft, the speed and angular velocity of the yo-yo are related by $v = r\omega$. Then we can eliminate ω and solve for v, giving

$$v = \sqrt{\frac{2gh}{1 + I/Mr^2}}\ .$$

This is smaller than the speed $v = \sqrt{2gh}$ of a single falling particle, because the yo-yo converts part of its potential energy into rotational kinetic energy.

Sample Problem 8. A rope is hung over a pulley of radius R and moment of inertia I. The rope does not slip on the pulley. Weights of mass m_1 and m_2 are attached to the ends of the rope, as shown. Released from rest, m_1 falls and m_2 rises. When m_1 has fallen a distance h, how fast is it moving? *Solve using translational and rotaional work-energy theorems!*

Solution: This is the same as *Sample Problem 2*, except we are supposed to solve it using work-energy theorems instead of energy conservation. Note that each weight is translating and the wheel is rotating, so we will use the translational work-energy theorem for the weights, and the rotational work-energy theorem for the pulley. Isolation diagrams for each of the weights and for the pulley are shown below. From the diagrams, note that the net force on m_1 is $m_1 g - T_1$, the net force on m_2 is $T_2 - m_2 g$,

and the net torque on the pulley is $(T_1 - T_2)R$. The translational work-energy theorem therefore gives

$$(i) \quad (m_1 g - T_1)h = \frac{1}{2}m_1 v^2 \quad \text{and} \quad (ii) \quad (T_2 - m_2 g)h = \frac{1}{2}m_2 v^2$$

for each of the weights, while the rotational work-energy theorem gives for the pulley

$$(T_1 - T_2)R\theta = \frac{1}{2}I\omega^2 \quad \text{or} \quad (iii) \quad (T_1 - T_2)h = \frac{1}{2}I(v/R)^2$$

since $R\theta = h$ and $\omega = v/R$, Now add Eqs. (i), (ii), and (iii). The cord tensions cancel out, leaving $(m_1 - m_2)gh = \frac{1}{2}\left(m_1 + m_2 + I/R^2\right)v^2$ which can be solved for v to give

$$v = \sqrt{\frac{2(m_1 - m_2)gh}{m_2 + m_1 + I/r^2}}$$

the same result found in *Sample Problem 2* using energy conservation.

19.4 Rolling Without Slipping

If a bowling ball slips as it rolls down the alley, there is no particular relationship between its translational velocity V, the velocity of the ball's center, and its angular velocity ω. But if the ball rolls without slipping, V is determined by ω. Suppose the ball turns through angle θ during some period of time, as shown in Fig. 3. A mark on the ball (point P) which was initially at the contact point with the alley has therefore moved around a distance $s = R\theta$, as shown in Fig. 3.

Fig. 3. If it does not slip, a ball of radius R turning through angle θ moves forward a distance $s = R\theta$..

The velocity and angular velocity are therefore similarly related, by

$$V = R\omega, \tag{14}$$

found from $V = \dfrac{ds}{dt} = R\dfrac{d\theta}{dt} \equiv R\omega$.

If we express the moment of inertia I of the rolling object (in its center-of-mass frame) in the form $I = fMR^2$, where f is the (dimensionless) shape factor ($f = 1/2$ for a solid cylinder about its axis of symmetry, for example), the total kinetic energy of the rolling object is then

$$\text{K.E.} = \frac{1}{2}MV^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}(1 + f)MV^2. \tag{15}$$

Sample Problem 9. A particular wheel of mass M and radius R has moment of inertia $I = \frac{2}{3}MR^2$ about its center. Starting from rest, the wheel rolls without slipping down an incline sloped at angle ϕ to the horizontal, as shown. By the time the wheel has rolled a distance D, (a) how many turns has the wheel made? (b) What is the speed of the wheel's center? Solve using energy conservation and using work-energy theorems.

Solution: (a) One full turn corresponds to a distance $2\pi R$ if the wheel doesn't slip, so the number of turns is $D/2\pi R$.

(b) *Using energy conservation*: Initially, the wheel has no kinetic energy, and potential energy $U = Mgh = Mg(D\sin\phi)$ above its endpoint at the bottom. At the endpoint, the wheel has no potential energy, and kinetic energy

$$\text{K.E.} = \frac{1}{2}MV^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}MV^2 + \frac{1}{2}\left(\frac{2}{3}MR^2\right) = \frac{5}{6}MV^2,$$

since $R\omega = V$. Conserving energy, we can solve for V to find $V = \sqrt{\frac{6}{5}gD\sin\phi}$.

(b) *Using work-energy theorems*: This method requires us to find the net force on the wheel, and the net torque about its center of mass. The isolation diagram of the wheel is shown. The net force *down* the slope is $F = Mg\sin\theta - F_f$ which is *constant,* so the work done on the wheel as it moves a distance D is $FD = (Mg\sin\theta - F_f)D$. The translational kinetic energy is $\frac{1}{2}MV^2$, where V is the velocity of the center of mass of the wheel. The translational work-energy theorem therefore gives

$$(i) \quad MgD\sin\theta - F_f D = \frac{1}{2}MV^2.$$

The torque on the wheel about its center of mass is due to the frictional force F_f; this is what causes the wheel to spin up. The torque is constant, so the rotational work-energy theorem is

$$(ii) \quad \tau\Delta\theta = (F_f R)\Delta\theta = F_f D = \frac{1}{2}I\omega^2$$

Adding Eqs. (i) and (ii) eliminates $F_f D$. Then using $\frac{1}{2}MV^2 + \frac{1}{2}I\omega^2 = \frac{5}{6}MV^2$ and solving for V gives $V = \sqrt{\frac{6}{5}gD\sin\phi}$ as before.

It is interesting to find the velocity vectors of different parts of a wheel as the wheel rolls without slipping along a horizontal surface. The velocity of various points as seen in the wheel's center of mass frame are shown in Fig. 4(a). This is the frame in which the center of the wheel stays at rest. Since the angular velocity ω is the same everywhere in the wheel, the speed of points increases linearly with radius; i.e., $v = \omega r$, as shown.

We can translate this picture into the rest-frame of the surface, by moving to the *left* at speed v relative to the CM frame. In this frame, the center of the wheel is moving to the *right* at speed v, so we can find the velocity of any point in the wheel by adding this right-pointing vector v to the vectors shown in Fig. 4(a). The velocity vectors of various points in the wheel in the rest-frame of the surface are shown in Fig. 4(b).

Fig. 4. (a) The velocity vectors of various points in a wheel in the frame in which the wheel's center is at rest. (b) The velocity vectors in the frame of the surface, if the wheel rolls without slipping. The velocities shown in (b) are found by adding a rightward-directed vector v to the velocity vectors shown in (a).

Note that the point of the wheel rim which happens to be touching the surface at a given time is instantaneously at rest relative to the surface. It does not slip relative to the surface, so even if there is friction between the wheel and surface, ideally the friction does no work on the wheel; that means that at least ideally, no heat is generated because there is no rubbing. (Inevitably there is a small amount of friction as points come in contact with the surface and then pull away, and perhaps as the wheel continuously flexes slightly while rolling, so heat is produced within the wheel itself. Often this rolling friction is very small, so a wheel can roll a long ways in the absence of air resistance before its kinetic energy is all converted to heat energy, and the wheel comes to rest.)

In our explorations so far, all of the wheels have either been rotating about a fixed axis (such as the spinning wheel of a car lifted off the ground on a rack), or they have been rotating about an axis that stays parallel to itself (like the axis of a wheel that rolls as the car moves along a straight road.) In all these cases it is sufficient to represent the angular velocity of the

wheel, its angular momentum, and the torque upon it, as either counterclockwise (positive) or clockwise (negative). If a car tire is rolling without slipping as the car travels to the right, for example, the tire rotates clockwise, so the angular velocity ω and angular momentum L of the wheel are both negative, according to our convention.

There are many interesting situations where the axis is neither fixed nor does it stay parallel to itself. If the car turns a corner, the direction of a wheel's rotation axis changes. If a single-engine airplane with a propeller flies straight, the propeller axis stays parallel to itself, but if the plane dives, the axis tilts. The rotation direction of a gyroscope can change continuously. In all these cases it is inadequate to talk about clockwise or counterclockwise rotation. We need a more general and powerful way to describe rotation. That will be the topic for tomorrow.

State of the Search

1. The kinetic energy of a wheel rotating about a fixed axis is

$$K = \frac{1}{2} I\omega^2 ,$$

where I is the moment of inertia of the wheel about the axis, and ω is its angular velocity. This formula is analogous to the equation for the kinetic energy of a particle, $\frac{1}{2}mv^2$. It is not only an analogy, however; $\frac{1}{2}I\omega^2$ really *is* the kinetic energy of the wheel, the sum of the kinetic energies of all the particles that make it up.

2. A work-energy theorem

$$(W_T)_{rot} = \frac{1}{2} I\,\omega^2 - \frac{1}{2} I\,\omega_0{}^2$$

for rotation shows that the net rotational work $(W_T)_{rot} = \int \tau_T \, d\theta$ done on a wheel is responsible for any change in its rotational kinetic energy. Here τ_T is the total torque acting on the wheel as the wheel rotates through an angle θ.

This is analogous to the work-energy theorem for translation.

3. The kinetic energy of a system of particles with masses m_i and total mass M whose center of mass moves at speed V_{CM} is

$$\text{K.E.} = \frac{1}{2}MV_{CM}^2 + \sum \frac{1}{2}m_i v_i'^2$$

where v_i' is the speed of the ith particle in the center-of-mass frame. The first term is the translational kinetic energy and the second term is the "internal" kinetic energy, the kinetic energy of the system in the CM frame.

4. If the system is a rigid body rotating about its CM with angular speed ω, the "internal" kinetic energy is the rotational kinetic energy $\frac{1}{2}I\omega^2$, where I is the moment of inertia of the body about its center of mass. In this case the total kinetic energy is

$$K = \frac{1}{2}MV_{CM}^2 + \frac{1}{2}I\omega^2,$$

the sum of the translational and rotational kinetic energy of the system.

If the system is a symmetrical rigid body of radius R that rolls without slipping along a surface, then V_{CM} and ω are related by $V_{CM} = R\omega$. Then if we express I in the form $I = fMR^2$, where f is the shape factor ($f = 1/2$ for a solid cylinder about its axis of symmetry, for example), the total kinetic energy of the rolling object is

$$K = \frac{1}{2}(1 + f)MV_{CM}^2.$$

PICTURE QUIZ 19

1. Two hoops, each having the same mass, rotate about their centers. Hoop A has radius R and rotates with angular speed ω, while hoop B has radius $2R$ and rotates with angular speed $\omega/2$. Which has the greater rotational kinetic energy?

(A) Hoop A (B) Hoop B (C) They have the same rotational kinetic energy.

2. A cord wound around a disk-like pulley of mass M and radius R is pulled keeping its tension constant, so as to allow the pulley to achieve angular speed ω after turning through one complete revolution. The required tension T is

(A) $\dfrac{M\omega}{2\pi R}$ (B) $\dfrac{2R\omega^2}{\pi M}$ (C) $\dfrac{MR\omega^2}{8\pi}$

(D) $\dfrac{R^2\omega}{4\pi M^2}$ (E) $\dfrac{4\pi M^2 R^2}{\omega^2}$

Check your answer is at least two ways!

3. A penny rolls without slipping along a table top. The penny's center has velocity v. The uppermost point on the penny has velocity

(A) 0 (B) $v/2$

(C) v (D) $3v/2$ (E) $2v$

4. A bicycle wheel rolls without slipping along the street. Which is greater: its translational kinetic energy or its rotational kinetic energy? (Idealize the rim and tire as infinitely thin.)

(A) Its translational K.E. is greater

B) Its rotational K.E. is greater

C) They are the same.

5. Two uniform disks have the same mass, but disk A has twice the radius of disk B. If, starting from rest, they both roll the same distance down two identical inclined planes, which has the greater translational kinetic energy when it reaches the bottom?

(A) Disk A

B) Disk B

(C) their translational kinetic energies are the same

PROBLEM QUIZ 19

1. A fan-shaped wheel of mass M rotates with angular speed ω_0 about its central axis. The wheel consists of two 90° segments of radius R, as shown. Find (a) its angular momentum (b) its kinetic energy.

2. A car typically wastes all of its kinetic energy when it stops at a stop light; the kinetic energy is turned into heat energy, mostly in the brake system, and cannot be recovered. A flywheel technology is under development in which kinetic energy of a moving car is fed into a rotating wheel within the car when the brakes are applied. The rotational kinetic energy of the wheel can then be used to start the car moving again when the light turns green. Suppose a car has mass $M = 1200$ kg, which includes an internal disk-like wheel of mass $m = 12$ kg and radius $R = 21$ cm. If the car is brought to rest from a speed of 25 miles/hour (11 m/s), what will be the rotation rate of the wheel in revolutions/minute (rpm) if the conversion from the car's translational kinetic energy to the rotational kinetic energy of the wheel is made with 75% efficiency?

3. A hoop and a disk of the same mass and radius are placed side-by-side on a plane inclined at angle θ to the horizontal. If they are started from rest at the same height, and they each roll without slipping down the plane, which reaches the bottom first?

4. A solid cylinder and a solid sphere are placed on a plane inclined at angle θ to the horizontal, as shown. There is a coefficient of static friction μ_s between the surface and each object. As the cylinder and sphere are released, they roll without slipping down the plane. The experiment is then repeated, with a plane pitched slightly steeper, with a slightly larger angle θ. When the experiment is repeated several times, with θ increased each time, one of the objects will be found to slip rather than roll without slipping. Which of the objects, the cylinder or the sphere, will start to slip first?

Questions and Problems from *Day Nineteen*

19Q1. The work done by a constant force F on a particle moving a distance x in the direction of F is $W = Fx$. What is the analogous result for the work done on a rotating wheel? What is an example in which the analog formula could be used?

19Q2. The power input to an object translating in one dimension with velocity v due to a force F is given by $P = Fv$. What is the analogous result for the power input to a rotating wheel? What is an example in which the analog result could be used?

19Q3. Cars lose energy when they slow down and stop at stoplights, and have to burn fuel to start up again. How could you couple an internal flywheel to the drive and brake systems, to allow much of the energy of braking to go into kinetic energy of the flywheel rather than heat in the brakes? If the flywheel has a mass $m = 5$ kg and radius $R = 0.5$ m, what typical angular velocities would be needed?

19Q4. A cylinder rolls without slipping along the floor. Is it possible to distribute the mass within the cylinder so that its rotational kinetic energy exceeds its translational kinetic energy?

19Q5. A pool ball is rolled along a straight track, from left to right, as shown below. Another pool ball is rolled along a track whose left and right endpoints are similar to those of the first track, except this second track has a dip in the middle, also as shown. If both balls begin their trip at the left with the same speed, which one reaches the right end first?

19Q6. A ball rolls without slipping back and forth in a semicircular track, as shown below on the left, while a small block slides back and forth without friction in a similar track. Which takes longer to oscillate back and forth, and why?

19Q7. There is a coefficient of static friction μ_s between a cylinder and slope. Is it possible to distribute the mass within the cylinder so that it will *not* roll down the slope?

19Q8. Is it possible to change the translational kinetic energy of an object without changing its rotational energy?

19Q9. Two buckets spin around in a horizontal circle on frictionless bearings, as shown. Suddenly, it starts to rain. As a result,

(a) The buckets continue to rotate at constant angular velocity because the rain is falling vertically while the buckets move in a horizontal plane.

(b) The buckets continue to rotate at constant angular velocity because the total mechanical energy of the bucket + rain system is conserved.

(c) The buckets speed up because the potential energy of the rain is transformed into kinetic energy.

(d) The buckets slow down because the angular momentum of the bucket + rain system is conserved.

(e) Both (a) and (b)

(f) None of the above.

(From *Physics* by Randall D. Knight)

19Q10. The center of mass of a pitched baseball of radius R moves at speed v, while the ball spins about an axis through its center of mass

with angular velocity ω. What is the ratio of its rotational and translational kinetic energies?

Section 19.1. Kinetic energy of a rotating wheel

19P1. A ring, a disk, and a solid sphere each have the same mass M and radius R. (a) If all three have the same angular speed about axes through their centers, which has the greatest rotational kinetic energy? Which has the least? (b) If the angular speed of the disk is ω_d, and all three have the same rotational kinetic energy, what are the angular speeds ω_r of the ring and ω_s of the sphere? (c) If each has the same kinetic energy, which has the greatest angular momentum? Which has the least?

19P2. A uniform stick of mass m and length d spins about an axis through its center with angular speed ω_1. An identical stick spins about an axis through one end of the stick instead. In terms of m, d, and ω_1, (a) if the second stick has the same rotational kinetic energy as the first, what is its angular momentum? (b) If the second stick has the same angular momentum as the first, what is its rotational kinetic energy?

19P3. The entire human race currently uses energy at the rate of about 15 Terawatts. (1 Watt = 1 Joule/s.) At that rate, after one year how long would the length of the day have increased, if for that year we were able to power our activities by harnessing Earth's rotational kinetic energy? (Earth has moment of inertia $I = 0.33MR^2$.)

19P4. Begin with a metal disk of mass M and radius R. Drill a hole of radius $R/2$ into the center of this disk. Attach spokes of negligible mass to the remaining annulus, so it can rotate about a small frictionless axle through the center, as shown below. If the outer edge of the resulting wheel is made to turn with speed v, find the wheels (a) kinetic energy (b) angular momentum, both in terms of M, R, and v.

19P5. One end of a string is attached to a block of mass m; the other end is strung over a disk-like pulley of mass M and radius R, which has a frictionless bearing, and then wound around the shaft of a propeller whose moment of inertia is I, as shown below. The radius of the shaft is r. (a) Neglecting air resistance, what is the angular velocity of the propeller when the block has fallen a distance h? (b) When air resistance is included, the propeller quickly approaches a terminal angular velocity ω_f. At $\omega = \omega_f$, how much power is going into frictional heating of the air and the propeller, in terms of given parameters and g?

19P6. One end of a uniform vertical pole of length ℓ is attached to the floor by a frictionless hinge, as shown below. The pole is nudged slightly sideways, and falls. How fast is the other end moving just before it hits the floor?

19P7. A flywheel has the form of a uniform disk of mass M and radius R. At time $t = 0$ the wheel is at rest. A constant force F_0 is then applied to the outer rim, tangent to the rim. As functions of time, find the wheels (a) kinetic energy (b) angular momentum.

19P8. A light cord is hung over a ring-like pulley of mass M, with essentially all of its mass a distance R from its rotation axis. The pulley can rotate with negligible friction. The ends of the cord are attached to weights of masses $2M$ and $3M$, as shown below. Released from rest, how far has the $3M$ weight fallen by the time it

achieves speed v ? Use conservation of energy.

19P9. Solve 19P8 using $F = ma$ and $\tau = I\alpha$ instead of energy conservation. Begin with isolation diagrams for the pulley and for each weight.

19P10. Three particles, each of mass m, are attached to one another and to a rotation axis by three light strings, each of length ℓ , as shown below. The combination rotates with angular velocity ω, so the particles remain in a straight line. (a) Calculate the rotational inertia of the combination about the rotation axis O. (b) Then find the kinetic energy of the system in two ways: (i) by adding up the kinetic energies of the individual particles; (ii) by finding the rotational kinetic energy $(1/2)I\omega^2$.

19P11. The three 200 g masses shown below are connected by massless, rigid rods to form a triangle. What is the triangle's rotational kinetic energy if it rotates at 5 rev/s about an axis through its center?

19P12. A drum major twirls a 96-cm long, 400 g baton about its center at 100 rpm. What is the baton's rotational kinetic energy?

19P13. A disk of mass M and radius R is rotating freely about its symmetry axis with angular velocity ω_0. A small bug of mass m is standing on the rim of the disk, as shown below. The bug then crawls inward to the center of the disk. (a) Find the angular speed ω_f of the disk when the bug has completed its journey. (b) Find the difference between the initial and final kinetic energies of the system of disk + bug. (The bug is pointlike, so has no final kinetic energy). (c) Explain any change in kinetic energy. That is, where does it come from or go to?

19P14. An iceskater, of total mass $M = 52$ kg, is twirling on the ice about her vertical axis, with angular velocity ω_0; her arms, each of mass $m = 3.5$ kg and length $\ell = 60$ cm, are fully extended horizontally, as shown below. She then brings her arms to her side, a distance $d = 20$ cm from the rotation axis. (a) Find her new angular velocity ω_f in terms of ω_0. (b) What is the ratio of her final kinetic energy to her initial kinetic energy? Where did the energy increase come from? (Suppose there is no friction with the ice, and model her body as a uniform vertical cylinder and her arms as uniform rods.)

19P15. A uniform rod of mass M and length L can turn in a horizontal plane about a frictionless vertical axle attached to one end of the rod. A bead of mass $m = M/3$ is threaded onto the rod, and is held in place a distance $d = L/3$ from the axle, as shown below, while the rod is made to spin at initial angular velocity ω_0. The bead is then released so that it can slide outward without friction along the rod. (a) What

quantities are conserved for the system of bead plus rod?(b) What is the angular velocity of the rod when the bead reaches the end of the rod? (c) What is the tangential velocity of the bead when it reaches the end of the rod? (This is the component of velocity perpendicular to the rod itself.) (d) What is the total speed of the bead?

19P16. In a supernova explosion, the core of a heavy star collapses and the outer layers are blown away. Before collapse, the core of a given star has mass $M = 2M_{\text{sun}}$ ($M_{\text{sun}} = 2\times10^{30}$ kg), radius $R_{\text{sun}} = 7\times10^5$ km, and rotates about its north-south pole axis with period 20 days. The core collapses in a few seconds to become a neutron star of radius 20 km. (a) What will be its period of rotation? (Pretend that the density of the core is uniform and remains so, and neglect any influence of outer layers on the core.) (b) Find the ratio of the final rotational kinetic energy of the core to its initial rotational kinetic energy. What could account for the change?

19P17. What is the magnitude of the torque (in Newton meters) acting upon the model pulsar of *Sample Problem 3*?

19P18. The Vela Pulsar has a rotational period $P = 0.089$ s. The period is slowly increasing, at the rate of 11 ns/day (1 ns $= 10^{-9}$ s).(a) Assuming the neutron star has one solar mass and a radius of 10 km, what is the stars power loss in Watts? (b) The age T of a pulsar, the time since its formation in a supernova explosion, can be roughly estimated by dividing its period by the time derivative of its period (in seconds/second). That is, T is of order $P/\dot{P}$. In this way, estimate the age of the Vela Pulsar. (Its actual age is probably between 10 and 30 thousand years.)

19P19. In some theoretical models of pulsars, the *braking torque* slowing the spin rate is proportional to the nth power of the pulsars angular velocity Ω, where n is known as the *braking index*. (a) Show that in such a case the time derivative of Ω is related to Ω itself by $\dot{\Omega} = -K\Omega^n$ where K is a constant. (b) Using the result of part (a), find a formula for the time rate of change of the pulsars period P in terms of P itself and the constants n and K. (c) Show that if the braking index $n = 3$, the product $P\dot{P}$ = constant. Both the Crab and Vela pulsars have values of $P\dot{P}$ close to 10^{-14} s.

Section 19.2. The work-energy theorem for rotation

19P20. What constant torque τ, acting on a wheel of mass M and radius R as it turns through one complete revolution, gives the wheel a rotational kinetic energy equal to the kinetic energy the wheel would gain if it dropped in uniform g a distance equal to its circumference?

19P21. A disk-like wheel of mass M and radius R is made to turn, starting at rest at angle $\theta = 0$, by a gradually increasing the torque $\tau = \tau_0\theta$, where τ_0 is a constant and θ is the rotation angle. Find the angular velocity of the wheel as a function of θ.

19P22. Using the rotational work-energy theorem along with $\tau = I\alpha$, Show that when a constant torque is exerted on a wheel, the wheels angular velocity obeys $\omega^2 = \omega_0^2 + 2\alpha\theta$, where ω_0 is the initial angular velocity and θ is the angle through which the wheel has turned.

19P23. A light cord is draped over a pulley of mass M, radius R, and shape-factor $f = 1/2$. Weights with masses M and $2M$ are hung from the two cord ends, similar to the situation in Sample Problem 5. The only friction is between the pulley and its axle. Released from rest, the heavier mass falls while the lighter rises; after falling a distance h, the heavier mass achieves only three-quarters the speed it would have had if there were no friction in the pulley. What is the (assumed constant) frictional torque on the pulley, in terms of any or all of M, R, g, and h?

19P24. A block of mass m sits at rest on a table. There is a coefficient of kinetic friction μ_k between the block and table. The block is attached to a light cord, and the cord is draped over a disk-like pulley of mass $M = 2m$ and radius R that has frictionless bearings. A second block of mass m is hung from the other end

of the cord, as shown below; when released it starts to fall. As it falls, the hanging mass has half the speed it would have if μ_k were equal to zero. Find the numerical value of μ_k.

19P25. Two blocks, each of mass m, are attached to opposite ends of a light cord. The cord is draped over a disk-like pulley of mass M and radius R, and one block is set on a table while the other hangs freely, as in the picture for problem 19P22. There is no friction between the tabletop and upper block, but there is friction between the pulley and its axle. Assuming that it is constant, find the frictional torque exerted on the pulley by the axle if, as the hanging block falls through some distance h it gains only half the speed it would if there were no frictional torque. Express the answer in terms of any or all of m, M, R, h, and g.

Section 19.3 Combined rotational and translational kinetic energy

19P26. The Moon always keeps the same face to Earth as it revolves in its orbit. Estimate the ratio of its kinetic energy of rotation about its own axis, to its average kinetic energy of motion around Earth. The Moon moves at the average speed $v = 1.02$ km/s around Earth; the average radius of its orbit is $r = 3.8 \times 10^5$ km. The Moon's radius is $R = 1738$ km.

19P27. Find the ratio of the kinetic energy of Earths rotation about its own axis to the kinetic energy of its motion around the Sun. Use $I = 0.33MR^2$ for the Earth about its axis, and the Earths speed as it orbits the Sun is v= 30 km/s.

19P28. A string is wrapped around a yo-yo with mass m and moment of inertia I. One end of the string is kept fixed while the yo-yo is dropped from rest, spinning about the string as it falls. (a) Draw an isolation diagram of the yo-yo showing the forces acting on it. (b) Use the work-energy theorem for linear motion $W_T = (1/2)mv^2$, where v is the velocity of the center of mass of the yo-yo, together with the rotational work-energy theorem $(W_T)_{rot} = (1/2)I\omega^2$, to show that $mgs = (1/2)mv^2 + (1/2)I\omega^2$, where s is the distance through which the CM of the yo-yo has fallen.

19P29. Starting at rest, a four-wheel vehicle coasts down an incline on the Moon that is tilted at angle θ to the horizontal. Each wheel has radius R, mass $M/4$, and shape-factor $f = 1/2$; the rest of the vehicle has mass M. After moving some distance down the slope, the vehicle has a speed only half what it would have been had there been no friction between the wheels and their axles. Find the frictional torque on each wheel, assuming it is constant and the same for each wheel, in terms of any or all of M, R, g_{moon}, and θ.

19P30. Two balls, each of mass m, are attached to opposite ends of a massless spring, and then made to oscillate in and out as shown below. In the CM frame, the position of the right-hand ball is given by $x_R = a + b\sin\omega t$ as a function of time, where a, b, and ω are constants. At all times the left-hand ball has the position $x_L = -x_R$, so the center of the spring stays at rest in the overall CM frame. As functions of time, find (a) the velocity of each ball in the CM frame, and (b) their total kinetic energy in the CM frame. From the point of view of an observer traveling to the right at velocity $V = b\omega$, find (as functions of time) (c) the velocity of each ball in the new frame; (d) their total kinetic energy in the new frame, using the velocities found in (c). Finally, (e) compare the kinetic energies found in (b) and (d).

19P31. A yo-yo is made of three solid cylinders of the same type of wood. The outer two have radius R and thickness $R/4$, and the inner one has radius $R/2$ and thickness $R/2$. A thin string is wound N times around the inner cylinder; all of the string is essentially at radius $R/2$ from the axis of symmetry. If the yo-yo falls from rest, how fast will its CM be moving when all the string is unwound?

Section 19.4 Rolling without slipping

19P32. An automobile tire has mass M, radius R, and moment of inertia $I = kMR^2$ about the cars axle. If the car is moving at velocity v along the street, and the tire is not skidding, find the kinetic energy of the tire in the frame of a pedestrian standing on the sidewalk, in terms of given parameters.

19P33. A wheel of radius R rolls without slipping along the floor. What fraction of its kinetic energy is rotational, if the wheels mass is (a) distributed in the form of a uniform disk? (b) all located at $r = R$, as for a hoop?

19P34. A pool ball of mass M and radius R rolls without slipping with angular velocity ?o along a level surface. (a) What fraction of its kinetic energy is translational? (b) The surface begins to slope upward. If the ball continues to roll without slipping, what altitude h does it achieve above the level surface by the time it instantaneously comes to rest, in terms of any or all of M, R, ω_0, and g?

19P35. A hoop, a disk, and a solid sphere, each of mass M and radius R, are placed side by side at the top of an inclined plane. They are released from rest, and roll without slipping down the slope. Which of the three reaches the bottom first? Explain why.

19P36. A hoop, a disk, and a solid sphere, with masses M_h, M_d, M_s, and radii R_h, R_d, R_s, respectively, are placed side by side at the top of an inclined plane, so that their contact points with the slope are at equal elevations. They are released from rest, and roll without slipping down the slope. In terms of given parameters, which reaches the bottom first? Explain why.

19P37. A hoop and a disk, of the same mass M, radius R, and initial velocity vo, independently roll without slipping up an inclined plane tilted at angle ? to the horizontal. Find the altitude achieved by (a) the hoop (b) the disk.

9P38. A ball bearing of mass m and radius r is released from rest at the top of a track at height h above the floor (where $h >> r$), and then rolls without slipping down the track, as shown below. At the bottom it starts rolling around a loop of radius $R = h/4$. Find the normal force exerted upon the ball bearing when it reaches (a) point A (b) point B

19P39. What is the minimum initial height h (expressed as a multiple of the loop radius R) required so the ball bearing described in 19P36 will reach point B, staying on the track?

19P40. A cart of mass M is supported by four disk-like wheels, each of mass $m = M/4$ and radius R. The cart moves down the track shown, with the wheels rolling without slipping, starting at rest at height h_0, until it reaches the end of the track at height $h_f = h_0/4$, at which point the track slopes upward at angle $\theta = 30^o$. How far to the right of this point (measured horizontally) will the cart strike the horizontal surface?

19P41. A pig and a pool ball are both placed on the top of a hemispherical dome of radius R, as shown below. The pig is given a slight nudge to the right, so it begins to slide down the frictionless right half of the dome. The ball is given a slight nudge to the left, so that it rolls without slipping down the left half of the dome, which has sufficient friction with the ball so that the ball rolls without slipping. Find the angle θ_{pig} at which the pig leaves the surface, and the angle θ_{poolball} at which the ball leaves the surface, both measured relative to the vertical. What is the physical reason why these angles are different?

19P42. An automobile tire of radius r, mass m, and moment of inertia $I = 0.75mr^2$, rolls without slipping down a track, starting from rest at elevation h, as shown. It rolls along a plateau of elevation R, and at point A begins rolling down a circular arc of radius R ($R >> r$). (a) Find the angle θ (with respect to the vertical) where the tire leaves the track, in terms of any or all of h, r, R, and g. (b) Find a maximum value of h for which the formula found in (a) is valid. What happens if h is larger than this?

19P43. A yo-yo consists of three solid wooden cylinders, all of the same density ρ and width w. The outer cylinders have radius R and the inner cylinder has radius r. The inner cylinder of the yo-yo is set on a rail tipped at angle θ to the horizontal. There is sufficient static friction between the inner cylinder and rail so that the yo-yo rolls without slipping down the incline. (a) Find the time required for the yo-yo to fall a vertical distance h. (b) What is this time in the limit $r \to 0$?

Day Twenty

Rotational Vectors

"Never put off till tomorrow what you can do the day after tomorrow." - Mark Twain

Every wheel we've looked at so far has either rotated about a fixed axis or has rolled so the axis never changes direction. In such cases we can take the angular velocity and angular momentum of the wheels to be *scalar* quantities, with magnitudes but no directions attached to them. This is also true of the net torques exerted on the wheels.

Wheels of course do not *always* rotate about fixed axes: The wheels of a car change orientation when the car turns; gyroscope wheels turn and wobble; even Earth's axis is not quite fixed in space. We can then no longer treat angular velocity, angular momentum, and torque as scalars. We need to specify the *orientation* of a wheel as well as how fast it spins. How can we do this?

20.1 Angular velocity and angular momentum vectors

We need a *vector* to define the orientation and rotation direction of a wheel. That much is clear. What is not so clear is which way the vector should *point*. We can't place the vector *in* the plane of rotation: There are an infinite number of such vectors, all pointing in different directions.

We can, however, orient the vector along the rotation axis, *perpendicular* to the plane of rotation. Given such a vector, the plane of rotation is clear; it has to be perpendicular to the vector. The only problem is that there are two possible choices for the perpendicular vector, as illustrated in Fig. 1(a): The vector could point out from *either side* of the plane.

Fig. 1. (a) Two vectors which could be used to define a rotation direction (b) The right- hand-rule convention

We choose between the two possible vectors by a "right-hand rule". Curling the fingers of the right hand in the direction of the wheel's rotation, the right thumb picks out the direction, as shown in Fig. 1(b). If the rotation plane stays the same, but the wheel rotates the opposite way, the rotation vector points in the opposite direction, according to the right-hand rule.

What is the physical quantity represented by this vector? We take it to be the *angular velocity vector* ω. This vector plays three roles at once. The *magnitude* of ω gives the angular speed of rotation, the rotation rate in radians/second. The *direction* of ω specifies both the axis of rotation and the direction in which the wheel is turning.

Sample Problem 1. A wheel rotates in the x, y plane once every 0.030 seconds, clockwise as viewed down along the positive z axis, as shown. Find the angular velocity vector of the wheel.

Solution: The period is $T = 0.030$ s, so the wheel's angular speed is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.030 \text{ s}} = 2.1 \cdot 10^2 \text{ s}^{-1} .$$

The direction of the angular velocity is along the z axis, perpendicular to the plane of rotation of the wheel. If we look down upon the wheel along the positive z axis, if the wheel is turning CW, the angular velocity is in the $-\hat{\mathbf{z}}$ direction, using the right hand rule. So the angular velocity vector is

$$\boldsymbol{\omega} = -2.1 \cdot 10^2 \, \hat{\mathbf{z}} \text{ s}^{-1} .$$

If the wheel were rotating counterclockwise instead, the vector ω would point in the opposite direction, i.e., in that case $\boldsymbol{\omega} = +2.1 \cdot 10^2 \, \hat{\mathbf{z}} \text{ s}^{-1}$.

How about angular momentum? In *Day Sixteen* we found that the angular momentum of a rigid wheel turning about a fixed axis is $L = I\omega$, in terms of the wheel's moment of inertia I and angular velocity ω. Now that angular velocity has become a vector $\boldsymbol{\omega}$, we can do the same for $\mathbf{L}$, by writing

$$\mathbf{L} = I\,\boldsymbol{\omega} \tag{1}$$

The magnitude of the angular momentum is the same as before, $L = I\omega$, but now $\mathbf{L}$ has a direction as well, along the fixed axis of rotation, in the same direction as $\boldsymbol{\omega}$.

Sample Problem 2. Our rotating space station consists of a cylindrical shell of mass M and radius R, with two endcaps, each of mass m and radius R, as shown. The station rotates around its axis of symmetry, which is oriented along the z-axis, so that people living on the inside rim of the cylindrical shell experience an effective gravity g_{eff}. Find the angular momentum vector of the station in terms of given parameters.

Solution: The moment of inertia of the cylindrical shell about the symmetry axis is $I_{\text{shell}} = MR^2$, and the moment of inertia of each endcap is $I_{\text{endcap}} = \frac{1}{2}mR^2$, so the total moment of inertia is

$$I = (M+m)R^2 .$$

The effective gravity is the inward acceleration of the rim, given by $g_{\text{eff}} = R\omega^2$, where ω is the angular velocity of the station. Therefore, $\omega = \sqrt{g_{\text{eff}}/R}$, so the angular momentum of the station is

$$\mathbf{L} = I\,\boldsymbol{\omega} = (M+m)\sqrt{g_{\text{eff}}R^3}\;\hat{\mathbf{z}} ,$$

if it is rotating as shown in the figure above.

20.2 Torque vectors and cross products

What causes a wheel to *change* its angular momentum? During *Day Seventeen* we found that it is the *net torque* that causes a wheel to spin up or spin down, where the torque due to a force $\mathbf{F}$ acting at distance $\mathbf{r}$ from the rotation axis is

$$\tau = F_{\perp} r = r_{\perp} F = F r \sin\phi. \qquad (2)$$

Here F and r are the magnitudes of the force vector and the radius vector from the origin to the point at which the force acts, and ϕ is the angle between these vectors. The quantity $F_{\perp}$ is the component of $\mathbf{F}$ perpendicular to $\mathbf{r}$, and $r_{\perp}$ is the component of $\mathbf{r}$ perpendicular to $\mathbf{F}$. If $\mathbf{F}$ acts along the same line as $\mathbf{r}$ (so the angle between them is 0 or π) there is no torque, and $\mathbf{F}$ is ineffective at changing the wheel's rotation rate.

In *Day Seventeen* the torques simply caused either counterclockwise or clockwise rotation, (i. e., either positive or negative rotation) so it was not necessary to give torque a *direction*. But for wheels or other objects that can turn in any direction they like, torques need a direction, just as angular velocity and angular momentum need a direction. How do we define the direction of a torque?

According to Eq. 2, torque is composed from the vectors $\mathbf{r}$ and $\mathbf{F}$. Is there some way to multiply together the vectors $\mathbf{r}$ and $\mathbf{F}$ to get the *direction* of the torque as well as the correct magnitude?

We have previously encountered a way to multiply two vectors like $\mathbf{r}$ and $\mathbf{F}$ using the *dot product* $\mathbf{r} \cdot \mathbf{F}$. (For example, the work W done on a particle by a constant force $\mathbf{F}$ is $W = \mathbf{F} \cdot \mathbf{d}$, where $\mathbf{d}$ is the particle's displacement.) However, we can't use the dot product for torque, because $\mathbf{r} \cdot \mathbf{F} = rF\cos\phi$, *not* $rF\sin\phi$. The dot product is also a *scalar* quantity, so cannot specify a direction.

We introduce a *new* product of $\mathbf{r}$ and $\mathbf{F}$, called the *cross product* $\mathbf{r} \times \mathbf{F}$, which has a direction as well as a magnitude. Let

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \qquad (3)$$

with the required magnitude

$$|\mathbf{r} \times \mathbf{F}| = r F \sin\phi, \qquad (4)$$

and a direction given by the right-hand rule, to be consistent with the right-hand rule for the direction of the angular velocity and angular momentum of a rotating wheel.

and a direction given by the right-hand rule, to be consistent with the right-hand rule for the direction of the angular velocity and angular momentum of a rotating wheel.

There are several ways to picture the right-hand rule for torques, as shown in Fig. 2. The particular order $\mathbf{r} \times \mathbf{F}$ (rather than the other possible choice $\mathbf{F} \times \mathbf{r}$) gives the correct direction for τ by the right-hand rule. If the thumb of the right hand is directed along $\mathbf{r}$ and the right forefinger is directed along $\mathbf{F}$, then the middle finger of the right hand points in the direction of the cross product $\mathbf{r} \times \mathbf{F}$, as shown in Fig. 2(a). Alternatively, if the fingers of the right hand are placed so that they curl in the direction in which we have to turn $\mathbf{r}$ so it points in the direction of $\mathbf{F}$, by the shortest route possible, then the right thumb points in the direction of the cross product, as illustrated in Fig. 2(b). A final alternative is shown in Fig. 2(c). Turn a conventional screw (with a right-handed thread) in the sense needed to twist $\mathbf{r}$ into the direction of $\mathbf{F}$ by the lesser of the two angles required. Then the screw propagates *into* the piece of wood if we turn the screw clockwise, and *out of* the piece of wood if we turn the screw counterclockwise. The direction in which the screw propagates is the direction of the cross product.

Fig. 2. The right-hand rule used to find the direction of a torque. (a) If the thumb of the right hand is put in the $\mathbf{r}$ direction and the forefinger in the $\mathbf{F}$ direction, the middle finger points in the $\mathbf{r} \times \mathbf{F}$ direction. (b) Place the tails of vectors $\mathbf{r}$ and $\mathbf{F}$ together and then rotate $\mathbf{r}$ toward $\mathbf{F}$ in the shorter of the two possible directions. Curl the fingers of the right hand in the rotation direction of $\mathbf{r}$. The thumb points in the direction of the torque. (c) Place the tails of $\mathbf{r}$ and $\mathbf{F}$ together, and turn $\mathbf{r}$ toward $\mathbf{F}$ as described in (b). A right-handed screw turned that way propagates in the $\mathbf{r} \times \mathbf{F}$ direction, into the wood in the case shown.

Sample Problem 3. A pole leans against a frictionless wall at a 45° angle, as shown.
(a) Find the directions of the torques acting on the pole due to (*i*) the floor (*ii*) gravity (*iii*) the wall, all with respect to the point where the pole rests on the floor.
(b) Then find the magnitudes of the normal force N, the frictional force F_f, and the wall's normal force F_W, all as multiples of the gravitational force mg.

Solution: (a) (*i*) The floor exerts both normal and frictional forces on the pole, but the torques due to each of these forces is *zero*, because the radius vector **r**, extending from the origin to the point at which these forces act, is zero. (*ii*) The radius vector **r** from the origin to the center of mass of the pole (which is the effective action point of gravity) is directed toward the upper right in the picture, and has length $\ell/2$, where ℓ is the length of the pole. If **r** is moved parallel to itself up along the pole until its tail reaches the CM of the pole, and then turned toward the downward gravitational force $m\mathbf{g}$, the right-hand rule shows that the gravitational torque is *into* the page. (*iii*) The radius vector **r** from the origin to the point at which the pole touches the wall is also directed toward the upper right, and has length ℓ. If **r** is moved parallel to itself up along the pole until its tail reaches the top of the pole, and then turned toward the normal force due to the wall, the right-hand rule shows that the gravitational torque is *out of* the page.

(b) If the pole is in equilibrium, the gravitational torque into the page must be balanced by the wall-torque out of the page. Therefore $(\ell/2)mg\cos 45^\circ = (\ell)F_W \sin 45^\circ$, so $F_W = mg/2$, since $\cos 45^\circ = \sin 45^\circ$. Balancing forces in the horizontal direction, we must have $F_f = F_W$, and balancing forces in the vertical direction we must have $N = mg$. Therefore the various forces are

$$N = mg, \qquad F_f = F_W = mg/2.$$

In *Sample Problem* 3, torques were calculated about the pole's point of contact with the floor. What if we used some other point instead, like the center of the pole or the point of contact with the wall? Torques due to individual forces would clearly be different, but would the total torque still be zero?

Consider a rigid body in equilibrium, and an origin O as shown. A number of forces $\mathbf{F}_i$ act on the body. The body is in equilibrium, so the sum of the forces on it must be zero,

$$\sum \mathbf{F}_i = 0.$$

The torque due to the ith force is $\tau_i = \mathbf{r}_i \times \mathbf{F}_i$, where $\mathbf{r}_i$ is the vector from the origin to the place where the force is applied. The body is in equilbrium, so the sum of the torques is also zero,

$$\sum \tau_i = \sum \mathbf{r}_i \times \mathbf{F}_i = 0 .$$

Now consider the same body, but with a different origin. The new origin O' is a distance $\mathbf{R}$ from O, and by vector addition, $\mathbf{r}_i = \mathbf{R} + \mathbf{r}_i'$ where $\mathbf{r}_i'$ is the vector distance from the new origin to the place where the ith force is applied.

The total torque about the old origin can therefore be written

$$\sum \tau_i = \sum \mathbf{r}_i \times \mathbf{F}_i = \sum \left(\mathbf{R} + \mathbf{r}_i'\right) \times \mathbf{F}_i$$

$$= \sum \mathbf{R} \times \mathbf{F}_i + \sum \mathbf{r}_i' \times \mathbf{F}_i = 0.$$

The first term is $\sum \mathbf{R} \times \mathbf{F}_i = \mathbf{R} \times \sum \mathbf{F}_i$, since $\mathbf{R}$ is the same for all forces, so can be brought outside the sum sign. The sum $\sum \mathbf{F}_i = 0$, however, since the body is in equilibrium. Therefore the total torque about the new origin is

$$\sum \tau_i' = \sum \mathbf{r}_i' \times \mathbf{F}_i = 0.$$

If a body is in equilibrium, so that the sum of the forces on it is zero and the sum of the torques about some origin is also zero, then the sum of the torques about any origin is zero.

The reader can easily repeat *Sample Problem* 3 using alternative origins, such as the center of the pole or the point of contact with the wall, for the torque calculations. The magnitudes and directions of individual torques will be different, but the magnitudes of N, F_f, and F_W will turn out to be exactly the same as before.

20.3 The relation between torque and angular momentum vectors

During *Day Seventeen* we showed that torque causes angular momentum to change according to the relationship $\tau = \frac{dL}{dt}$. Now that we have promoted angular momentum and torque into the class of vectors, we can write the equation in vector form,

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}. \tag{5}$$

If the torque on a wheel is due to the force shown, for example, the torque direction is out of the page, so the angular momentum will change in a direction out of the page as well. If the wheel starts at rest, it will turn faster and faster in the counterclockwise direction consistent with the right-hand rule. We designate the direction "out of the page" by the symbol $\odot$ (supposed to represent the tip of an arrow approaching you). Similarly, we designate "into the page" by the symbol $\otimes$ (supposed to represent the feathers at the rear of the arrow as it departs from you). We take both $\odot$ and $\otimes$ to be *unit* vectors, designating directions and having unit length.

The cross-product definition of torque and the relationship $\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$ allows us to write an expression for the angular momentum of a *single particle* in a particularly simple way. During *Day Sixteen* we found that the moment of inertia of a single particle of mass m is $I = mr^2$, where r is the perpendicular distance of the particle from the axis of rotation. The magnitude of the angular momentum of a single particle is therefore

$$L = I\omega = mr^2\omega, \tag{6}$$

and the vector is directed perpendicular to the plane of rotation of the particle, by the right-hand rule.

The tangential velocity of the particle, the component of $\mathbf{v}$ perpendicular to the radius vector $\mathbf{r}$, is $v_{\tan} = v_{\perp} = r\omega$, so the magnitude of the angular momentum can also be written

$$L = mr\ v_{\perp} = r\ p_{\perp}, \tag{7}$$

where $p_\perp$ is the tangential momentum, the component of the momentum $\mathbf{p}$ perpendicular to the radius vector $\mathbf{r}$. If ϕ is the angle between the two vectors, then the component of $\mathbf{p}$ perpendicular to $\mathbf{r}$ is $p \sin\phi$, so the magnitude of the angular momentum of a single particle can also be written

$$L = r\,p\,\sin\phi \tag{8}$$

which is exactly the magnitude of the vector

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}\,. \tag{9}$$

We can check that the angular momentum of a single particle really is $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ (rather than the other possible choice $\mathbf{p} \times \mathbf{r}$) using $\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$ to relate the change in angular momentum to the torque on the particle. The time derivative of $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \left(\frac{d\mathbf{r}}{dt}\right) \times \mathbf{p} + \mathbf{r} \times \left(\frac{d\mathbf{p}}{dt}\right), \tag{10}$$

using the product rule for derivatives.* Here $\frac{d\mathbf{r}}{dt}$ is the velocity $\mathbf{v}$ of the particle (which is obviously always parallel to its momentum $\mathbf{p} = m\mathbf{v}$), so the first cross product on the right is *zero*. According to Newton's second law the time rate of change of the particle's momentum is given by $\frac{d\mathbf{p}}{dt} = \mathbf{F}$, where $\mathbf{F}$ is the net force on the particle, so Eq. 10 becomes

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F}\,. \tag{11}$$

The torque vector is $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$, so we have verified that $\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$ is consistent with $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ for a single particle.

* See Problem 20P23 for a suggestion on how to prove that the product rule for vector cross products, $\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \frac{d\mathbf{A}}{dt} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{dt}$, holds for any differentiable vectors $\mathbf{A}$ and $\mathbf{B}$.

Sample Problem 4. Find the angular momentum of a single particle moving in a straight line with constant momentum **p**, for various choices of the origin.

Solution: It may seem strange at first, but a particle moving in a straight line can have angular momentum. The angular momentum of a particle depends on the particle's position vector **r** as well as its linear momentum **p**, so **L** depends on the choice of origin. Suppose for example the particle is traveling to the right, as shown below. Then if we choose the origin to be directly behind (or directly in front of) the particle, the position vector **r** is parallel (or antiparallel) to the momentum vector **p**, so the angular momentum is zero, as shown in Fig. A below. If we instead choose the origin to be on one side of the velocity line, the angular momentum $\mathbf{r} \times \mathbf{p}$ points into the page, as shown in Fig. B, and if the origin is on the other side of the velocity line, the angular momentum points out of the page, as shown in Fig. C.

The angular momentum of a particle moving to the right (A) if the origin is directly behind the particle; (B) if the origin is on one side of the particle's velocity line; (C) the origin is on the other side of the velocity line.

20.4 Cross products in general

We write the cross product of two arbitrary vectors $\mathbf{A}$ and $\mathbf{B}$ as $\mathbf{A} \times \mathbf{B}$, with *magnitude*

$$|\mathbf{A} \times \mathbf{B}| = AB \sin\phi, \tag{12}$$

where A and B are the magnitudes of the vectors $\mathbf{A}$ and $\mathbf{B}$, and ϕ is the angle between $\mathbf{A}$ and $\mathbf{B}$. The *direction* of $\mathbf{A} \times \mathbf{B}$ is determined by the right-hand rule, as illustrated for $\mathbf{A} = \mathbf{r}$ and $\mathbf{B} = \mathbf{F}$ in Fig. 2. If $\mathbf{A}$ and $\mathbf{B}$ are parallel to one another the cross product is zero, since the angle ϕ between them is zero, so $\sin\phi = 0$. For given magnitudes A and B, the cross product has a maximum magnitude if $\mathbf{A}$ and $\mathbf{B}$ are perpendicular to one another, because then $\sin\phi = \pm 1$. Reversing the order of multiplication,

$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B} \tag{13}$$

because even though the magnitudes of the cross products are the same, the directions are opposite according to the right-hand rule. Cross products therefore do not *commute*; the order of multiplication makes a difference.

We can also find the cross products of the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, vectors of unit length in the x, y, and z directions. Any one of these vectors crossed into itself must be zero, because each is parallel to itself. That is,

$$\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0. \tag{14a}$$

The cross product of two different unit vectors is the third unit vector (or its negative):

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \qquad \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}} \qquad \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \tag{14b}$$

$$\hat{\mathbf{y}} \times \hat{\mathbf{x}} = -\hat{\mathbf{z}} \qquad \hat{\mathbf{z}} \times \hat{\mathbf{y}} = -\hat{\mathbf{x}} \qquad \hat{\mathbf{x}} \times \hat{\mathbf{z}} = -\hat{\mathbf{y}} \tag{14c}$$

For example, the cross product of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ is $\hat{\mathbf{z}}$, as shown, while the cross product of $\hat{\mathbf{y}}$ and $\hat{\mathbf{x}}$ is $-\hat{\mathbf{z}}$.

Using these rules for unit vectors, we can find an expression for the cross product of two arbitrary vectors $\mathbf{A}$ and $\mathbf{B}$ expressed in terms of Cartesian components. That is, if

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}} \quad \text{and} \quad \mathbf{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}},$$

then

$$\mathbf{A}\times\mathbf{B} = \hat{\mathbf{x}}(A_yB_z - A_zB_y) + \hat{\mathbf{y}}(A_zB_x - A_xB_z) + \hat{\mathbf{z}}(A_xB_y - A_yB_x). \tag{15}$$

For example, if $\mathbf{A} = A_x\,\hat{\mathbf{x}}$ and $\mathbf{B} = B_y\,\hat{\mathbf{y}}$, then

$$\mathbf{A}\times\mathbf{B} = \hat{\mathbf{x}}(0) + \hat{\mathbf{y}}(0) + \hat{\mathbf{z}}(A_xB_y) = A_xB_y\hat{\mathbf{z}},$$

which has the magnitude and direction we would expect.

There is a compact way to write Eq. 15. Expanding the determinant shown below about the first row, we get

$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{\mathbf{x}}(A_yB_z - A_zB_y) - \hat{\mathbf{y}}(A_xB_z - A_zB_x) + \hat{\mathbf{z}}(A_xB_y - A_yB_x), \tag{16}$$

which is the same as the right-hand side of Eq. 15. The cross product may therefore be written as a determinant:

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}, \tag{17}$$

which is easier to remember than Eq. 15.

It is straightforward to show that the cross product of two vectors obeys the "distributive law"

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C} \tag{18}$$

as can be shown using either Eq. 15 or Eq. 17. (See Problem 20P21)

Sample Problem 5.

Find the cross product $\mathbf{A} \times \mathbf{B}$ where $\mathbf{A} = 2\hat{\mathbf{x}} - 3\hat{\mathbf{y}} + \hat{\mathbf{z}}$ and $\mathbf{B} = \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$.

Solution:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 2 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \hat{\mathbf{x}} \begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix} - \hat{\mathbf{y}} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -4\hat{\mathbf{x}} - \hat{\mathbf{y}} + 5\hat{\mathbf{z}}.$$

Sample Problem 6: Find the torque exerted on a flagpole by a cable attached at height h up the pole, and attached also to the ground at point $(x, y) = (0, h)$, as shown. The origin of coordinates is at the base of the pole, and the tension in the cable is T.

Solution: The force due to the cable is directed along the cable, so the force triangle is similar to a triangle formed from the distances involved, as shown. The vertical component of the cable's force on the pole is therefore $F_z = -T \cos 45^\circ = -T/\sqrt{2}$, negative because it is downward. The y-component of the force is $F_y = T/\sqrt{2}$, and the x-component is zero. The torque exerted on the pole by the cable is therefore

$$\mathbf{r} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & h \\ 0 & T/\sqrt{2} & -T/\sqrt{2} \end{vmatrix} = \hat{\mathbf{x}} \begin{vmatrix} 0 & h \\ \frac{T}{\sqrt{2}} & -\frac{T}{\sqrt{2}} \end{vmatrix} - \hat{\mathbf{y}} \begin{vmatrix} 0 & h \\ 0 & -\frac{T}{\sqrt{2}} \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} 0 & 0 \\ 0 & \frac{T}{\sqrt{2}} \end{vmatrix} = -\frac{hT}{\sqrt{2}}\hat{\mathbf{x}}.$$

The torque is in the negative x-direction, as is readily seen from the right-hand rule, and has the correct magnitude $\tau = h\,F_\perp = hT/\sqrt{2}$.

Sample Problem 7: A uniform cube with edge-length d is held upright by hinges at A and B, as shown. The cube has weight $\mathbf{W}$. Although the force $\boldsymbol{A}$ exerted by hinge A has a vertical component, the force $\boldsymbol{B}$ exerted by hinge B does not. Find $\boldsymbol{A}$ and $\boldsymbol{B}$.

Solution: First we give a systematic solution, and then a more informal solution.

Systematic Solution: Using the Cartesian coordinates shown in the diagram, the forces exerted on the cube are

$$\mathbf{A} = (A_x, A_y, A_z),\ \mathbf{B} = (B_x, B_y, 0), \text{ and the weight } \mathbf{W} = (0, 0, -W).$$

The radius vectors from the origin to the points where the forces are applied are

$$\mathbf{r}_A = (0, 0, 0),\ \ \mathbf{r}_B = (0, 0, d)\text{, and } \mathbf{r}_W = \left(\frac{d}{2}, \frac{d}{2}, \frac{d}{2}\right)\text{, the location of the CM.}$$

The torque exerted by each force $\mathbf{F}$ has the form $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$, so $\boldsymbol{\tau}_A = 0$ (since $\mathbf{r}_A = 0$), and

$$\boldsymbol{\tau}_B = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x_B & y_B & z_B \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & d \\ B_x & B_y & 0 \end{vmatrix} = \hat{\mathbf{x}}\begin{vmatrix} 0 & d \\ B_y & 0 \end{vmatrix} - \hat{\mathbf{y}}\begin{vmatrix} 0 & d \\ B_x & 0 \end{vmatrix} + \hat{\mathbf{z}}\begin{vmatrix} 0 & 0 \\ B_x & B_y \end{vmatrix}$$

$$= d\,[-B_y\hat{\mathbf{x}} + B_x\hat{\mathbf{y}}],\text{ and similarly } \boldsymbol{\tau}_W = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ d/2 & d/2 & d/2 \\ 0 & 0 & -W \end{vmatrix} = W\,d/2\,(\hat{\mathbf{y}} - \hat{\mathbf{x}}).$$

For the cube to remain in equilibrium, the net force in each direction must be zero:

F_x: $\quad A_x + B_x = 0$

F_y: $\quad A_y + B_y = 0$

F_z: $\quad A_z - W = 0,$

so $A_z = W$, with two equations remaining in the four unknowns A_x, B_x, A_y, B_y.

The net *torque* in each direction must also be zero. There are no torques in the z-direction, and

$$\tau_x: \qquad -d(B_y) - d\,W/2 = 0$$

$$\tau_y: \qquad d(B_x) + d\,W/2 = 0,$$

so $B_x = B_y = -W/2$. Therefore also $A_x = A_y = +W/2$ from the force equations. Altogether,

$$\mathbf{A} = (W/2,\ W/2,\ W) \text{ lbs, and } \mathbf{B} = (-W/2, -W/2,\ 0) \text{ lbs.}$$

Informal Solution:

We can see right away that the vertical (z) component of $\mathbf{A}$ has to be $A_z = W$, to support the cube's weight. From the symmetry of the problem, the x and y directions are equivalent, so $A_x = A_y$ and $B_x = B_y$. To balance forces in the x and y directions, we must have $B_x = -A_x$ and $B_y = -A_y$.

If we compute torques about the origin of coordinates, the torque due to $\mathbf{A}$ is zero, and the torque due to $\mathbf{B}$ has the magnitude $\tau_B = B \cdot d$, since the radius to the point where $\mathbf{B}$ is applied is d, and is perpendicular to $\mathbf{B}$. This must balance the torque due to the cube's weight, which is $\tau_W = W(d/\sqrt{2})$ in magnitude, since the component of the radius vector which is perpendicular to the $\mathbf{W}$ vector is $d/\sqrt{2}$. Equating the two magnitudes, we get $B = W/\sqrt{2}$.

The vector $\mathbf{B}$ has equal components in the $-x$ and $-y$ direction, so we find

$$\mathbf{B} = (-W/2, -W/2, 0), \text{ so then } \mathbf{A} = (W/2,\ W/2,\ W),$$

using equations in the first paragraph. With practice, the informal approach is often faster and more interesting, because one can use physical insight. The systematic approach is especially valuable for problems without symmetry, or as a straightforward if somewhat tedious approach requiring a minimum of thought.

20.5 Orbital angular momentum and spin angular momentum

A pool ball rolls along a table. We already know, from *Day Nineteen*, that its kinetic energy can be split into two parts. Its *translational* kinetic energy is the energy it would have if it were simply skidding without turning as it translates along the pool table. Its *rotational* kinetic energy is the energy it would have if its center were at rest, while the ball rotates about it. The sum of these kinetic energies is the total kinetic energy of the ball relative to the table.

$$E = E_{\text{trans}} + E_{\text{rot}}$$

Can we similarly split the *angular momentum* of an object into two parts? Can we say that a rolling pool ball has an angular momentum due to its translational motion along the table, plus an angular momentum due to its rotation about its own center of mass?

$$\mathbf{L} \stackrel{?}{=} \mathbf{L}_{\text{trans}} + \mathbf{L}_{\text{rot}}$$

Pick a point on the pool table as origin. The angular momentum of a single particle (atom or molecule, say) in a pool ball about the origin is $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. We can sum up the angular momenta of all the single particles to get the total angular momentum of the ball.

The location of the ith particle relative to the origin is the vector sum $\mathbf{r}_i = \mathbf{R}_{\text{CM}} + \mathbf{r}_i'$, the sum of the vector distance to the center of mass of the ball and the vector distance from the center of mass to the particle, as shown. Similarly, the velocity of the particle relative to the origin is the vector sum $\mathbf{v}_i = \mathbf{V}_{\text{CM}} + \mathbf{v}_i'$, the sum of the velocity of the center of mass relative to the table and the velocity of the particle relative to the center of mass. The total sum is therefore

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i = \sum_i (\mathbf{R}_{\text{CM}} + \mathbf{r}_i') \times m_i (\mathbf{V}_{\text{CM}} + \mathbf{v}_i'), \tag{19}$$

the vector sum of the angular momenta of all of the particles in the ball. This expression contains four terms when we multiply out the cross product.

(i) The first term in Eq.19 is $\mathbf{R}_{CM} \times \mathbf{P}_{CM}$, since $\mathbf{P}_{CM} = M\mathbf{V}_{CM}$, where $M = \sum m_i$ is the total mass of the ball. It is the cross product of the position vector of the center of mass with the total momentum of the system.

(ii) The term $\mathbf{R}_{CM} \times \sum_i m_i \mathbf{v}_i' = 0$, since by definition the total momentum $\sum_i m_i \mathbf{v}_i'$ of the system in the CM frame is *zero*.

(iii) The term $\left(\sum_i m_i \mathbf{r}_i'\right) \times \mathbf{V}_{CM} = 0$ as well, because the sum $\sum_i m_i \mathbf{r}_i' = 0$ in the center of mass frame, by the definition of center of mass. That is, the location of the center of mass of a system, measured from any origin, is by definition $\mathbf{R}_{CM} = \sum_i m_i \mathbf{r}_i / M$, where M is the total mass of the system. If the origin is located at the center of mass itself, then $\mathbf{R}_{CM} = 0$, so $\sum_i m_i \mathbf{r}_i = 0$ as well.

(iv) The final term is $\sum_i \mathbf{r}_i' \times \mathbf{p}_i'$, the sum of the angular momenta of the particles in the CM frame. This is the "internal" angular momentum of the system, independent of the origin of coordinates we used to define $\mathbf{P}_{CM}$.

In summary, we've found that the total angular momentum of the system of particles splits neatly into two parts:

$$\mathbf{L} = \mathbf{R}_{CM} \times \mathbf{P}_{CM} + \sum_i \mathbf{r}_i' \times \mathbf{p}_i' \quad (20)$$

The first term in Eq. 20 is the angular momentum the system would have if all of its mass were placed at the CM. It is similar to the angular momentum of a single particle, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. We call this term the *orbital* angular momentum. If $\mathbf{P}_{CM}$ is the momentum of Earth relative to the Sun, for example, then $\mathbf{R}_{CM} \times \mathbf{P}_{CM}$ is the orbital angular momentum of Earth around the Sun, where $\mathbf{R}_{CM}$ extends from the Sun to the center of Earth.

The second term, the angular momentum about the CM, is called the *spin* angular momentum. Suppose the object is rigid, so that the momentum of a single particle is

perpendicular to its radius vector, as shown. Therefore the magnitude of $\mathbf{r}_i \times m_i\mathbf{v}_i$ is just $m_i(r_i)^2\omega$, since $v_i = r_i\omega$, where the angular velocity ω is the same for all particles in the object. The spin angular momentum is therefore

$$\sum_i \mathbf{r}_i \times \mathbf{p}_i = \left[\sum_i m_i(r_i)^2\right]\boldsymbol{\omega} \equiv I'\boldsymbol{\omega}, \tag{21}$$

where I' is the moment of inertia of the system about its center of mass. So the total angular momentum of a rigid body is

$$\mathbf{L} = \mathbf{R}_{CM} \times \mathbf{P}_{CM} + I'\boldsymbol{\omega}, \tag{22}$$

the sum of its orbital and spin angular momenta.

Sample Problem 8. Viewed from an inertial frame centered on the Sun, Earth both translates and rotates. Find the ratio of its spin and orbital angular momenta.

Solution: As mentioned in *Sample Problem* 1 of *Day Nineteen*, our planet has rotational inertia $I = 0.33MR^2$ about its center of mass, where M is its mass and R its radius. The rotational angular velocity about its north-south pole axis has the magnitude

$$\omega = \frac{2\pi}{24 \text{ hrs}} = 7.3 \cdot 10^{-5} \text{ s}^{-1},$$

and Earth's speed around the Sun is

$$v_e = \frac{2\pi r}{1\text{ year}} = \frac{2\pi(1.50 \cdot 10^8\text{ km})}{3.16 \cdot 10^7\text{ s}} = 30\text{ km/s}.$$

The ratio of the magnitudes of the spin and orbital angular momenta is therefore

$$\frac{L_{\text{spin}}}{L_{\text{orbital}}} = \frac{I'\omega}{r\,p_{CM}} = \frac{0.33\,R^2\omega}{r\,v_e} = \frac{0.33\,(6.4\cdot 10^3\text{ km})^2\,(7.3\cdot 10^{-5}\text{ s}^{-1})}{1.50\cdot 10^8\text{ km}\,(30\text{ km/s})} = 2.2\cdot 10^{-7}.$$

Relative to the Sun, nearly all of Earth's angular momentum is orbital.

Sample Problem 9. A penny of mass M and radius R rolls without slipping on a table. What fraction of its angular momentum is *spin* angular momentum?

Solution: The penny is (to a good approximation) a disk, with moment of inertia $\frac{1}{2}MR^2$. Its spin angular momentum magnitude is $L_{\text{spin}} = I\omega = \frac{1}{2}MR^2\omega = \frac{1}{2}MRv$, since it rolls without slipping, so the speed v of its center is related to its angular velocity ω by $v = R\omega$. The vector $\mathbf{L}_{\text{spin}}$ points into the page as the penny rolls to the right, as shown above.

The penny's *orbital* angular momentum $\mathbf{L}_{\text{orb}} = \mathbf{R}_{CM} \times \mathbf{P}_{CM}$ *depends upon which point is chosen as origin*, because $\mathbf{R}_{CM}$ extends from the origin to the center of the penny. If the origin is chosen to be at some point on the table, then the component of $\mathbf{R}_{CM}$ perpendicular to $\mathbf{P}_{CM}$ is just the radius of the penny, as shown. The orbital angular momentum in this case has the magnitude RMv pointed into the page, twice the spin angular momentum; for this origin, the spin angular momentum is *one third* of the total angular momentum. If instead the origin is chosen to be at some point of height R above the table, then $\mathbf{R}_{CM}$ is parallel (or antiparallel) to $\mathbf{P}_{CM}$, so the orbital angular momentum is *zero*. For such an origin, 100% of the angular momentum is spin angular momentum.

While batting a baseball, there is a spot on the bat where the batter would *like* the ball to hit, because the impact then feels especially easy on the hands and wrists, and the ball is likely to travel farther. This point is called the "sweet spot", or (more formally) the "center of percussion". There is a similar spot on a tennis racquet, where the impact on the player's hand is minimized and the ball is more likely to be hit accurately.

Suppose a uniform rod likes horizontally on a frictionless sheet of ice, and a ball skids along the ice and strikes the rod. What happens to the rod?

If the ball hits the center of the rod, the rod will be pushed forward, as shown, but it will not rotate.

If the ball strikes a different point, the center of the rod will still move forward, and the rod will now also rotate, as shown. The farther toward the end of the rod the ball strikes, the more the rod will rotate when struck.

Note that it is possible that if the ball strikes the rod in just the right place, then the far end of the rod (the "upper" end in the pictures) will *not move* during the collision! The far end *will* move later, as the rod translates and rotates. But briefly, during the actual contact between rod and ball, that end is stationary. So if a batter were holding the rod at that end, and the ball struck that particular spot, the batter would not feel a "sting" as the ball struck.

The point where the ball strikes is the sweet spot for such a batter. If the batter held the bat at a different point, slightly away from the end, as batters really do, then the sweet spot would be slightly different as well.

Sample Problem 10. Locate the sweet spot on a uniform rod of length d, if the rod is held at one end.

Solution: We can find the motion of the rod using the fundamental laws $F = \frac{dp}{dt}$ and $\tau = \frac{dL}{dt}$. Let the sweet spot be some unknown distance ℓ from the held end, as shown.

The impact of the ball is so great during the brief encounter that we can neglect the force due to the batter's hand. In any case, we choose d to minimize that force.

Integrating $F = \frac{dp}{dt}$ over time, we have the impulse equation (from Day Four)

$$I = \int F\,dt = \int dp = \Delta p = MV_{\text{CM}}$$

where M is the mass of the rod and V_{CM} is the velocity of its center of mass after the impact. Integrating $\tau = \frac{dL}{dt}$ in a similar way,

$$\int \tau\,dt = \Delta L = L_{\text{orbital}} + L_{\text{spin}}$$

$$= M\frac{d}{2}V_{\text{CM}} + I'\omega = M\frac{d}{2}V_{\text{CM}} + \frac{1}{12}Md^2\omega$$

where we have taken the origin to be at the "upper" end of the rod, where the batter's hands are located. About that point the ball exerts the torque $\tau = \ell F$, since the ball strikes a distance ℓ from the origin. Therefore

$$\int \tau\,dt = \int \ell F\,dt = \ell\int F\,dt = M\frac{d}{2}V_{\text{CM}} + \frac{1}{12}Md^2\omega.$$

The upper end of the rod will not move instantaneously if we require that $V_{\text{CM}} = \frac{d}{2}\omega$; that is, $\frac{d}{2}\omega$ is the speed a point on the bat would have if the bat were rotating about a fixed end, a distance $d/2$ away. Thus

$$\ell\int F\,dt = M\frac{d}{2}V_{\text{CM}} + \frac{1}{12}Md^2\frac{2V_{\text{CM}}}{d} = MdV_{\text{CM}}\left(\frac{1}{2}+\frac{1}{6}\right) = \frac{2}{3}MdV_{\text{CM}}$$

The impulse equation gave $\int F\,dt = MV_{\text{CM}}$, so $\ell = \frac{2}{3}d$.

If one end of a uniform rod remains instantaneously stationary when the rod is struck by a ball, the ball strikes a point two-thirds of the way from that end.

20.6 A summary of conservation laws

We are now in a position to summarize all the fundamental conservation laws of mechanics.

(1) *Mass*

In classical mechanics mass is always conserved. Mass is conserved in collisions, for example; the final bodies may be different than the initial bodies, but the total mass after the collision is the same as it was before.

(2) *Momentum*

The momentum of a particle is $\mathbf{p} = m\mathbf{v}$, the product of its mass and velocity. The total momentum $\mathbf{P}$ (sometimes called the total *linear* momentum) of a system is the vector sum of the momenta of all the particles in the system. The total momentum is conserved if there is *no net external force* on the system. There may be internal forces between two parts of the system, but these cancel out (by Newton's third law) for the system as a whole. If the net external force is not zero, the total momentum changes according to Newton's second law $\mathbf{F}_{\text{ext}} = \dfrac{d\mathbf{P}}{dt}$.

(3) *Angular Momentum*

The angular momentum of a particle about some origin is $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, the product of its position vector measured from the origin, with its momentum $\mathbf{p} = m\mathbf{v}$. The total angular momentum of a system is the sum of the angular momenta of all the particles in the system. If the system is a rigid body, for example, its angular momentum is $\mathbf{L} = \mathbf{R}_{\text{CM}} \times \mathbf{P}_{\text{CM}} + I\,\omega$, the sum of its orbital and spin angular momenta. The total angular momentum of a system is conserved if there is *no net external torque* on the system. There may be internal torques due to forces between two parts of the system, but these cancel out for the system as a whole. If the net external torque is not zero, the total angular momentum changes according to $\tau_{\text{ext}} = \dfrac{d\mathbf{L}}{dt}$.

(4) *Energy*

The "mechanical energy" of a single particle is the sum of its kinetic energy $\frac{1}{2}mv^2$ and its potential energy U. The mechanical energy of a system is the sum of the kinetic and potential energies. The kinetic energy of a rigid body is $\frac{1}{2}M v_{\text{CM}}{}^2 + \frac{1}{2}I\omega^2$, the sum of its translational and rotational kinetic energies. The mechanical energy of a system is conserved if (a) *no net external work* is done on the system, and if (b) *there is no frictional energy loss* within the system. In a larger sense, if we count the heat produced by internal friction as a form of energy, the overall energy of the system is conserved if no net external work is done. Kinetic energy is conserved during ideal, *elastic* collisions. Kinetic energy is not conserved in inelastic collisions, in which heat is produced or there is energy involved in deforming the colliding objects.

Sample Problem 11. A thin rod of mass M and length ℓ lies at rest on a frictionless sheet of ice. A small ball of putty, also of mass M, moves along the ice with velocity v_0 perpendicular to the rod, aimed to hit one end of the rod, as shown. When they meet, the putty sticks to the rod in a completely inelastic collision.

(a) What quantities are conserved in the collision?
(b) After the collision, describe the motion of the rod + putty system, find the center-of-mass velocity of the system, and find the angular velocity of the system about its CM.

Solution: (a) There are no horizontal external forces on rod-plus-putty system, and the gravitational and normal force from the ice cancel one another in the vertical direction. There is therefore no net external force on the system, so the total momentum of the system is conserved. Also there is no net external torque on the system, so the total angular momentum of the system is conserved as well. *Kinetic energy is not conserved, because the collision is inelastic; we expect that some of the kinetic energy will be converted into internal energy of the ball and rod.*

(b) Conservation of momentum means that the CM of the system continues to move to the right after the collision, with the same velocity it had before the collision. The CM is located halfway between the putty and the center of the rod, because the putty and rod have equal mass. Conservation of angular momentum means that the system will rotate about its CM after the collision, as shown.

Before the collision, the total momentum of the system is Mv_0, the momentum of the putty alone. After the collision the momentum is $2MV_{CM}$, where V_{CM} is the velocity of the center of mass of the system. Conservation of momentum then tells us that $V_{CM} = v_0 / 2$.

We can find the angular velocity of the system after the collision by conserving overall angular momentum. The angular momentum of the system depends upon which point we choose as origin, but the value found for the final angular velocity cannot depend upon this arbitrary choice.

We need to select some origin which remains fixed relative to the ice, so we can calculate consistently in the same inertial frame. If we choose an origin located on the ice just underneath the end of the rod which will be struck by the putty, the initial angular momentum $\mathbf{r} \times \mathbf{p}$ of the putty is *zero*, because the position and momentum vectors of the putty are antiparallel to one another. The rod has no initial angular momentum either, so the total angular momentum of the system is zero initially, about this fixed point on the ice.

After the collision, the system has both an orbital and a spin angular momentum; these angular momenta must exactly cancel one another to conserve overall angular momentum. The CM of the system, which is a distance $\ell/4$ from the origin when the collision takes place, moves with speed $v_O/2$, as shown above. The orbital angular momentum of the system about the chosen origin is therefore

$$\mathbf{L}_{orb} = \mathbf{R}_{CM} \times \mathbf{P}_{CM} = \frac{\ell}{4}\, 2MV_{CM}\otimes \;=\; \frac{\ell M v_O}{4}\otimes$$

directed *into* the page. The spin angular momentum of the system is $I'\omega$, its moment of inertia about the CM multiplied by the angular velocity of rotation about the CM. The moment of inertia of the system about the CM is the moment of inertia of the putty plus the moment of inertia of the rod. The moment of inertia of the putty is $M(\ell/4)^2$, the product of its mass and the square of its distance from the system CM, assuming the putty is essentially a point mass. The moment of inertia of the rod about its center is $M\ell^2/12$, so by the parallel-axis theorem, its moment of inertia about the system CM, a distance $\ell/4$ from the rod's center, is $M\ell^2/12 + M(\ell/4)^2$. Adding in the moment of inertia of the putty $I_p = M(\ell/4)^2$ about the system CM, we find that the overall moment of inertia of the system is $I' = \frac{1}{12}M\ell^2 + 2M\left(\frac{\ell}{4}\right)^2 = \frac{5}{24}M\ell^2$. The spin angular momentum is therefore

$$\mathbf{L}_{spin} = \frac{5}{24}M\ell^2\,\boldsymbol{\omega}$$

directed *out* of the page, corresponding to a counterclockwise rotation of the system about its center of mass. We know that $\mathbf{L}_{spin}$ is directed out of the page for two reasons:
(i) It makes sense that the rod rotate counterclockwise if it is hit from the left at its lower end.
(ii) The spin angular momentum must be opposite to the orbital angular momentum, so they can cancel one another, conserving overall angular momentum in the collision. Because they must cancel one another, their magnitudes have to be equal:

$$\frac{\ell M v_O}{4} = \frac{5}{24}M\ell^2\omega,$$

The angular velocity of the system after the collision is therefore

$$\omega = \frac{6}{5}\frac{v_O}{\ell}.$$

State of the Search

1. The angular velocity vector $\boldsymbol{\omega}$ specifies the orientation direction of a spinning wheel, as well as the magnitude of the angular velocity and the rotation direction. The direction of $\boldsymbol{\omega}$ is found by a *right hand rule*, curling the fingers of the right hand in the direction of rotation, with the thumb in the direction of $\boldsymbol{\omega}$.

2. The angular momentum vector of an object is $\mathbf{L} = I\boldsymbol{\omega}$, where I is the object's moment of inertia. The angular momentum of a single particle can be written as a cross product $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, in terms of its location $\mathbf{r}$ from an origin, and its momentum $\mathbf{p}$. The cross product of arbitrary vectors $\mathbf{A}$ and $\mathbf{B}$ has the magnitude $AB\sin\phi$ (where ϕ is the angle between $\mathbf{A}$ and $\mathbf{B}$), and a direction perpendicular to both $\mathbf{A}$ and $\mathbf{B}$ given by the right-hand rule.

3. Torque can also be written as a vector. The torque due to a force $\mathbf{F}$ acting at a point $\mathbf{r}$ from the origin is $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. The net torque on an object causes its angular momentum to change, as given by $\boldsymbol{\tau} = \dfrac{d\mathbf{L}}{dt}$.

4. An object is in equilibrium if and only if both the net torque and the net force on it are zero. If the net force is zero and the net torque is zero about some particular axis, the net torque is zero about *any* axis. If there *is* a net torque the object starts to rotate; if there *is* a net force the center of mass of the object starts to move.

5. The total angular momentum of a system of particles is

$$\mathbf{L} = \mathbf{R}_{\mathrm{CM}} \times \mathbf{P}_{\mathrm{CM}} + \sum_i \mathbf{r}_i' \times \mathbf{p}_i' .$$

If the system is a rigid body, this becomes

$$\mathbf{L} = \mathbf{R}_{\mathrm{CM}} \times \mathbf{P}_{\mathrm{CM}} + I'\boldsymbol{\omega},$$

the sum of the *orbital* angular momentum and the *spin* angular momentum of the body about its center of mass. The total angular momentum of a body is conserved if no net external torque acts upon the body.

PICTURE QUIZ 20

1. The propeller of a single-engine private plane turns counterclockwise as seen by the pilot seated behind it. If the plane is heading east, what is the direction of the propeller's angular momentum vector?

(A) north (B) south (C) east (D) west (E) none of these

2. Electromagnetic energy flows in the $\mathbf{E} \times \mathbf{B}$ direction, where $\mathbf{E}$ is the electric field and $\mathbf{B}$ is the magnetic field. If $\mathbf{E}$ points in the z direction and $\mathbf{B}$ points in the negative y direction, in which direction is the energy flowing?

(A) $+x$ (B) $-x$ (C) $+y$ (D) $-y$ (E) $+z$ (F) $-z$

3. A force $\mathbf{F}$ acts on a flagpole, as shown. We want to find the torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ exerted by $\mathbf{F}$ on the flagpole. If the origin of coordinates is chosen to be at the *top* of the flagpole, the directions of the $\mathbf{r}$ vector and the torque vector are, respectively,

(A) down, to the right (B) up, into the page

(C) down, out of the page (D) up, to the left

(E) down, into the page.

4. A soccer ball rolls without slipping along the field. The ratio of its spin angular momentum to its orbital angular momentum about a point on the field is closest to which one of the following?

(A) $\frac{1}{3}$ (B) $\frac{2}{5}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$ (E) 1

5. A wheel is attached to one end of an axle; the other end is supported by a long, heavy, vertical chain attached to the ceiling of a garage, as shown. Holding the axle in a horizontal position, we spin the wheel very fast in a clockwise direction, as seen by a viewer at the far right. Now the axle is released, so it is supported only by the chain. What is the principal direction in which the center of the wheel is first seen by the viewer to move?

(A) into the page (B) out of the page (C) up (D) down

PROBLEM QUIZ 20

1. The force on a particle of electric charge q moving in a magnetic field $\mathbf{B}$ with velocity $\mathbf{v}$ is $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$. If the force is in the positive x direction and $\mathbf{B}$ lies in the y, z plane at 45° to the positive y axis and 45° to the negative z axis, which of the following is a possible velocity $\mathbf{v}$ of the particle?

(a) $\mathbf{v} = (3\ \hat{\mathbf{x}} - 4\ \hat{\mathbf{y}})$ m/s (b) $\mathbf{v} = (2\ \hat{\mathbf{y}} + \hat{\mathbf{z}})$ m/s (c) $\mathbf{v} = (4\ \hat{\mathbf{x}} + 2\ \hat{\mathbf{z}})$ m/s

2. A uniform pole of height $h = 10$ m and mass $M = 50$ kg is erected vertically on Mars. A ball at the bottom of the pole is held fixed in a socket on the ground. A horizontal rope extending in the $-x$ direction is attached to the top end of the pole; the rope is under tension $T_0 = 25$ N. Two wires are attached to the center of the pole. The first wire is also attached to the ground at a point 3 m from the positive x axis and 4 m from the positive y axis. The second wire is attached to the ground at a point 3 m from the positive x axis, and 4 m from the *negative* y axis, as shown. Find the tension in each of the wires.

3. A bowling ball of mass M and radius R is launched with velocity v_0 along the alley with no initial spin, as shown below. There is a coefficient of sliding friction μ_k between the ball and alley. As functions of time, find the ball's (a) spin angular momentum, (b) orbital angular momentum about a point P on the alley surface, and (c) total angular momentum about P.

Problems 4 and 5 involve gyroscopes, as discussed in the Special Topic.

4. The oceanliner Queen Elizabeth II (the "QE2") has a vane extending out each side of the ship, used to help stabilize the ship against roll in heavy seas. Each vane is mounted on an axle so it can be tilted up or down. If the ship starts to roll to starboard, for example, the vane on the

right side is tilted upward, as shown in a side view of the ship, so that the front of the vane is raised while the rear is lowered. The resulting angle of attack tends to raise the right side of the ship, because water flowing by it pushes upward on the vane. A rapidly-spinning gyroscope within the ship is used to detect the beginning of a roll, and the resulting response of the gyro is read by sensors and conveyed to motors which tilt the vanes. Explain how the gyro might be mounted, and how it would respond to a roll to starboard and a roll to port.

5. A motorcyclist sees that the road ahead curves to the right. To maintain stability in rounding the curve, the motorcycle must lean to the right (as shown in a rear view); otherwise the rider would fall over to the left. To make the motorcycle lean to the right, the biker briefly turns the handlebars to the *left*, which has the effect of making the bike lean to the right. Discuss the gyroscopic properties of the front wheel as the biker turns the handlebars to the left. Does the gyroscopic behavior *help* or *hinder* the bike's lean to the right?

Special Topic I: Gyroscopes

Gyroscopes are one of the strangest, most interesting, and most useful examples of rotational motion. They are strange because they seem to defy gravity. They are interesting, among other reasons, because a full understanding of their motion requires much of Newtonian mechanics. They are useful in gyro-compasses, in stabilizing ships, and especially in guidance systems for aircraft, missiles, and spacecraft.

A. The Effect of Gravity on a *Non-Spinning* Gyroscope

Start with a bicycle wheel mounted on an axle. Place one end of the axle on a tall stand, as shown. Without spinning the wheel, hold the axle horizontal and then release it. The apparatus falls to the floor.

Why does it fall? Because of gravity, obviously. If we turned off gravity, the wheel would stay where it is. However, that it should fall *with* gravity is not completely obvious. One end of the axle sits on the stand, after all, which exerts an *upward* force on it. Could the upward normal force N exerted by the stand be enough to support the wheel so it won't fall? That is, the net force on the wheel would be zero if $N = mg$, so according to $F_{\text{net}} = ma_{\text{CM}}$, Newton's second law for a system of particles, the center of mass of wheel would not accelerate, and so would stay in place. Why doesn't that happen?

Sample Problem S1. *Show why the apparatus as described above* cannot *be in equilibrium.*

Solution: Equilibrium requires both no net force and no net torque. Even though the net force on the apparatus could in principle be zero, if the upward normal force were $N = mg$, the net torque is *not* zero. Gravity exerts a torque on the apparatus about the point of support on the stand as origin. Gravity exerts the *only* torque about that point, because the normal force of the stand, acting as the origin, can exert a force but no torque. The net torque is therefore $\tau = Mgd$ directed into the page when the axle is horizontal, where d is the distance from the point of support to the wheel's center of mass.

According to the law $\tau = \frac{d\mathbf{L}}{dt}$, in a short time Δt this net torque gives the wheel an angular momentum

$$\Delta \mathbf{L} = \tau \Delta t \tag{S1}$$

directed into the page, which means the apparatus starts to rotate about the point of support; that is, it begins to fall. The torque persists in the same direction, so the apparatus continues to rotate about the point of support, which means that it continues to fall.

The CM of the wheel accelerates downward, so the net force on it must be downward as well. That is, the normal force upward due to the stand has to be *less* than the gravitational force downward.

B. the effect of gravity on a *spinning* gyroscope: precession

There is a more interesting experiment. Place one end of the axle on the stand, and firmly hold the other end so the axle is horizontal. Then spin up the bicycle wheel by slapping it tangentially over and over until it really spins fast. Then release the axle. The apparatus now does *not* fall, but instead slowly turns sideways. The axle remains horizontal, slowly turning in the horizontal plane.

Why doesn't the apparatus, which we call a *gyroscope*, fall? Why does it matter that the wheel is now spinning?

The torque on the gyroscope is the same as before, $\tau = Mgd$, directed into the page; the torque exerted by gravity doesn't depend upon whether or not the wheel is spinning. Therefore ΔL, the change in the gyro's angular momentum, is also the same as before, with magnitude

$$\Delta L = \tau \Delta t = Mgd\,\Delta t, \tag{S2}$$

directed into the page. Why then does the gyro remain horizontal rather than fall?

The only difference is that because the wheel is now spinning, the gyro has an *initial* angular momentum $\mathbf{L}_0 = I\boldsymbol{\omega}_0$ pointing along the axle, as shown in Fig. S1. Here I is the wheel's moment of inertia and $\boldsymbol{\omega}_0$ is its initial angular velocity. The direction of $\boldsymbol{\omega}_0$ and of $\mathbf{L}_0$ is found from the right-hand rule. So now after gravity has pulled on the wheel for a short time Δt, its angular momentum is

$$\mathbf{L} = \mathbf{L}_0 + \Delta\mathbf{L} = \mathbf{L}_0 + \tau\,\Delta t, \tag{S3}$$

the sum of the vector $\mathbf{L}_0$ and a very small vector $\tau\Delta t$ perpendicular to $\mathbf{L}_0$. By vector addition, the angular momentum of the gyro has now turned through a small horizontal angle, as shown from above in Fig. S1.

Fig. S1. The angular momentum initially and after time Δt. The gyro is viewed from above.

Although the direction of $\mathbf{L}$ has changed, its magnitude has remained constant. The reason for this is that when we add an infinitesimal perpendicular vector to an initial vector, the resultant vector has the same magnitude but a slightly different direction. That is, if $\Delta\mathbf{L}$ is infinitesimal, the length of the hypotenuse $\mathbf{L}$ of the triangle shown in Fig. S1 is the same as the length of the finite side $\mathbf{L}_0$. The proof of this assertion is as follows.

> At all times the angular momentum vector $\mathbf{L}$ is parallel to itself, so the square of its magnitude is $L^2 = \mathbf{L}\cdot\mathbf{L}$. The time derivative of L^2 is therefore
>
> $$\frac{d(L^2)}{dt} = \frac{d}{dt}(\mathbf{L}\cdot\mathbf{L}) = \frac{d\mathbf{L}}{dt}\cdot\mathbf{L} + \mathbf{L}\cdot\frac{d\mathbf{L}}{dt} = 2\mathbf{L}\cdot\frac{d\mathbf{L}}{dt} = 2\mathbf{L}\cdot\tau = 0$$
>
> since $\tau = \dfrac{d\mathbf{L}}{dt}$, and we have assumed that the torque τ is $\perp$ to $\mathbf{L}$ at all times. If the *square* of $\mathbf{L}$'s magnitude stays fixed, then the magnitude $|\mathbf{L}|$ stays fixed as well.

The gravitational torque keeps on acting, so the axis of rotation of the wheel keeps turning horizontally. Looked at from above, the gyro *precesses* in the counterclockwise direction at constant rate. It is also true that the center of mass of the wheel doesn't fall, so the net vertical force on it must be zero. Therefore now that the wheel is spinning, the upward normal force of the support on one end of the axle must be equal in magnitude to the downward force of gravity on the entire gyro.

Fig. S2. Precession in the short time interval Δt. The vectors are viewed from above.

How fast is this precession? Let $\Delta\phi$ be the angle of precession during the short time interval Δt, as shown in Fig. S2, where the axle is viewed from above. By definition of angles in radians, we know that

$$\Delta\phi = \frac{\Delta L}{L} = \frac{\tau\,\Delta t}{L}, \tag{S4}$$

using the narrow triangle shown (angles are the arclength subtended on the circle divided by the radius of the circle). We have used the fact that $\Delta L = \tau\,\Delta t$. The angular velocity of precession therefore has the magnitude

$$\Omega = \frac{\Delta\phi}{\Delta t} = \frac{\tau}{L} = \frac{Mgd}{I\omega} \tag{S5}$$

where I is the moment of inertia of the wheel and ω is the angular velocity of the wheel turning about the axle. As long as the wheel keeps spinning, the gyroscope keeps precessing, remaining horizontal.

It is interesting that the precession rate Ω is inversely proportional to the wheel spin rate ω. We note that as the wheel spins down due to friction, Eq. S5 predicts that the precession rate increases, and that is what we observe as well. The gyro precesses faster and faster as the wheel spins down.

Sample Problem S2. Show why a rapidly spinning gyroscope *can't* simply fall.

Solution: The initial angular momentum $\mathbf{L}_0$ of the gyroscope shown is directed to the right. If it fell as shown, its angular momentum would mostly be down and to the right, since it is rapidly spinning. (If it fell there would also be a little angular momentum directed into the page, due to the falling motion of the gyroscope, using the right hand rule.) The *primary* change $\Delta\mathbf{L}$ in angular momentum would be directed downward. However, the torque about the point of support, due entirely to gravity, is directed *into* the page, not downward. There is no torque which could make the angular momentum change *downward*. So the rapidly spinning gyroscope *can't* simply fall.

Sample Problem S3. A gyroscope wheel is a disk of mass $M = 0.12$ kg and radius $R = 6.8$ cm mounted on an axle of negligible mass, a distance $d = 9.5$ cm from the supported end of the axle. The wheel is set spinning with angular velocity $\omega = 350\ \text{s}^{-1}$, and the axle is set at angle θ to the vertical and then released. Find the gyroscope's period of precession.

Solution: The gravitational torque acting on the gyroscope is directed horizontally, and has the magnitude

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{d} \times M\mathbf{g} = dMg \sin\theta$$

$$|d \times Mg| = dMg \sin(\pi - \theta) = dMg \sin\theta$$

as shown, since θ is the angle between $\mathbf{d}$ and $M\mathbf{g}$. Projecting the angular momentum vectors at time t and at $t + \Delta t$ down onto the horizontal plane, each has magnitude $L \sin\theta$, and they are separated by the precession angle $\Delta\phi$. Therefore the precession angle is

$$\Delta\phi = \frac{\Delta L}{L \sin\theta} = \frac{\tau \Delta t}{L \sin\theta} = \frac{Mgd \sin\theta\, \Delta t}{L \sin\theta} = \frac{Mgd\, \Delta t}{L}.$$

The angle θ has cancelled out; the precession angular velocity $\Omega = \dfrac{\Delta\phi}{\Delta t} = \dfrac{\tau}{L} = \dfrac{Mgd}{I\omega}$ does not depend upon the angle the axle makes with the vertical! Numerically,

$$\Omega = \frac{Mgd}{I\omega} = \frac{Mgd}{0.5MR^2\omega} = \frac{(9.8\ \text{m/s}^2)(.095\ \text{m})}{0.5(.068\ \text{m})^2(350\ \text{s}^{-1})} = 1.15\ \text{s}^{-1}.$$

The period of precession is $T = 2\pi/\Omega = 5.5$ s.

C. Slow Rotation; Nutation

Spinning gyros precess while non-spinning gyros fall. Why do they behave so differently, even though the torque is the same on each?

There has to be an intermediate behavior. We somehow have to bridge the gap between the case where the wheel is not spinning and the gyro falls, and the case where the wheel is spinning fast, and the gyro precesses around without falling. What happens when the wheel is spinning *slowly*?

Set the gyro with one end of the horizontal axle on the stand, and spin the wheel more slowly than before. Observe that the axle starts to fall, just as though the wheel weren't spinning at all, but it then stops falling and moves sideways, i.e. starts to precess. Then the axle actually rises again, almost up to its initial height, then falls again, precessing as it goes. The motion of the free end of the axle is something like the curve shown in Fig. S3(a). Eventually

the up-and-down part of the motion damps out, but the precession remains.*

Fig. S3. Motion of the tip of a gyro if it is (a) slowly spinning (b) spinning a bit faster

Repeat the experiment with the wheel spinning a bit more rapidly; the same thing happens except the dips are shallower, as shown in Fig. S3(b), and the axle stays more nearly horizontal. That helps explain what happens when the wheel is spinning very fast, as it was in the earlier experiments. In the limit of a very fast wheel rotation rate the dips are so small as to be unnoticeable, so the top stays essentially horizontal, precessing only. This is how the gyro makes the transition between the two very different behaviors, pure falling with no precession (if $\omega = 0$), and pure precession with no falling (if ω is very large), where ω is the angular velocity of the wheel on the axle. Both of the extremes, as well as the intermediate behaviors, occur for the same gravitational torque law, but for different values of ω.

Summary

A gyroscope is a wheel on an axle free to move in three dimensions. When the wheel is spun up, and one end of the axle is supported on a stand, the subsequent strange motion is a beautiful application of the fact that a net torque $\boldsymbol{\tau}$ causes angular momentum to change, as given by

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}.$$

A torque can be exerted on the gyroscope by gravity or other forces. The response of a fast-spinning gyroscope is to precess, i.e., to turn in a direction perpendicular to the direction of the force. The angular velocity of precession of a gyroscope of mass M in a gravitational field g, supported a distance d from the center of mass of the gyro, is

$$\Omega = \frac{Mgd}{I\omega},$$

where I is the moment of inertia of the wheel and ω is its angular velocity.

In general a gyroscope exhibits both precession and nutation. For a gyro moving under the influence of gravity, precession is the horizontal rotation of the gyro's axle; nutation is the axle's alternate falling and rising.

*The vertical dips are called "nutation"; they are superimposed upon the horizontal precession.

Special Topic II. Pseudoforces in Rotating Reference Frames

Two pseudoforces act on objects viewed from the standpoint of a rotating reference frame, such as the rotating frame of the Space Station. As shown in *Special Topic* II of *Day Fifteen*, there is the *centrifugal* pseudoforce, which acts like an effective gravity pushing things outward, away from the axis of rotation. There is also the *Coriolis* pseudoforce, which acts to bend the paths of particles moving in the rotating frame. Now that we have discussed the technique of cross products, we can derive general formulas for these two pseudoforces and understand their properties better than before.

Newton's second law $\mathbf{F} = m\mathbf{a}$ is invalid in rotating reference frames, since such frames are noninertial. However, we can still use $\mathbf{F} = m\mathbf{a}$ if we sneakily add the two pseudoforces in with the real forces in calculating the total force $\mathbf{F}$. We need to show how this comes about.

We will start by finding the velocity $\mathbf{v}_{\text{in}}$ of a particle as viewed from an inertial frame, in terms of its velocity $\mathbf{v}_{\text{rot}}$ as viewed in a rotating frame. We then similarly relate the particle's acceleration in the two frames. Finally we will find an equation $\mathbf{F}_{\text{rot}} = m\mathbf{a}_{\text{rot}}$ valid in the rotating frame, where $\mathbf{F}_{\text{rot}}$ includes the two pseudoforces as well as all of the real forces.

Suppose some arbitrary vector $\mathbf{A}$ is unchanging in a rotating frame. We take $\mathbf{A}$ to be perpendicular to the rotation axis, with its tail located at the rotation axis, as shown in Fig. S1.

Fig. S1. A vector $\mathbf{A}$ is at rest in the rotating frame, so its direction changes in the inertial frame as the rotating frame turns.

Then if the rotating frame turns through some small angle $\Delta\theta$, the change in $\mathbf{A}$ is $\Delta\mathbf{A}$ as shown, perpendicular to $\mathbf{A}$ in the limit of very small $\Delta\theta$. The magnitude of $\Delta\mathbf{A}$ is $\Delta A = A\,\Delta\theta$, since A is the "radius" of the circle defined by the tip of $\mathbf{A}$. In vector form,

$$\Delta\mathbf{A} = \Delta\boldsymbol{\theta} \times \mathbf{A}, \tag{S1}$$

where the vector $\Delta\boldsymbol{\theta}$ has magnitude $\Delta\theta$ and direction pointing out of the page for counterclockwise rotation, and into the page for clockwise rotation. Equation S1 holds whether $\mathbf{A}$ is perpendicular to the rotation axis or not, and whether its tail is located at the rotation axis

or not. For example, if $\mathbf{A}$ is parallel to the rotation axis, then $\Delta\mathbf{A} = \Delta\boldsymbol{\theta} \times \mathbf{A} = 0$; $\mathbf{A}$ does not change in this case.

Now suppose some vector $\mathbf{A}$ *does* change even in the rotating frame, by the amount $\Delta\mathbf{A}_{\text{rot}}$. Then we have to add $\Delta\mathbf{A}_{\text{rot}}$ to the change due to frame rotation. That is,

$$\Delta\mathbf{A}_{\text{in}} = \Delta\mathbf{A}_{\text{rot}} + \Delta\boldsymbol{\theta} \times \mathbf{A} \tag{S2}$$

is the total change in $\mathbf{A}$ as observed in the inertial frame. Dividing by the time interval Δt during which the rotating frame turns by $\Delta\theta$, we have

$$\Delta\mathbf{A}/\Delta t|_{\text{in}} = \Delta\mathbf{A}/\Delta t|_{\text{rot}} + (\Delta\boldsymbol{\theta}/\Delta t)\times\mathbf{A}. \tag{S3}$$

Now take the limit $\Delta t \to 0$. The ratios become derivatives, so

$$d\mathbf{A}/dt|_{\text{in}} = d\mathbf{A}/dt|_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{A} \tag{S4}$$

where $\boldsymbol{\omega} = d\boldsymbol{\theta}/dt$ is the instantaneous angular velocity of the rotating frame. Equation relates the time rate of change of any arbitrary vector $\mathbf{A}$ between the inertial and rotating frames.

In particular, let $\mathbf{A} = \mathbf{r}$, the position vector of a particle. Then the derivative $d\mathbf{r}/dt|_{\text{in}} = \mathbf{v}_{\text{in}}$ is the velocity of the particle measured in the inertial frame, and $d\mathbf{r}/dt|_{\text{rot}} = \mathbf{v}_{\text{rot}}$ is its velocity in the rotating frame, so

$$\mathbf{v}_{\text{in}} = \mathbf{v}_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{r}. \tag{S5}$$

The second term on the right is the velocity the particle would have in the inertial frame if it were at rest in the rotating frame, a distance $\mathbf{r}$ from the axis.

Now choose $\mathbf{A} = \mathbf{v}_{\text{in}}$ instead. Equation S4 then gives

$$d\mathbf{v}_{\text{in}}/dt\big|_{\text{in}} = d\mathbf{v}_{\text{in}}/dt\big|_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{v}_{\text{in}}. \tag{S6}$$

The left-hand side of this equation is the acceleration $\mathbf{a}_{\text{in}}$ of the particle viewed in the inertial frame.

The first term on the right of Eq. S6 is a mixture of inertial and rotating frame quantities,

and has to be unmixed to interpret properly. That's easy, however, because we already know that $\mathbf{v}_{in} = \mathbf{v}_{rot} + \boldsymbol{\omega} \times \mathbf{r}$, so

$$d\mathbf{v}_{in}/dt\big|_{rot} = d(\mathbf{v}_{rot} + \boldsymbol{\omega} \times \mathbf{r})/dt\big|_{rot} = \mathbf{a}_{rot} + (d/dt)\ (\boldsymbol{\omega} \times \mathbf{r})\big|_{rot}. \tag{S7}$$

Therefore Eq. S6 gives

$$\begin{aligned} \mathbf{a}_{in} &= \mathbf{a}_{rot} + (d/dt)\ (\boldsymbol{\omega} \times \mathbf{r})\big|_{rot} + \boldsymbol{\omega} \times \mathbf{v}_{in} \\ &= \mathbf{a}_{rot} + \boldsymbol{\omega} \times \mathbf{v}_{rot} + \boldsymbol{\omega} \times (\mathbf{v}_{rot} + \boldsymbol{\omega} \times \mathbf{r}) \end{aligned} \tag{S8}$$

since $(\boldsymbol{\omega} \times d\mathbf{r}/dt)\big|_{rot} = (\boldsymbol{\omega} \times \mathbf{r})\big|_{rot}$. We have assumed that the rotating frame has constant angular velocity $\boldsymbol{\omega}$, so $d\boldsymbol{\omega}/dt = 0$.

Now bring all terms except $\mathbf{a}_{rot}$ to the left side of Eq. S8, multiply through by the mass of the particle, and use Newton's second law $\mathbf{F} = m\mathbf{a}_{in}$ where $\mathbf{F}$ is the sum of all the *real* forces acting on the particle. This gives

$$\mathbf{F}_{rot} = m\mathbf{a}_{rot}, \tag{S9a}$$

where

$$\mathbf{F}_{rot} \equiv \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\boldsymbol{\omega} \times \mathbf{v}_{rot} \tag{S9b}$$

is the sum of the real forces $\mathbf{F}$ and two pseudoforces. We have succeeded in our quest to write $\mathbf{F} = m\mathbf{a}$ in rotating frames, but we have to add two pseudoforces to the real forces in order to do so.

The first pseudoforce is called the centrifugal pseudoforce

$$\mathbf{F}_{centrifugal} = -\ m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{S10}$$

because in the rotating frame it pushes the particle away from the axis of rotation. Suppose for example the rotating frame turns clockwise, so $\boldsymbol{\omega}$ is out of the page in Fig. S-2.

Fig. S-2. The directions of $\boldsymbol{\omega} \times \mathbf{r}$ and $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ if $\boldsymbol{\omega}$ is out of the page and $\mathbf{r}$ is directed as shown.

If $\mathbf{r}$ is measured directly out from the rotation axis, then $\mathbf{r}$ is perpendicular to $\boldsymbol{\omega}$, so $(\boldsymbol{\omega} \times \mathbf{r})$ has the magnitude ωr and points to the right as shown. Then since $\boldsymbol{\omega}$ is necessarily perpendicular to $(\boldsymbol{\omega} \times \mathbf{r})$, the cross product $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ has the magnitude $\omega^2 r$ and points toward the rotation axis, as shown in the second figure. Finally we have to multiply by the mass m, so $\mathbf{F}_{\text{centrifugal}}$ has magnitude $m\omega^2 r$ and points away from the rotation axis.

In vector form, since we have taken $\mathbf{r}$ to be perpendicular to the rotation axis, we can write

$$\mathbf{F}_{\text{centrifugal}} = m\omega^2 \mathbf{r}. \qquad (\mathbf{r} \perp \mathrm{w}) \qquad \text{(S11)}$$

This is the familiar "centrifugal force" mv^2/r in the rotating frame, since the tangential velocity of a partiacle at rest in the rotating frame is $v = r\omega$. Note that if $\mathbf{r}$ is *not* perpendicular to the rotation axis we can use the general formula of Eq. S10, taking $\mathbf{r}$ to be that component of $\mathbf{r}$ that is perpendicular to the rotation axis.

The second pseudoforce is the Coriolis pseudoforce

$$\mathbf{F}_{\text{Cor}} = -\ 2m\boldsymbol{\omega} \times \mathbf{v}_{\text{rot}} \qquad \text{(S12)}$$

which depends upon the velocity but not the position of the particle in the rotating frame, in contrast to the centrifugal pseudoforce which depends upon the positon of the particle but not its velocity. The Coriolis pseudoforce acts only when the particle is moving in the rotating frame, in any direction that is not parallel to $\boldsymbol{\omega}$. It tends to deflect the particle's path, because $\boldsymbol{\omega} \times \mathbf{v}_{\text{rot}}$ is perpendicular to the direction of motion at any instant.

In summary, in uniformly rotating reference frames we have to add two pseudoforces

$$\mathbf{F}_{\text{centrifugal}} = m\omega^2 \mathbf{r} \qquad \text{(S13)}$$

and

$$\mathbf{F}_{\text{Cor}} = -\ 2m\,\boldsymbol{\omega} \times \mathbf{v}_{\text{rot}} \tag{S14}$$

to the real forces when using $\mathbf{F} = m\mathbf{a}$ in the rotating frame. The centrifugal pseudoforce $\mathbf{F}_{\text{centrifugal}}$ depends on the particle's position but not its velocity, and pushes the particle away from the axis of rotation. The Coriolis pseudoforce $\mathbf{F}_{\text{Cor}}$ depends upon the particle's velocity but not its position, and pushes the particle in a direction perpendicular to its direction of motion, unless $\mathbf{v}_{\text{rot}}$ the particle's velocity in the rotating frame, is parallel to ω; in that case, there is no Coriolis force.

Centrifugal and Coriolis pseudoforces have to be used anytime we are describing motion from the standpoint of a rotating frame, including the rotating frame of Earth. The centrifugal pseudoforce makes Earth bulge out at the equator, so Earth's radius at the equator is greater than its polar radius. When added vectorially to Earth's real gravity, the centrifugal pseudoforce decreases the effective gravity felt at the equator, because real gravity is inward while the centrifugal pseudogravity is outward.

The Coriolis pseudoforces has profound effects on weather. If rising air in some region leaves a local area of low pressure, surrounding air flows in from the sides to try to equilibrate pressures. As the air flows in, it is deflected to the right (in the northern hemisphere), giving the air a twist in the counterclockwise sense, as shown. Therefore all hurricanes circulate in the counterclockwise sense in the northern hemisphere. In the southern hemisphere, the Coriolis pseudoforce makes air currents bend to the left, so hurricanes circulate clockwise in the southern hemisphere.

Coriolis effects are also important in the motion of long-range aircraft, missiles, or artillery shells. This was not fully appreciated by British ships in a World-War I naval battle off the Falkland Islands in the southern hemisphere. They found that their shells were falling in the ocean to the left of their targets, which were German warships. The reason was that the sights on their large guns had been set to include a typical Coriolis deflection to the right at the latitudes corresponding to those of northern Europe. Therefore shells were landing in the Falklands battle by twice the Coriolis deflection away from their targets.

Questions and Problems from *Day Twenty*

20Q1. A ball is held at rest relative to Earth's surface, at some height directly above a point P on the equator. The ball is then dropped. Does the ball land at P, or slightly to the west or east of P? *Hint*: View the motion of the ball and the motion of Earth's surface from the point of view of an inertial frame in which the planet is seen to rotate toward the east. The angular momentum of the ball is conserved about Earths center.

20Q2. A ball is thrown directly upward from the point of view of a person standing on the equator of Earth. Does the ball fall back exactly to the point from which it was thrown, or slightly to the east or west of it? (See the hint in 20Q1.)

20Q3. The angular velocity vector ω of Earth points from south to north along Earths rotation axis. Let the unit vector $\hat{\mathbf{r}}$ point outward from the Earths center toward and beyond your location on the surface; that is, $\hat{\mathbf{r}}$ is locally vertically upward. What is the direction of the vector $\omega \times \hat{\mathbf{r}}$? What is the direction of $\omega \times (\omega \times \hat{\mathbf{r}})$?

20Q4. A merry-go-round is turning about its vertical axis with angular velocity ω. Let a radius vector r extend from the rotation axis to a person standing on the merry-go-round. Is the velocity of the person equal to $r \times \omega$ or $\omega \times r$?

20Q5. Suppose that the sum of the forces and the sum of the torques on an object are both zero. Does this imply the object is not moving?

20Q6. A bicycle wheel rolls without slipping along the street. Which is greater, the wheels orbital angular momentum or its spin angular momentum? Does the result depend upon a choice of origin?

20Q7. A cylinder rolls without slipping along the floor. Is it possible to distribute the mass within the cylinder so that its spin angular momentum exceeds its orbital angular momentum, calculated about an origin on the floor?

20Q8. The shortstop throws a baseball toward first base. While in flight, is the orbital angular momentum of the ball conserved about an origin located at the shortstops feet? Why or why not?

20Q9. If you hit a tennis ball with horizontal velocity v_0 without giving it any spin, friction with the court surface slows down the forward motion when the ball bounces. What top-spin ω_0 would you have to give the ball (of radius R) so that friction with the surface would not change its forward velocity during the bounce?

20Q10. In a game of pool, the cue ball can be struck so that the ball has no initial spin. As it slides along the table why does it gain a spin angular momentum? Is this angular momentum horizontal or vertical? How can it gain a spin angular momentum in the other direction when it strikes another ball?

20Q11. Where does a gyroscope get the energy needed to precess? *Hint*: See the last part of the Special Topic.

20Q12. A cement truck carries a heavy load of concrete in a large cylindrical tank at the rear of the truck, which turns constantly in a clockwise sense as seen by the driver of a car following the truck. If the road ahead suddenly has a downgrade, will the truck tend to veer to the right or to the left? Explain.

20Q13. The axis of rotation of the crankshaft is mounted fore and aft in most automobiles; in some it is mounted transversely. In each case what would be the effect on a car turning left or right?

20Q14. If you spin a hard-boiled egg that sits originally on its bigger end, it will turn around and stand on the more pointed end. Why is this? (The behavior is similar to that of a Tippy-Top, which has a large spherical surface on one end and a small stem on the other end, which can be twisted between fingers to spin up the top. If the top is sitting on the large end, it will flip over and spin around the stem. See

an article by George D. Freier in *The Physics Teacher*, January 1967, p. 36.)

20Q15. A tennis ball on a string circles quickly around in a horizontal plane, as shown below. If the ball is given a sharp blow downward (in the negative direction), in which direction does the axis of rotation tilt? *Hint*: In what direction is the new velocity vector, immediately after the blow?

20Q16. A physicist once mounted a rapidly-spinning flywheel inside a suitcase, with the axle ends attached to the centers of the opposite large sides of the suitcase, as shown below. He walked in a straight line into a hotel with the suitcase, and handed it to a hotel porter. What happened as the porter carried the suitcase around a corner?

20Q17. Some helicopters have two sets of primary rotors, rotating in opposite directions about a vertical axis. What is the advantage in having two sets of rotors?

20Q18. Place a bicycle upright on the street; it falls over. But if you give the bike a push forward, it rolls for quite awhile before eventually falling over. Why does it stay up longer when it is rolling?

20Q19. If a real, nonuniform baseball bat is used to bat a ball, do you think the "sweet spot" is less than 2/3 or more than 2/3 the length of the bat, measured from the bat's small end?

Section 20.1 Angular velocity and angular momentum vectors

20P1. A flywheel centered at the origin rotates in the x, z plane with moment of inertia $I = 3.5$ kg m^2 and with period $T = 0.42$ s, clockwise from the perspective of an observer stationed on the negative y axis. Find the magnitude and direction of the flywheels (a) angular velocity ω (b) angular momentum $\mathbf{L}$.

20P2. A bicycle is traveling due south at 25 km/hr. Each tire has mass 0.50 kg and radius 33 cm. Pretending the tires mass is all at the same radius, find the magnitude and direction (east, west, north, south) of each tires (a) angular velocity ω (b) angular momentum $\mathbf{L}$ about its axle. Assume the tires are rolling without slipping.

20P3. A cement truck is traveling due west along a straight horizontal road. The large bin at the rear of the truck is turning counterclockwise with period $T = 11$ s from the point of view of the driver of a car following the truck. The axis of rotation of the bin is tilted at angle $\theta = 15^o$ to the horizontal, tipped downward in front, as shown below. The moment of inertia of the empty bin is $I = 9.5 \times 10^3$ kg m^2. Using unit vectors, find the magnitude and direction of the bin's (a) angular velocity ω (b) angular momentum $\mathbf{L}$ due to its rotation. Let the positive x axis point west and the positive y axis point vertically upward.

20P4. The bin of a rotary clothes dryer is a cylinder of radius $R = 35$ cm and moment of inertia $I = 2.1$ kg m^2; its rotation axis is tilted at angle $\phi = 30^o$ to the horizontal, with coordinates as shown below. If a point on the rim is turning with speed $v = 2.0$ m/s, find the magnitude and direction of the bin's (a) angular velocity ω (b) angular momentum L. Use unit vectors.

Section 20.2 The torque vector and cross products

20P5. A horizontal uniform rod of mass M and length d is supported at a point a distance $d/4$ from the left end of the rod, as shown below. (a) A weight of what mass m can be hung from the left end of the rod so the system will be in equilibrium? (b) With the support point as origin, what is the direction of the torque vector on the rod due to the weight? (c) What is the direction of the torque vector on the rod due to gravity?

20P6. A uniform triangular sign is supported by a pin at the lower left, and by a wire attached to the upper vertex as shown below. If the origin is at the pin, what is the direction of the torque vector on the sign due to (a) gravity? (b) the wire? If instead the origin is taken at the center of mass of the sign, what is the direction of the torque vector on the sign due to (c) the wire? (d) the pin? (e) gravity?

20P7. A small straight hole is drilled through a solid object so that the hole passes through the CM of the object. A small frictionless, rigid rod is slid through the hole, so the object is free to turn about the rod as an axle. (a) Prove that if the object is subjected only to uniform gravity and the support of the rod, if released from rest in any orientation the object will remain in the same orientation, without starting to turn. (b) Use the result of part (a) to suggest a quick (and nondestructive) technique for locating the CM of a broom or a tennis racquet.

20P8. Prove that the total torque on an arbitrary object exerted by a uniform gravitational field g about a given axis is the same as if the entire mass of the object were concentrated at its center of mass.

20P9. (a) Prove that if an object is in equilibrium under the action of three forces which are not parallel to one another, then the forces must be coplanar and their lines of action (i.e., the lines along the force directions) must meet at a single point. (b) Show by example that if all three forces are parallel (or antiparallel) to one another, their lines of action need not meet at a single point when the object is in equilibrium.

20P10. Repeat *Sample Problem 3*, taking as origin (a) the center of the pole (b) the point at which the pole touches the wall.

Section 20.3. The relation between torque and angular momentum vectors

20P11. A communications satellite of mass m = 1200 kg is placed in circular orbit in the equatorial plane of Earth. It is at an altitude such that its orbital period is 24 hours, so it always stays above the same point on the equator. (a) Find the distance of the satellite from the center of Earth, in kilometers. (b) Find the magnitude and direction of the satellites angular momentum.

20P12. A small stone of mass m is tied to a string of length d in gravity-free space, and whirled around in a circle at speed v, in a clockwise direction as viewed from the positive y axis, as shown below. (a) What is the magnitude and direction of the stones angular momentum about the rotation axis? (b) If the string breaks, what is the magnitude and direction of the stone's subsequent angular momentum about the same axis?

20P13. A bowling ball of mass m = 4.4 kg and radius 12 cm is launched down the bowling alley with no spin. The ball slides for a time

without rotating, in a straight line due west at 15 m/s. For each choice of origin (a), (b), (c), find the magnitude and direction of the ball's angular momentum vector.

Section 20.4 Cross products in general

20P14. If $\mathbf{A} = 3\hat{\mathbf{x}} - 2\hat{\mathbf{y}} + \hat{\mathbf{z}}$ and $\mathbf{B} = 2\hat{\mathbf{x}} + \hat{\mathbf{y}} - 3\hat{\mathbf{z}}$, find (a) $\mathbf{A} \times \mathbf{B}$ (b) $\mathbf{B} \times \mathbf{A}$ (c) $(\mathbf{A} + \mathbf{B}) \times (\mathbf{A} - \mathbf{B})$

20P15. If $\mathbf{A} = \hat{\mathbf{x}} + 2\hat{\mathbf{y}} - 3\hat{\mathbf{z}}$, $\mathbf{B} = 2\hat{\mathbf{x}} - 4\hat{\mathbf{y}} + 3\hat{\mathbf{z}}$, and $\mathbf{C} = -\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$, find (a) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ (b) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$. Are the results the same?

20P16. Given the same three vectors $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ as in 20P15, find (a) $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$ (b) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$. Are the results the same?

20P17. Show that the area of (a) a parallelogram with sides $\mathbf{A}$ and $\mathbf{B}$ is $|\mathbf{A} \times \mathbf{B}|$ (b) a triangle, two sides of which are $\mathbf{A}$ and $\mathbf{B}$, is $(1/2)$ $|\mathbf{A} \times \mathbf{B}|$.

20P18. Show that the volume of a parallelepiped with sides $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ is equal to $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$.

20P19. Show that

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

in terms of Cartesian components of $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

20P20. Prove the vector identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$.

20P21. Prove that the cross product is distributive with respect to addition: That is, show that $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ for any vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

20P22. A rigid body rotates about a fixed axis with angular velocity ω. Select an origin at some point on the rotation axis, and let $\mathbf{r}$ be the vector from the origin to a particular particle in the body. Show then that the velocity vector of the particle in the reference frame of the origin is given by $\mathbf{v} = \omega \times \mathbf{r}$. (Note that this equation reduces to the well-known relationship $v = r\omega$ if the vector $\mathbf{r}$ is perpendicular to the rotation axis.)

20P23. Prove that the product rule

$(d/dt)(\mathbf{A} \times \mathbf{B}) = d\mathbf{A}/dt \times \mathbf{B} + \mathbf{A} \times d\mathbf{B}/dt$

is valid for any differentiable vectors $\mathbf{A}$ and $\mathbf{B}$. *Hint*: Expand the cross product in terms of Cartesian components.

20P24. A triangle has sides represented by the vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$, as shown; the interior angles are α, β, and γ , also as shown. (a) Explain why $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$. (b) Prove the law of sines by taking the cross product of the equation in part (a) by the vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$, in turn.

20P25. A wheel whose center is at the origin is free to rotate about the z axis, with moment of inertia $I = 5.0$ kg m^2. A force $\mathbf{F} = 3.0\hat{\mathbf{x}} - 1.5\hat{\mathbf{y}}$ N acts continuously on the wheel at a point which is a distance $\mathbf{r} = 4.5\hat{\mathbf{x}} + 2.0\hat{\mathbf{y}}$ cm from the origin. (a) What is the torque acting on the wheel? If the wheel starts at rest at time $t = 0$, find (b) the angular speed, and (c) the magnitude of the angular momentum, both at time $t = 6.9$ s.

20P26. A particle of mass $m = 2.5$ g is initially at rest at position $\mathbf{r} = 4.5\hat{\mathbf{x}} + 2.0\hat{\mathbf{y}} - 3.0\hat{\mathbf{z}}$ meters. At time $t = 0$ it is subjected to a force $\mathbf{F} = -1.5\hat{\mathbf{x}} + 2.0\hat{\mathbf{y}} - 3.0\hat{\mathbf{z}}$ N. (a) Find the instantaneous acceleration a of the particle at $t = 0$. Relative to the origin, find (b) the torque τ exerted on the particle at $t = 0$; (c) the particle's angular momentum $\mathbf{L}$ as a function of time, for times so short that neither $\mathbf{r}$ nor $\mathbf{F}$ has changed appreciably.

20P27. The position of a particle of mass $m = 4.0$ kg is given by $\mathbf{r} = 3.0t^2\hat{\mathbf{x}} - 2.0\hat{\mathbf{y}}$ in meters, where t is measured in seconds, and find (a) the particle's angular momentum, and (b) the net torque on the particle, both as functions of time and relative to the origin.

20P28. The center of mass of a 400 lb rectangular sign is in the center of the sign. There are hinges at A and B, and the sign is prevented from swinging under wind loads by chains OC and OD, as shown below. The two lengths of chain together are slightly longer than the dimension from C to D, so that only one of the two lengths is taut under wind load. Determine the forces exerted by hinges A and B during a wind loading of 300 lbs normal to the sign and acting at its center. Assume that neither A nor B has components in the direction of AB.

20P29. A vertical 1250 lb mast is supported by a ball-and-socket joint at its base, and is restrained by the two cables shown below. A horizontal force of 2450 lb is exerted on the top of the mast, in the negative x direction, as shown below. Find the tensions $\mathbf{F}_1$ and $\mathbf{F}_2$ and the force $\mathbf{A}$ exerted on the mast by the ball-and-socket joint. Use unit vectors.

20P30. A uniform bar of mass 120 kg and length 5.2 meters is held in a horizontal position by cables having tensions $\mathbf{P}, \mathbf{Q}$, and $\mathbf{F}$. $\mathbf{Q}$ is in the xz plane and makes a 30° angle with the z axis, as shown below. $\mathbf{P}$ makes a 45° angle with the z axis, and the horizontal projection of $\mathbf{P}$ makes a 30° angle with the x axis. Nothing is known about the direction of $\mathbf{F}$. Find $\mathbf{P}, \mathbf{Q}$, and $\mathbf{F}$.

20P31. A 7.4 m pole of weight 32 N is held in a vertical position by forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{Q}$, and $\mathbf{A}$. $\mathbf{F}_1$ and $\mathbf{F}_2$ are tensions supplied by cables attached at P and having the indicated directions. The force $\mathbf{Q}$ acts at B, the midpoint of the pole, and has a magnitude of 23 N. The horizontal component of $\mathbf{Q}$ makes an angle of 30° with a line drawn through the point B parallel to the x axis, and $\mathbf{Q}$ itself is 45° above its horizontal component. Nothing is known about $\mathbf{A}$ except that it is applied at the base of the pole. Find the magnitudes F_1 and F_2, and find the force $\mathbf{A}$ in terms of unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$.

Section 20.5 Orbital angular momentum and spin angular momentum

20P32. A hoop of mass M and radius R rolls without slipping with speed v along the floor. Find (a) its orbital angular momentum about a point on the floor; (b) its spin angular momentum. (c) What fraction of its angular momentum is orbital?

20P33. A uniform-density cylinder of mass M, radius R, and length D rolls without slipping with speed v along the floor. Find the magnitudes of (a) its orbital angular momentum about a point on the floor; (b) its spin angular momentum. (c) What fraction of its angular

momentum is orbital?

20P34. A solid pool ball rolls without slipping along the table. (a) What fraction of its total angular momentum is orbital, about a point at rest on the table? (b) Repeat part (a) for a hollow tennis ball, with all of its mass concentrated at its outer radius.

20P35. Viewed from Earth, one always sees the same side of the Moon. Estimate the ratio of the Moon's spin angular momentum to its orbital angular momentum around Earth, assuming the Moons mass density is uniform. The Moon's radius is $R = 1740$ km and the average distance between the centers of the Moon and Earth is $D = 3.83 \times 10^5$ km.

20P36. Tidal effects of the Moon on Earth have caused Earth's rotation rate to slow, thus reducing Earths spin angular momentum. This reduction has been made up by an increase in the orbital angular momentum of the Moon around Earth. The length of Earth's day caused by this effect has been an increase of 0.1 seconds in the past 3800 years. How much farther is the Moon now from Earth than it was 3800 years ago? (Take Earth's moment of inertia about its axis to be $I_e = 0.33 M_e R_e^2$. Also $M_e = 6.0 \times 10^{24}$ kg, $M_m = 7.4 \times 10^{22}$ kg, mean Earth-Moon distance $= 3.8 \times 10^5$ km.)

20P37. A uniform disk-like merry-go-round of mass M and radius R is at rest when a child of mass m and velocity v jumps on the platform. The child lands a distance r from the axle, as shown, and then remains at rest relative to the platform. (Note that v is perpendicular to the initial radius vector r.) Find the subsequent angular speed of the merry-go-round, assuming negligible friction at the axle.

20P38. A thin wooden rod of length and mass M is suspended vertically from one end and rotates freely about O, the point of suspension. A bullet of mass m and unknown speed is fired horizontally into the lower end of the rod, where it becomes embedded. Assume that the rod does not move appreciably during the collision. The pendulum with embedded bullet then swings up to a maximum angle θ, as shown below. Consider the system bullet plus rod. (a) Is angular momentum of the system about the point of suspension necessarily conserved in the collision between bullet and rod? Explain. (b) Is linear momentum of the system necessarily conserved in the collision? Explain. (c) Is kinetic energy of the system necessarily conserved in the collision? Explain. (d) What is the initial speed v of the bullet in terms of the given parameters and g, the gravitational field?

20P39. A thin uniform rod of mass M and length L lies at rest on a frictionless sheet of ice. A small ball of putty of mass $m = M/2$ moves along the ice with velocity v_o perpendicular to the rod. It strikes the rod at a point $L/4$ from one end of the rod, as shown below, sticking to the rod as a result of the completely inelastic collision. (a) Explain why the momentum and angular momentum are each conserved in the collision, for the system of the rod + putty. (b) Describe the motion of the system after the collision. (c) Find the center-of-mass velocity of the system, and find the angular velocity of the system about its CM. (d) Find the loss in kinetic energy during the collision.

20P40. A thin uniform rod of mass M and length L lies at rest on a frictionless sheet of ice. A small ball, also of mass M, moves along the ice with velocity v_o perpendicular to the rod. It strikes very near one end of the rod, undergoing a completely elastic collision; afterwards the ball continues to move along its original direction, more slowly than before. (a) What quantities are conserved in the collision, for the system of ball plus rod? (b) After the collision, find the velocity of the ball, the velocity of the center of the rod, and the angular velocity of the rod about its center.

20P41. A hoop of mass M and radius R is given a back-spin of magnitude ω_o , as shown below, and launched to the right so that its center has initial velocity v_o. There is sliding friction between the hoop and floor. For each of the following cases, when the wheel stops skidding and begins to roll without slipping, find the velocity of its center of mass in terms of v_o, including both magnitude and direction. (a)$\omega_o = v_o/(2R)$ (b) $\omega_o = v_o/R$ (c) $\omega_o = 2v_o/R$. *Hint*: Show that the total angular momentum of the hoop is conserved about an origin at a fixed point on the floor.

20P42. A hoop of mass M and radius R is given a back-spin of magnitude ω_o as shown below, and launched to the right. The initial velocity of the center of mass of the hoop is v_o. The floor is smooth at the launching point, but starting at P it becomes rough so that the hoop eventually stops skidding and starts moving to the left. By the time it returns to P, the hoop is rolling without skidding. What is now the velocity of the center of mass of the hoop? (It is given that $\omega_o = 40$ rad/s, $R = 0.40$ m, and $v_o = 12$ m/s, but M is not known.)

20P43. A bowling ball is launched down an alley with initial velocity v_o and with no initial spin. Friction slows it down to what velocity (in terms of v_o) by the time it begins to roll without slipping?

20P44. A tennis ball of mass m is given an initial horizontal velocity v_o but no spin, as shown below, a distance h above point P on the court surface. Relative to P, (a) what is the balls initial orbital angular momentum $\mathbf{L}_o$? (b) Does $\mathbf{L}_{\text{orbital}}$ increase or decrease while the ball is falling toward point Q on the surface? What is responsible for the change? (c) Find $\mathbf{L}_{\text{orbital}}$ as a function of time while the ball travels toward point Q, in terms of m, v_o, and h. (Neglect air resistance). (d) In terms of $\mathbf{L}_o$, what is $\mathbf{L}_{\text{orbital}}$ just before the ball reaches point Q? (e) At Q the ball bounces back up. What happens to $\mathbf{L}_{\text{orb}}$ during the bounce? What is responsible for the change? (f) There is friction between the ball and surface during the brief time they are in contact. Show that the ball gains a spin angular momentum $\mathbf{L}_{\text{spin}}$. Is $\mathbf{L}_{\text{spin}}$ directed into or out of the page?

20P45. A billiard ball at rest is given a sharp, horizontal impulse by a cue, acting at a distance h above the midpoint of the ball, as shown below. (a) In terms of h, find the ratio of the balls spin angular momentum to its linear momentum p immediately following the impulse. (b) For what value of h will the ball roll without

slipping immediately following the impulse, in terms of R, the balls radius? (c) Suppose the ball is on a table with friction. If $h = 4R/5$ and v_o is the speed of the ball immediately following the impulse, what is the speed of the ball by the time it starts rolling without slipping? (d) How much kinetic energy does the ball lose to frictional heating, for the experiment described in (c)?

20P46. A yo-yo of outer radius R and spindle radius r rests on a horizontal surface. Its string is pulled gently in a direction that makes an angle ϕ with the vertical. (a) Assuming sufficient static friction with the table so that the yo-yo will not slip, find the angle $\phi = \phi_0$ (in terms of r and R) so that the yo-yo will not start to roll either to the right or to the left. (b) If the string is gently pulled with $\phi > \phi_0$, will the yo-yo start to roll to the right or to the left? Explain. (c) If instead the string is gently pulled with $\phi < \phi_0$, will the yo-yo start to roll to the right or to the left? Explain.

20P47. A yo-yo of mass M, moment of inertia I (about its symmetry axis), outer radius R, and spindle radius $R/2$ rests on a horizontal surface. Its string is pulled gently in the horizontal direction with tension T, as shown. Assuming sufficient static friction with the table so that the yo-yo rolls without slipping, find the yo-yos (a) angular acceleration about its symmetry axis (b center-of-mass acceleration relative to the surface.

20P48. Where is the sweet spot of a uniform bat located, if the bat is held 1/3 of the way from one end? Guess an answer and then derive it.

20P49. A rod of length d is nonuniform. A length αd has negligible mass, and the length $(1-\alpha)d$ has uniform non-zero mass. If the rod is held at the light end, find the sweet spot of the rod as a function of the parameter α. Check the result in the limits $\alpha \to 0$ and $\alpha \to 1$.

Special Topic I: Gyroscopes

20P50. In the gyroscope experiments described, the angular momentum of the wheel was outward, away from the point of support. Suppose instead the wheel were spun the opposite way, so its angular momentum was inward, toward the point of support. Describe its subsequent motion.

20P51. A particular bicycle wheel is essentially a ring of mass $M = 2.80$ kg and radius $R = 31$ cm. It turns around the middle of an axle consisting of a uniform rod of length $L = 1.20$ m and mass $m = 450$ g. The axle is oriented horizontally with one end supported in place. The wheel is spun to have an angular velocity of $? = 32\ \mathrm{s}^{-1}$, and the unsupported end of the axle is released from rest. (a) What is the period of precession of the gyro? (b) When the wheel has slowed due to friction, so that it has an angular velocity of $12\ \mathrm{s}^{-1}$, what then is its period of precession?

20P52. The wheel of a gyroscope is a solid disk of mass $M = 220$ g and radius $R = 5.0$ cm, mounted at the center of the gyro axle, a uniform rod of length $L = 12$ cm and mass $m = 35$ g. The axle is oriented horizontally with one end supported by a stand. The wheel is spun to have a rotation period of $T = 0.010$ s, and the unsupported end of the axle is released from rest. (a) What is the period of precession of the gyro? (b) When the wheel has slowed due to friction, so that it has a period of $T = 0.030$ s, what then is the precession period of the gyro?

20P53. The precessional angular velocity Ω of a gyroscope depends upon the wheels mass M, moment of inertia I, spin angular velocity ω, distance from the support point d, and gravity g, as given in Eq. S5. Find three *incorrect* for-

mulas for Ω in terms of any or all of M, I, ω, d, and g only, which are nevertheless *dimensionally* consistent.

20P54. A detached bicycle wheel is set rolling along the ground. One finds that if the wheel starts leaning to the right, the wheel also turns to the right, which tends to stabilize the wheel, i.e., to keep it from falling. In terms of the torque on the wheel as it leans, show that the wheel precesses by turning to the *right*. What if the wheel leans to the left?

20P55. It has been proposed to build flywheels into cars, which could spin with very high angular velocity. When a car stops at a stop sign some of the cars kinetic energy is diverted into the kinetic energy of the rotating wheel, instead of all into heat energy in the brakes. Then the wheels energy will be used to help propel the car up to speed again. The proposal is to mount the wheel horizontally, so its axis is vertical, attached to the frame of the car. If there is just a single wheel, what effect would it have on the car, if any, if the car (a) turned right; (b) turned left; (c) started up a hill; (d) started down a hill? (e) What advantage might there be in mounting two counterrotating wheels on the same axle?

20P56. A bicyclist rides "no hands", and finds that at sufficiently high speeds the bicycle remains stable, and does not tip over. Show that if the bicycle tilts slightly to the right, the gyroscopic precession of the front wheel is such as to turn the bicycle in the direction which tends to make it stay upright. Why does the bicycle have to be moving fairly fast?

20P57. An airplane has a propeller in the front that rotates in the clockwise sense as seen by the pilot. Flying horizontally, the pilot causes the tail rudder to extend out to the left, as shown below. (a) If the airplane is ultralight and the propeller is large, heavy, and rotates very fast, which one of the following is the largest response of the plane? (i) the plane turns to the right (ii) the plane turns to the left (iii) the nose of the plane rises (iv) the nose of the plane falls. Explain why your answer is correct. (b) Now suppose instead that the plane is heavy and the propeller is small, light, and rotates slowly. Which of the answers (i), (ii), (iii), (iv) is the largest effect on the plane in this case? Explain.

20P58. Show that whatever the orientation of a gyro axis relative to the horizontal, the magnitudes and directions of the precessional angular velocity vector Ω (whose direction is given by the right-hand rule), the torque vector τ and angular momentum vector $\mathbf{L}$ are related by $\tau = \mathbf{\Omega} \times \mathbf{L}$.

20P59. The rotation of Earth makes it bulge somewhat at the equator. The Moon and Sun then exert torques on Earth, since gravity pulls on the bulges in an unbalanced way, as illustrated in the diagrams for the Moons pull on Earth, when the Moon is on opposite sides of Earth. The torque on Earth is out of the page in both diagrams. Earth therefore behaves like a gyroscope, and its rotation axis slowly precesses. (a) Earths axis is tilted at angle 23.5^o to the plane of the orbits of the Sun and Moon. The star Polaris is currently nearly at the zenith as seen by an observer at the North Pole. What is the maximum angle Polaris will ultimately achieve relative to the north-south pole axis? (This will require about 13,000 years.) (b) The vernal and autumnal equinoxes are the two points in Earth's orbit around the Sun where Earth's spin angular velocity vector is perpendicular to the radius vector from the Sun to Earth. Will Earth's precession affect the position of the equinoxes in Earth's orbit? Draw a picture to help explain what will happen.

Day Twenty-One

This is the end of our third week of exploration. We began the week by experiencing artificial gravity, first in a uniformly accelerating spaceship and then in a uniformly rotating frame. The "effective gravity" within a uniformly accelerating frame is

$$\mathbf{g}_{\text{eff}} = -\mathbf{a}_{\text{F}},$$

where $\mathbf{a}_{\text{F}}$ is the acceleration of the frame as viewed by an inertial observer. If the ship is accelerating to the *right,* the effective gravity inside the ship is to the *left.*

We can still use Newton's law $\mathbf{F} = m\mathbf{a}$ to predict the motion of a particle in the uniformly accelerating frame, but only if we add the effective gravitational pseudoforce $\mathbf{F}_{\text{eff}} = -m\mathbf{a}_{\text{F}}$ to the real forces acting on an object of mass m.

We defined two kinds of mass an object can have, its *inertial mass* m_{I} and its *gravitational mass* m_{G}. Inertial mass is that intrinsic property of an object that resists acceleration. The gravitational mass of an object is the mass in $F_{\text{grav}} = m_{\text{G}}g$, the gravitational force on the object when it is in a gravitational field g. Gravitational mass is the intrinsic property of an object that is pulled upon by gravity. It is remarkable that the ratio of these two conceptually very different kinds of mass is the same for all kinds of materials, steel, wood, gold, ..., according to all experiments done to date.

Einstein's *principle of equivalence* states that physics in a uniformly accelerating frame, with no gravity, is the same as physics in an inertial frame with uniform gravity. One consequence of the principle of equivalence is that the ratio of the inertial and gravitational masses of all particles must be the same. The principle of equivalence is supposed to be valid for all kinds of experiments, including experiments with optics, atomic nuclei, electromagnetism, biological organisms, ... No violations of the principle of equivalence have been found experimentally.

Artificial gravity occurs also in *rotating* frames. People living on the inside rim of a rotating cylindrical colony, for example, would experience an effective gravity directed outward, away from the axis of rotation. The magnitude of the effective gravity is $g_{\text{eff}} = r\omega^2$, where r is the radius of the colony and ω is the angular velocity of rotation.

We then turned to a study of rotational motion. Consider a point particle of mass m a distance $\mathbf{r}$ from an arbitrary origin. The *angular momentum* of the particle about an origin is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

the *cross product* of $\mathbf{r}$ with the momentum $\mathbf{p} = m\mathbf{v}$ of the particle, where $\mathbf{v}$ is the particle's velocity. Note that $\mathbf{L}$ depends upon the location of the origin as well as properties of the particle itself.

The cross product of *arbitrary* vectors $\mathbf{A}$ and $\mathbf{B}$ is defined as follows: The *magnitude* of $\mathbf{A} \times \mathbf{B}$ is $AB\sin\theta$ (where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$). The quantity $AB\sin\theta$ can be thought of as the magnitude A multiplied by the component of $\mathbf{B}$ perpendicular to $\mathbf{A}$; alternatively, it is the magnitude B multiplied by the component of $\mathbf{A}$ perpendicular to $\mathbf{B}$. The *direction* of $\mathbf{A} \times \mathbf{B}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$ and obeys a right hand rule: If the tails of $\mathbf{A}$ and $\mathbf{B}$ are brought together, and one imagines $\mathbf{A}$ rotated about this point towards $\mathbf{B}$, the vector $\mathbf{A} \times \mathbf{B}$ is in the direction in which a right-hand screw would propagate.

The *torque* acting on a particle due to a force $\mathbf{F}$ is defined to be

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}.$$

From the nature of the cross product, the torque is zero if $\mathbf{F}$ is parallel to $\mathbf{r}$, and maximized (for given r and F) if $\mathbf{F}$ is perpendicular to $\mathbf{r}$. The net torque on the particle causes the particle's angular momentum to change with time according to

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}.$$

Therefore the angular momentum of the particle is conserved (*i. e.*, is constant in time) if no net torque acts on the particle.

Many particles, such as those in a wheel, can move together as a *rigid body*. Given a rotation axis, an object is rigid if all particles in the object have the same angular velocity ω. The magnitude of the angular velocity, the rotational frequency ν, and the period of rotation T are related by

$$\omega = 2\pi\nu = 2\pi/T.$$

The total angular momentum of a wheel is the sum of the angular momenta of all the particles making up the wheel. It may be written in the from

$$\mathbf{L} = I\boldsymbol{\omega}$$

where I is the *moment of inertia* (or rotational inertia) of the wheel. The moment of inertia of a point-mass m a distance r from the rotation axis is

$$I_{\text{point mass}} = mr^2 ,$$

and the moment of inertia of a rigid body composed of more than one point mass is just the sum of the point-mass moments of inertia,

$$I = \sum_i m_i r_i^2 .$$

For a continuous mass distribution, the sum becomes an integral,

$$I = \int r^2 dm.$$

The integral can be carried out for a variety of circular wheels of total mass M and radius R(or length ℓ) and expressed in the form

$$I = fMR^2$$

where f is the "shape factor" of the wheel, which depends both upon how the mass is distributed in the wheel, and upon the axis of rotation used.

In particular,

$I = MR^2$ for a ring (or hoop) about its axis of symmetry

$= \frac{1}{2}MR^2$ for a disk (or solid cylinder) about its axis of symmetry

$= \frac{2}{3}MR^2$ for a spherical shell about any axis through the center

$= \frac{2}{5}MR^2$ for a uniform solid sphere about any axis through the center

$= \frac{1}{3}M\ell^2$ for a uniform stick about a perpendicular axis through one end of the stick

$= \frac{1}{12}M\ell^2$ for a uniform stick about a perpendicular axis through the center of the stick

The *parallel axis theorem*

$$I = I' + Md^2,$$

where I' is the moment of inertia of a rigid body about an axis passing through its center of mass and d is the distance to a parallel axis, is often useful. If one knows I' it is easy to get I.

As with a point mass, the angular momentum of a body changes if a net external *torque* is exerted upon it. That is,

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt},$$

where τ is the net torque, the sum of the torques due to all external forces acting on the body. The *internal* torques, due to the action of one particle on another within the body, all cancel out.

If the net external torque on an object is zero, then the angular momentum of the object stays constant, *i. e.* is conserved. The conservation of angular momentum is one of the three conservation laws of classical dynamics, which includes also conservation of momentum and conservation of energy. Note that the conservation of momentum and the conservation of angular momentum are two totally distinct conservation laws. The momentum of a system is

conserved if there is no net external *force*; angular momentum of a system is conserved if there is no net external *torque*.

For a wheel of moment of inertia I about some fixed axis of rotation, the equation $\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$ can be reduced to

$$\tau = I\,\alpha$$

where $\alpha = \ddot{\theta}$ is the angular acceleration of the wheel. Here each dot means a time derivative, so $\ddot{\theta}$ is the second derivative of the rotation angle of the wheel.

The *rotational kinetic energy* of the wheel is

$$K = \frac{1}{2} I\omega^2$$

where ω is the angular velocity of the wheel as it rotates about a fixed axis. For the *general* motion of a rigid body, as it both translates and rotates, the kinetic energy is

$$K = \frac{1}{2} M V_{\mathrm{CM}}^2 + \sum \frac{1}{2} m_i v_i'^2 = \frac{1}{2} M V_{\mathrm{CM}}^2 + \frac{1}{2} I\omega^2,$$

the sum of translational and rotational kinetic energies. Here I is the moment of inertia about the center of mass.

If a wheel of radius R rolls along a surface without slipping, the distance s it moves is related to the angle θ through which it turns by

$$s = R\theta.$$

Similarly, the velocity of the wheel's center is related to its angular velocity by

$$V = R\omega,$$

and the acceleration a of the wheel's center is related to the wheel's angular acceleration α by

$$a = R\alpha.$$

The angular momentum of a system of particles can be decomposed into two parts,

$$\mathbf{L} = \mathbf{R}_{CM} \times \mathbf{P}_{CM} + \sum_i \mathbf{r}_i' \times \mathbf{p}_i' .$$

The first term is the angular momentum of a point particle located at the center of mass of the system, whose mass is the total mass of the system and whose velocity is that of the center of mass; here $\mathbf{R}_{CM}$ is the vector from the origin to the center of mass of the system and $\mathbf{P}_{CM}$ is the total momentum of the system. The second term is the total angular momentum of the system measured in the center of mass frame; here $\mathbf{r}_i'$ is the vector from the center of mass to the ith particle, and $\mathbf{p}_i'$ is the momentum of the ith particle in the center of mass frame.

If the system is a rigid body, the angular momentum in the CM frame is $I'\omega$, so the total angular momentum is

$$\mathbf{L} = \mathbf{R}_{CM} \times \mathbf{P}_{CM} + I'\boldsymbol{\omega}$$

where the first term is called the *orbital* angular momentum, and the second term is called the *spin* angular momentum.

With the end of our third week, we have completed our search for the fundamental laws of motion, including Newton's laws, the three fundamental conservation laws, and much more.

One can go much more deeply into classical mechanics. There are the methods of Lagrange and Hamilton, for example, and the *Principle of Least Action.* These methods give insight into the meaning of mechanics, and are often very useful in solving problems as well.

One can also investigate the bounds of validity of the laws of motion, finding that they must be modified for objects moving at appreciable speeds compared with the speed of light, and that they must be replaced entirely by a deeper and more general mechanics, *quantum* mechanics, particularly for particles that are very small.

Instead, from here on we will investigate some interesting and important *applications* of classical mechanics, in particular *orbital* motion, *oscillatory* motion, and the motion of *waves*. These additional topics introduce no new fundamental laws. They can be understood using the laws we have already uncovered, as spelled out in Chapters A - F.

APPLICATIONS:

ORBITS

OSCILLATION

& WAVES

A. Orbits

"Not the least achievement of Newton was to spot the Three Laws in Kepler's writings, hidden away as they were like forget-me-nots in a tropical flowerbed. ... The three laws are the pillars on which the edifice of modern cosmology rests; but to Kepler they meant no more than bricks among other bricks for the construction of his baroque temple, designed by a moon-struck architect. He never realized their real importance. In his earliest book he had remarked that "Copernicus did not know how rich he was"; the same remark applies to Kepler himself." - Arthur Koestler, in *The Sleepwalkers*

The orbital motion of particles about a central point is a beautiful and important example of rotational motion. Examples are the orbits of planets around the Sun and satellites around Earth. We begin this chapter with ideal circular orbits, and then discuss the more general elliptical orbits.

Plane polar coordinates r, θ are often used to describe orbits, so we will also find expressions for the velocity and acceleration of particles in polar coordinates, and then give several examples. If a planet, moon, or particle orbits around a fixed point in a plane, for example, it is often convenient to describe the motion using the polar coordinates shown.

In terms of Cartesian coordinates x and y, the polar coordinates r and θ are

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta \tag{1}$$

as shown in Fig. 1. Just as unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are associated with Cartesian coordinates, the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are associated with polar coordinates. These vectors have unit length and point in the directions of increasing r and θ, respectively. Any vector in the plane can be expressed either in terms of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ or in terms of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$.

Fig. 1. Polar coordinates

A.1 Circular orbits

A satellite circles planet Earth. Why does it move as it does? How long does it take to orbit once around?

Earth's gravity attracts the satellite, pulling it inward. Generally the satellite is too far from Earth's surface simply to use $\mathbf{F} = m\mathbf{g}$ with $\mathbf{g}$ a constant vector. In fact, according to Newton's law of gravity, $\mathbf{g}$ varies inversely as the square of the distance r from Earth's center,

$$\mathbf{F} = m\mathbf{g} = -\frac{GMm}{r^2}\hat{\mathbf{r}} \qquad (2)$$

in terms of Newton's gravitational constant G, Earth's mass M, the mass of the satellite m, and the unit vector $\hat{\mathbf{r}}$ pointing outwards. The unit vector does not change the magnitude of the force; it only indicates a direction. The minus sign in the expression for the force indicates that the force is *inward*, in the $-\hat{\mathbf{r}}$ direction.

Gravity around Earth is exactly inverse-square only if our planet is exactly spherical, which of course it isn't. In fact, Earth's center bulges at the equator because of its 24-hour rotation, with an equatorial radius about 21 km greater than the polar radius. It is also slightly pear-shaped, fatter in the southern hemisphere than in the north, as shown wildly exaggerated. Even though the inverse-square law is not exactly right for the nonspherical Earth, it is good enough for many applications and provides a pretty good idea of what a satellite will do over a few orbits. If a satellite orbits many times, however, it gradually wanders away from the motion predicted by Eq. 2.

We can use $\mathbf{F} = m\mathbf{a}$ to find the motion of a satellite in response to the gravitational force. For a circular orbit the acceleration is

$$\mathbf{a}_{\text{centripetal}} = -r\omega^2\hat{\mathbf{r}}, \qquad (3)$$

where the minus sign again indicates that the acceleration is inward, in the $-\hat{\mathbf{r}}$ direction. The angular speed is $\omega = 2\pi/T$ in terms of the period T, so $\mathbf{F} = m\mathbf{a}$ gives

$$\frac{GMm}{r^2} = mr\left(\frac{2\pi}{T}\right)^2,$$

cancelling the minus signs. Solving for T, we find

$$T = \frac{2\pi}{\sqrt{GM}} r^{3/2}. \tag{4}$$

The period is independent of the satellite mass, depends inversely on the square root of Earth's mass, and increases as the three-halves power of the orbital radius. Satellites at larger radii take longer to circle around.

Sample Problem 1. Find the period of a satellite in a low-Earth orbit, with altitude $h = 100$ km above Earth's surface.

Solution: Earth's mass and radius are $M = 6.0 \cdot 10^{24}$ kg and $r = 6.4 \cdot 10^3$ km, so using Eq. 4 the satellite's period is

$$T = \frac{2\pi}{\sqrt{\left(6.67 \cdot 10^{-11}\ \mathrm{N \cdot m^2/kg^2}\right)\left(6.0 \cdot 10^{24}\ \mathrm{kg}\right)}}\left[(6.4 + 0.1) \cdot 10^3\ \mathrm{km}\right]^{3/2} = 5200\ \mathrm{s} = 87\ \mathrm{min}.$$

Satellites in low-Earth orbit (with altitudes much less than Earth's radius) have orbital periods that aren't very sensitive to altitude. They all circle once around in about an hour and a half.

A.2 Kepler's Laws and Planetary Orbits

In the late sixteenth century the Danish astronomer Tycho Brahe (1546-1601) made careful observations of the positions of the planets as a function of time. Using Tycho's data, his younger associate, the German astronomer Johannes Kepler (1571-1630), worked for many years to understand planetary orbits. Unlike Tycho, Kepler believed in the Copernican theory, that the Sun is at the center and the planets orbit around it. He asked himself: What is the shape of the orbits? Are there any regularities in their motions?

Buried in Kepler's writings are three important laws:

KI. *Planets move in elliptical orbits, with the Sun at one focus.*

KII. *Planetary orbits sweep out equal areas in equal times.*

KIII. *The periods squared of the planetary orbits are proportional to their semimajor axes cubed.*

(a) *Kepler's first law*

An ellipse can be drawn on a piece of paper using two pins and a loop of string. Stick the pins into the paper at two points called the *foci* of the ellipse, and drop the loop of string over them. Pull the loop taut with a pencil point, and draw the closed figure constrained by the taut loop. Note that the sum of the distances d_1 and d_2 from the two foci is the same for all points of the ellipse.

Call the long axis of the ellipse the major axis, with length $2a$, where a is the *semimajor axis*. Therefore $d_1 + d_2 = 2a$, as shown. Let the distance between the two foci be $2ae$, where e is the *eccentricity* of the ellipse. That is, the eccentricity is defined to be the ratio of the distance between the foci to the entire length of the major axis.

In Cartesian coordinates, the equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b are the semimajor and semiminor axes. Note that $x = \pm a$ if $y = 0$, and $y = \pm b$ if $x = 0$. In polar coordinates, the equation of an ellipse is $r = \frac{a(1-e^2)}{1+e\cos\theta}$, in terms of the semimajor axis a and eccentricity e. Here the radius r is measured from one of the foci, and the angle θ is measured from a point on the ellipse which is closest to that focus. (A derivation of the polar-coordinate equation is given in the appendix at the end of this chapter.)

Kepler found that each planet he studied moves in an elliptical orbit, with the Sun located at one of the two foci. The eccentricity and semimajor axis of each planet, as they are now known, is given in Table 1, along with the orbital periods.

planet	semimajor axis $a(10^8\text{km})$	eccentricity e	period T (years)
Mercury	0.580	0.2056	0.241
Venus	1.081	0.0068	0.615
Earth	1.496	0.0167	1.000
Mars	2.278	0.0934	1.881
Jupiter	7.781	0.0484	11.86
Saturn	14.27	0.0543	29.46
Uranus	28.70	0.0460	84.00
Neptune	45.00	0.0082	164.8
Pluto	59.09	0.2481	248.3

Table 1. The semimajor axes a, eccentricities e, and periods T of the Sun's planets

Note that Mercury and Pluto have the most eccentric orbits and that Venus is the least eccentric. Mercury's orbit is sketched at the right. The perihelion is the closest point on the orbit to the Sun, and the aphelion is the farthest point. In terms of the semimajor axis a and eccentricity e, the distances of a planet from the Sun at perihelion and at aphelion are

$$r_p = a(1 - e) \qquad \text{and} \qquad r_a = a(1 + e)\,. \tag{5}$$

(b) *Kepler's second law*

As a planet orbits the Sun its radius vector **r** sweeps out an area ΔA during some period of time Δt, as illustrated. According to Kepler's second law, this swept-out area ΔA is the same for any two identical time intervals Δt as the planet moves along its trajectory.

For example, according to KII the area swept out by the planet Earth during January must be the same as during July, because these months have equal length. The orbital radius is smaller during January, so the width of the pie-shaped wedge must be wider during January than during July to compensate. The pie-shaped wedge is shorter but wider in January, with the same area as the longer but narrower wedge during July.

Why is this true? Take a very short time interval Δt, so that the narrow pie-shaped wedge is essentially a triangle with area

$$\Delta A = \frac{1}{2}\,(\text{ base x height }) = \frac{1}{2}\, r\, \Delta s_{\perp}, \qquad (6)$$

where the "height" $\Delta s_{\perp}$ is the component of the planet's displacement perpendicular to the position vector **r**. This component of the displacement is $\Delta s_{\perp} = v_{\perp}\, \Delta t$, the tangential velocity of the planet times the short time interval.

The area is therefore

$$\Delta A = \frac{1}{2}\, r v_{\perp}\, \Delta t. \qquad (7)$$

The right-hand side of this equation is closely related to the angular momentum L of the planet about the Sun. Recall that the moment of inertia of a particle of mass m is $I = mr^2$ if it is a distance r from a rotation axis, so the angular momentum of the particle is

$$L = I\omega = mr^2\omega = m\,r\,v_\perp\ , \tag{8}$$

where $r\omega = v_\perp$ is the tangential velocity. The infinitesimal area swept out by a planet's orbit during a short time Δt is therefore

$$\Delta A = \frac{1}{2}\,r\,v_\perp\,\Delta t = \frac{L}{2m}\,\Delta t, \tag{9}$$

where L is the planet's angular momentum about the Sun and m is the planet's mass.

Is the angular momentum L *conserved*? We know that the angular momentum of an object is conserved if and only if the net torque on it is zero. The torque on the planet due to the Sun, which exerts the only appreciable force, is $\tau = r\,F_\perp$, where r is the distance to the Sun and $F_\perp$ is the tangential force, the component of force perpendicular to the radius vector. The Sun's gravity is directed exactly inward, however, with $F_\perp = 0$ so the torque $\tau = 0$. Therefore L is conserved, which makes it easy to find the area swept out during a finite time interval by summing Eq. 9 over all the small ΔA's to get a large area A:

$$A = \int dA = \int \frac{L}{2m}\,dt = \frac{L}{2m}\,t \tag{10}$$

where $\frac{L}{2m}$ can be brought outside the integral because it is constant. In fact, the quantity $\frac{L}{2m}$ remains constant during *any* time interval t, so the area A swept out during any two identical time intervals t is the same; the particle sweeps out "equal areas in equal times." Kepler's second law is simply a consequence of the conservation of angular momentum.

One interesting consequence of conservation of angular momentum for planetary orbits is that there is a simple relationship between the speeds of a planet at perihelion and aphelion. At the nearest and farthest points to the Sun the velocity of the planet is purely tangential, so can be written $v = r\omega$, where ω is the angular velocity. Therefore the angular momentum of the planet at these two points is $L = mr^2\omega = mrv$. Conservation of angular momentum tells us that the product rv is the same at aphelion as at perihelion, or equivalently

$$\frac{v_a}{v_p} = \frac{r_p}{r_a}. \tag{11}$$

If the aphelion of an orbit is twice as far from the Sun as the perihelion, the planet moves twice as fast at perihelion as it does at aphelion.

(c) *Kepler's third law*

When he compared the orbits of the known planets, Kepler found that their periods T and semimajor axes a are related by

$$T^2 \propto a^3; \tag{12}$$

the squares of the planets' orbital periods are proportional to their semimajor axes cubed. This relationship can be verified from the data in Table 1, by taking the ratio of the periods of any two planets and squaring the result; then showing that this is equal to the ratio of the corresponding semimajor axes cubed.

If T_e and a_e are the period and semimajor axis of Earth's orbit, for example, and T_m and a_m are the period and semimajor axis of Mars' orbit, then

$$\left(\frac{T_m}{T_e}\right)^2 = \left(\frac{a_m}{a_e}\right)^3 \tag{13}$$

according to Kepler III. Is it true? For greater accuracy we use observational results given in the CRC tables (53rd edition), which gives the semimajor axes as $a_e = 1.4957 \pm 0.0007 \cdot 10^8$ km and $a_m = 2.2784 \pm 0.0022 \cdot 10^8$ km. Then KIII predicts that

$$\frac{T_m}{T_e} = \left(\frac{2.2784}{1.4957}\right)^{3/2} = (1.5233)^{3/2} = 1.8801 .$$

The *observed* ratio of periods (also from the CRC tables) is

$$\frac{T_m}{T_e} = \frac{(5.93553 \pm .00016)\cdot 10^7 \text{s}}{(3.1558150)\cdot 10^7 \text{s}} = 1.8808,$$

just as predicted, within observational error. (The data in Table 1 can also be used, but the fewer significant figures given there leads to agreement to fewer significant figures.)

Why is $T^2 \propto a^3$? We already showed in Section A.1 that Newton's second law $\mathbf{F} = m\mathbf{a}$ together with the inverse-square gravitational force law gives KIII for circular orbits, where the semimajor axis a is simply the radius r. That is, we showed that

$$T^2 = \frac{4\pi^2}{GM} r^3, \tag{14a}$$

which confirms KIII for circular orbits. Using $\mathbf{F} = m\mathbf{a}$ for *elliptical* orbits shows that the correct general relationship is simply

$$T^2 = \frac{4\pi^2}{GM} a^3 \tag{14b}$$

in terms of the semimajor axis a. (See the appendix at the end of the chapter.) It is interesting that the period depends upon the semimajor axis of the orbit, but *not* on the eccentricity. Two orbits with the same semimajor axis have the same period, even though their eccentricities are different, as illustrated at the right.

Sample Problem 2. Halley's comet has a period of approximately 76 years. About how far does the comet get from the Sun? (Note: At its near point the comet comes inside the radius of Earth's orbit.)

Solution: Earth and Halley's comet both orbit the Sun, so their periods and semimajor axes are related by $\left(\frac{T_H}{T_E}\right)^2 = \left(\frac{a_H}{a_E}\right)^3$. The semimajor axis of Halley's comet is

$$a_H = a_E \left(\frac{T_H}{T_E}\right)^{2/3} = a_E \left(\frac{76\text{ yr}}{1\text{ yr}}\right)^{2/3} = 1\text{ A.U. }(18) = 18\text{ A.U.}$$

where the mean radius of Earth's orbit is 1 A. U. = $1.49 \cdot 10^8$ km. The comet comes inside Earth's orbit, so the farthest distance from the Sun must be between 35 and 36 A. U., as illustrated. (Astronomical tables give the distance as 35.3 A. U.)*

* Kenneth R. Lang Astrophysical Data: Planets and Stars Springer Verlag 1992

A.3 Velocity and acceleration in polar coordinates

We now need to describe position, velocity, and acceleration vectors in polar coordinates r, θ, useful for describing orbits and for many other applications as well.

(a) *Unit vectors and their time derivatives*

The location of any point P in a plane can be represented by the position vector $\mathbf{r} = r\hat{\mathbf{r}}$, illustrated in Fig. 2. The magnitude r tells how far P is from the origin, and the unit vector $\hat{\mathbf{r}}$ tells the direction in which P is located.

If P represents the position of a particle (or the center of mass of a planet, for example), usually both the magnitude r and direction $\hat{\mathbf{r}}$ of P change with time. Although the unit vector $\hat{\mathbf{r}}$ does not change in length, it does change in direction unless the particle moves exactly toward or away from the origin. The unit vector $\hat{\boldsymbol{\theta}}$ is always perpendicular to $\hat{\mathbf{r}}$, so $\hat{\boldsymbol{\theta}}$ also generally changes as the particle moves.

Fig. 2. The position vector of a particle $\mathbf{r} = r\hat{\mathbf{r}}$

To find the velocity and acceleration of the particle at position $\mathbf{r} = r\hat{\mathbf{r}}$, we will first need to find the time derivatives of the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. An easy way to do this is to begin by expressing $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ in terms of Cartesian unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$. The reason is that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ do *not* change as the particle moves; they not only have constant length but their directions never change either. From Fig. 3, note that by vector addition,

$$\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}} \tag{15a}$$

and

$$\hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}. \tag{15b}$$

If $\theta = 0$, for example, then $\hat{\mathbf{r}} = \hat{\mathbf{x}}$ (both vectors point to the right) and $\hat{\theta} = \hat{\mathbf{y}}$ (both point upward). If $\theta = \frac{\pi}{2}$ then $\hat{\mathbf{r}} = \hat{\mathbf{y}}$ (both point upward) and $\hat{\theta} = -\hat{\mathbf{x}}$ (both point to the left).

Fig. 3. Polar coordinate unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ in terms of Cartesian unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$.

We can now differentiate Eqs. 15(a, b) to find $\frac{d\hat{\mathbf{r}}}{dt}$ and $\frac{d\hat{\boldsymbol{\theta}}}{dt}$. This is not difficult, because the Cartesian unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ stay constant not only in magnitude but in direction, always pointing in the x-direction and the y-direction, respectively. Using the chain rule,

$$\frac{d\hat{\mathbf{r}}}{dt} = -\hat{\mathbf{x}}(\sin\theta)\,\dot{\theta} + \hat{\mathbf{y}}(\cos\theta)\,\dot{\theta} = (-\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta)\dot{\theta} = \dot{\theta}\,\hat{\boldsymbol{\theta}} \tag{16a}$$

$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\hat{\mathbf{x}}(\cos\theta)\,\dot{\theta} - \hat{\mathbf{y}}(\sin\theta)\,\dot{\theta} = -(\hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta)\dot{\theta} = -\dot{\theta}\,\hat{\mathbf{r}}. \tag{16b}$$

We will now use these unit-vector derivatives $\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\,\hat{\boldsymbol{\theta}}$ and $\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\dot{\theta}\,\hat{\mathbf{r}}$ to find the velocity and acceleration vectors of a particle in polar coordinates.

(b) *Velocity in polar coordinates*

A particle is located at the point $\mathbf{r} = r\,\hat{\mathbf{r}}$. The velocity of the particle is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d(r\,\hat{\mathbf{r}})}{dt} = \frac{dr}{dt}\,\hat{\mathbf{r}} + r\,\frac{d\hat{\mathbf{r}}}{dt}$$

$$= \dot{r}\,\hat{\mathbf{r}} + r\,\dot{\theta}\,\hat{\boldsymbol{\theta}} = v_r\,\hat{\mathbf{r}} + v_\theta\,\hat{\boldsymbol{\theta}} \tag{17}$$

using the product rule in differentiation, and the time derivative of $\hat{\mathbf{r}}$ given in Eq. 16(a). Note that the particle's radial and tangential velocity components are

$$v_r = \dot{r} \quad \text{and} \quad v_\theta = r\,\dot{\theta}\,. \tag{18}$$

If the particle is moving in a circle, for example, then $\dot{r} = 0$ because r never changes. There is then no radial component of velocity, and only a tangential component $v_\theta = r\,\dot{\theta} \equiv r\omega$.

Sample Problem 3: A spiral wire has the shape $r = \ell\left(\frac{\theta}{2\pi}\right)$, where r is the distance from one end of the wire, θ is the angle from the horizontal, and ℓ is a constant length. If a bead, starting from the origin, moves along the wire at constant speed v, find its velocity vector $\mathbf{v}$ in polar coordinates, in terms of ℓ, r, and the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$.

Solution: The velocity vector in polar coordinates is $\mathbf{v} = \dot{r}\,\hat{\mathbf{r}} + r\,\dot{\theta}\,\hat{\boldsymbol{\theta}} = \dot{r}\,\hat{\mathbf{r}} + r\left(\frac{2\pi}{\ell}\right)\dot{r}\,\hat{\boldsymbol{\theta}}$, using $\dot{\theta} = \frac{2\pi}{\ell}\,\dot{r}$ (found by differentiating the wire shape with respect to time). The magnitude of the bead's velocity is the square root of the sum of squares of its components,

$$v = \sqrt{\dot{r}^2 + \left(\frac{2\pi r}{\ell}\right)^2 \dot{r}^2} = \dot{r}\sqrt{1 + \left(\frac{2\pi r}{\ell}\right)^2}\,,$$

which is the constant speed of the bead. Therefore

$$\dot{r} = \frac{v}{\sqrt{1 + \left(\frac{2\pi r}{\ell}\right)^2}} .$$

We can substitute this back into the formula for $\mathbf{v}$ to get

$$\mathbf{v} = \frac{\left[\hat{\mathbf{r}} + \left(\frac{2\pi r}{\ell}\right)\hat{\boldsymbol{\theta}}\right] v}{\sqrt{1 + \left(\frac{2\pi r}{\ell}\right)^2}}$$

in terms of the given quantities. Note that the initial velocity of the bead (at $r = 0$) is $\mathbf{v} = v\hat{\mathbf{r}}$, directed radially (which makes sense), and that as r grows, an ever-increasing fraction of the velocity is tangential (which also makes sense).

(c) *Acceleration in polar coordinates*

We are now prepared to differentiate the velocity of the particle to find its acceleration. Starting with Eq. 17, and using also the product rule and Eqs. 16, the acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}) = \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d\hat{\mathbf{r}}}{dt} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d\hat{\boldsymbol{\theta}}}{dt}$$

$$= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\left(-\dot{\theta}\hat{\mathbf{r}}\right)$$

$$= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\boldsymbol{\theta}}$$

$$= a_r\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}} . \qquad (19)$$

The radial and tangential components of acceleration are therefore

$$a_r = \ddot{r} - r\dot{\theta}^2 \qquad \text{and} \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\,\dot{\theta}. \tag{20}$$

If the particle moves in a circle at constant speed, for example, then $\dot{r} = 0 = \ddot{r}$ and $\ddot{\theta} = 0$, so the only component of acceleration is $a_r = -r\dot{\theta}^2$, the well-known centripetal acceleration $a_{\text{centripetal}} = -r\omega^2$, the minus sign indicating that the acceleration is *inward*.

Sample Problem 4. A straight radial line is drawn from the center of a merry-go-round to the rim; the merry-go-round is then caused to rotate with angular speed ω. A child, starting at the center at time $t = 0$, walks at constant speed v along the line. According to outside inertial observers, find the radial and tangential components of the child's (a) velocity (b) acceleration.

Solution: The child's distance from the center is $r = vt$, so $\dot{r} = v$ and $\ddot{r} = 0$. The child's angular speed is $\dot{\theta} = \omega$, a constant, so $\ddot{\theta} = 0$. (a) the radial and tangential components of velocity are therefore

$$v_r = \dot{r} = v, \qquad \text{and} \qquad v_\theta = r\,\dot{\theta} = r\omega = \omega\, vt.$$

The child has a radial velocity as well as a tangential velocity due to the rotation of the merry-go-round.

(b) the radial and tangential components of acceleration are

$$a_r = \ddot{r} - r\dot{\theta}^2 = -\omega^2 v\, t \text{ and } a_\theta = r\ddot{\theta} + 2\dot{r}\,\dot{\theta} = 2\omega\, v.$$

Note that because of Newton's second law $\mathbf{F} = m\mathbf{a}$, both radial and tangential friction forces $F_r = -m\omega^2 vt$ and $F_\theta = 2m\omega\, v$ must be exerted on the child by the merry-go-round surface as the child walks.

Sample Problem 5. A block of mass m is attached to one end of a string. The block is set on a frictionless table and the other end of the string is threaded through a small hole in the tabletop and held in place. The block is then given a tangential velocity v_O so that it orbits the hole in a circle of radius r_O, as shown.

(a) Find the tension T_O in the string.

(b) The free end of the string is then pulled at constant velocity v_S downward. Find then the tension T in the string as a function of time, valid until the block reaches the hole or the string breaks.

Solution: (a) The centripetal acceleration of the block is $a_O = v_O^2/r_O$, so the net inward force, which is the tension in the string, is

$$T_O = ma_O = mv_O^2/r_O \,.$$

(b) The net force due to the string tension is purely radial and in the negative $\hat{\mathbf{r}}$ direction, so

$$\mathbf{F} = -T\hat{\mathbf{r}} = m\mathbf{a} = m\left[\left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(r\ddot{\theta} + 2\dot{r}\,\dot{\theta}\right)\hat{\boldsymbol{\theta}}\right].$$

There is no force in the $\hat{\boldsymbol{\theta}}$ direction, so the tangential acceleration $\left(r\ddot{\theta} + 2\dot{r}\,\dot{\theta}\right) = 0$. This is equivalent to angular momentum conservation, since

$$\frac{dL}{dt} = \frac{d}{dt}mr^2\dot{\theta} = m\left(r^2\ddot{\theta} + 2r\,\dot{r}\,\dot{\theta}\right) = mr\left(\ddot{r} - r\dot{\theta}^2\right) = 0.$$

Therefore the angular velocity is $\dot{\theta} = \dfrac{L}{mr^2}$, where L is a constant. Eliminating $\dot{\theta}$ in the radial part of $\mathbf{F} = m\mathbf{a}$, we get

$$-T = m\left(\ddot{r} - r\dot{\theta}^2\right) = m\left(\ddot{r} - \frac{L^2}{m^2r^3}\right).$$

The string velocity v_S is constant, so the radius r of the block's orbit as a function of time is $r = r_O - v_S t$, whose second derivative with respect to time is *zero*. The string tension is therefore

$$T = \frac{L^2}{mr^3} = \frac{L^2}{m(r_o - v_s t)^3} = \frac{m v_o^2 r_o^2}{(r_o - v_s t)^3}$$

since the angular momentum is $L = L_o = mr_o^2\dot{\theta}_o = mv_o r_o$. Note that at time $t = 0$ the tension has the value found in part (a), and that as the orbital radius $r = r_o - v_s t$ goes to zero the tension goes to infinity, meaning that the string must break when the radius becomes small enough. As the radius shrinks the block has to orbit faster and faster to keep the angular momentum constant, so the inward force has to grow larger and larger to be consistent with the increasing acceleration.

Sample Problem 6. An airplane buzzes an airfield by flying in a straight line at constant speed v and at a constant low elevation h above the field, traveling from left to right as shown. At time $t = 0$ the plane is directly above the air control tower. A radar set on the tower tracks the plane by measuring r (the distance to the plane) and θ (the angle from the vertical) both as functions of time. A tower computer evaluates also the time derivatives $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, $\ddot{\theta}$, all as functions of time.

(a) Find r and θ in terms of h, v, and time t.
(b) Find v_r and v_θ, the radial and tangential components of the plane's velocity.
(c) Find a_r and a_θ, the radial and tangential components of the plane's acceleration.

Solution: (a) The x- and y-coordinates of the plane are $x = vt$ and $y = h$, so

$$r = \sqrt{x^2 + y^2} = \sqrt{v^2 t^2 + h^2}. \quad \text{Also}$$

$$\tan\theta = \frac{x}{y} = \frac{v\,t}{h}, \text{ so } \theta = \tan^{-1}\left(\frac{v\,t}{h}\right).$$

(b) The radial velocity is $v_r = \dot{r} = \dfrac{v^2 t}{\sqrt{v^2 t^2 + h^2}}$, differentiating r as given in part (a).

Also from part (a), the time derivative of $\tan\theta$ is $\sec^2\theta\,\dot{\theta} = \dfrac{v}{h}$, so

$$\dot{\theta} = \frac{v/h}{\sec^2\theta} = \frac{v/h}{1 + \tan^2\theta} = \frac{v/h}{1 + (v\,t/h)^2} = \frac{v\,h}{h^2 + (v\,t)^2}.$$

The tangential velocity is therefore $v_\theta = r\,\dot{\theta} = \dfrac{v\,h}{\sqrt{(v\,t)^2 + h^2}}$.

$v_r = \dot{r} = 0$

$v_\theta = r\dot{\theta} = v$.

Note that $v_r = 0$ and $v_\theta = v$ at $t = 0$, when the plane is directly over the radar set and therefore traveling purely tangentially, and that $v_r \to v$ and $v_\theta \to 0$ as $t \to \infty$, which makes sense because in the limit the plane is going directly away from the tower.

(c) Differentiating once again the results for $\dot{r}$ and $\dot{\theta}$ given in part (b), we find that

$$\ddot{r} = \frac{v^2 h^2}{\left[v^2 t^2 + h^2\right]^{3/2}} \qquad \text{and} \qquad \ddot{\theta} = \frac{-2h\,v^3\,t}{\left[v^2 t^2 + h^2\right]^2}.$$

Therefore the radial and tangential components of acceleration are

$$a_r = \ddot{r} - r\dot{\theta}^2 = \frac{v^2 h^2}{\left[v^2 t^2 + h^2\right]^{3/2}} - \sqrt{v^2 t^2 + h^2}\left[\frac{v\,h}{\sqrt{(v\,t)^2 + h^2}}\right]^2 = 0,$$

and

$$a_\theta = r\ddot{\theta} + 2\dot{r}\,\dot{\theta}$$

$$= \sqrt{v^2 t^2 + h^2}\left(\frac{-2h\,v^3\,t}{\left[v^2 t^2 + h^2\right]^2}\right) + 2\left(\frac{v^2\,t}{\sqrt{v^2 t^2 + h^2}}\right)\left(\frac{v\,h}{h^2 + (v\,t)^2}\right) = 0.$$

Both acceleration components are zero! We could have seen this coming, because the airplane has no acceleration in Cartesian coordinates, so it can have no acceleration in polar coordinates either, or in any coordinate system in any inertial frame. The components of a vector of zero length are necessarily zero.

State of the Search

1. The gravitational force on an object of mass m outside a spherically symmetric body of mass M is given by Newton's gravitational force law

$$\mathbf{F} = m\mathbf{g} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

where $G = 6.67 \cdot 10^{-11}$ N m^2/kg^2 is Newton's constant of gravity and r is the distance of the object from the center of the spherical body.

2. Properties of *circular* orbits about (approximately) spherical objects like the Sun and Earth can be found using $\mathbf{F} = m\mathbf{a}$ with Newton's force law and the centripetal acceleration

$$a = \frac{v^2}{r} = r\omega^2 = r\left(\frac{2\pi}{T}\right)^2.$$

3. Satellites can also be in *elliptical* orbits, as discovered centuries ago by J. Kepler, who studied the motion of planets around the Sun. Kepler's three laws, all of which were later derived by I. Newton from his laws of motion and the inverse-square law gravitational force given above, are

   KI. *Planets move in elliptical orbits, with the Sun at one focus.*
   KII. *Planetary orbits sweep out equal areas in equal times.*
   KIII. *The periods squared of the planetary orbits are proportional to their semimajor axes cubed.*

   Kepler's second law is equivalent to the conservation of angular momentum, valid because the gravitational force is central, so the Sun exerts no torque on the planets.

4. Plane polar coordinates r, θ are often used to describe orbits and other trajectories. In such coordinates there are corresponding unit vectors $\hat{\mathbf{r}}$ and $\hat{\theta}$. The velocity of an object moving in a plane has radial and tangential components

$$\mathbf{v} = \dot{r}\,\hat{\mathbf{r}} + r\,\dot{\theta}\,\hat{\theta}$$

and the acceleration has radial and tangential components

$$\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(r\ddot{\theta} + 2\dot{r}\,\dot{\theta}\right)\hat{\theta}.$$

PICTURE QUIZ A

1. Two small satellites are in circular orbits about a spherical asteroid. The inner satellite has orbital radius R about the center of the asteroid, and it moves with speed v_o. If the outer satellite has orbital radius $4R$, how fast does it move, as a multiple of v_o?

(A) $4v_o$ (B) $2v_o$ (C) v_o (D) $v_o/2$ (E) $v_o/4$

2. If the inner satellite in quiz problem 1 has angular momentum L_o and twice the mass of the outer satellite, what is the angular momentum of the outer satellite, as a multiple of L_o?

(A) $4L_o$ (B) $2L_o$ (C) L_o (D) $L_o/2$ (E) $L_o/4$

3. A comet orbiting the Sun has its perihelion at radius 1.0 A.U. and aphelion at radius 17.0 A.U. The period of its orbit is

(A) 9 years (B) 18 years (C) 27 years (D) 36 years (E) 54 years

4. If the speed of the comet described in quiz problem 3 is v_o at perihelion, its speed at aphelion is

(A) $v_o/17$ (B) $v_o/\sqrt{17}$ (C) v_o (D) $\sqrt{17}\, v_o$ (E) $17\, v_o$

PROBLEM QUIZ A

1. Jupiter has more mass than all the other planets combined, so the solar system is primarily the Sun-Jupiter system. The mass of Jupiter is about 0.1% that of the Sun, and the orbital radius of Jupiter is about 1000 times the radius of the Sun. The rotational period of the Sun's equator is about 25 days, and Jupiter takes about 12 years to orbit the Sun. Show that Jupiter has a much larger fraction of the solar system's angular momentum than the Sun has.

2. The space shuttle is in a circular low-Earth orbit of period T_O and radius R_O about Earth's center. A satellite deployed from the shuttle bay is given a boost velocity Δv relative to the Shuttle and in the Shuttle's direction of motion relative to Earth's surface, so the satellite enters an elliptical orbit with perigee R_O and apogee $3R_O$. (a) Sketch the satellite's orbit (b) Find the period of this orbit as a multiple of T_O. (c) Find how fast the satellite is moving relative to Earth when it is at apogee, in terms of T_O, R_O, and Δv.

3. A cylindrical space colony of radius R rotates about its axis of symmetry with angular speed ω. An elevator moving within a frictionless tube at constant speed v relative to the colony transports people from the axis to the colony rim along a radial line, as shown, starting at time $t = 0$. In terms of given parameters and as functions of time, find (a) the radial velocity (b) the tangential velocity of the elevator relative to an outside inertial frame. The elevator is supported by cables. If the elevator and occupants have total mass M, find the force on the elevator exerted by (c) the cables (d) the tube, both as functions of time.

4. The speed of a racecar increases at a constant rate from 25 km/hour to 150 km/hr along a circular track of radius 250 m, moving a total distance 120 m during this period. Find (a) the tangential acceleration (b) the radial acceleration (c) the magnitude of the total acceleration of the car, all at the end of this period.

Appendix: Elliptical Orbits and the Laws of Newton

Johann Kepler claimed that planets move in elliptical orbits, with the Sun at one focus of the ellipse. Nearly a hundred years later Isaac Newton showed that his laws of motion and gravity predict that the paths are indeed elliptical. How can we demonstrate this?

(A) First, we can find the equation for an ellipse in polar coordinates (r, θ), placing the origin at the right-hand focus, as shown. Then the distance $d_2 = r$, and the angle α within the triangle is $\alpha = \pi - \theta$. Using the law of cosines $A^2 = B^2 + C^2 - 2BC\cos\alpha$ for triangles with sides A, B, C, where α is the angle opposite the side of length A, we find

$$d_1{}^2 = r^2 + (2ae)^2 + 4ae\, r\cos\theta \tag{A1}$$

since $\cos(\pi - \theta) = -\cos\theta$. Here a and e are the semimajor axis and eccentricity of the ellipse (See Section A.2 (a)) Now $d_1 = 2a - d_2 = 2a - r$, so

$$\sqrt{r^2 + (2ae)^2 + 4aer\cos\theta} = 2a - r\,. \tag{A2}$$

Squaring,

$$r^2 + (2ae)^2 + 4aer\cos\theta = 4a^2 - 4ar + r^2. \tag{A3}$$

Cancelling the r^2 terms and then the common factor $4a$, we get $a\,e^2 + e\,r\cos\theta = a - r$; then solving for r,

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}\,. \tag{A4}$$

This is the shape $r(\theta)$ of an ellipse in terms of its eccentricity e and semimajor axis a.* The circle is the limiting case of an ellipse as the eccentricity goes to zero. As the two foci come together, $d_1 = d_2 = R$ in the limit, where R is the radius of the circle.

* Note from Eq. A4 that the smallest r (at perihelion) corresponds to the largest value of $\cos\theta$, i. e. $\cos\theta = 1$ at $\theta = 0$. Therefore $r_p = a(1 - e^2)/(1 + e) = a(1 - e)$. Similarly, the largest r (at aphelion) has $\theta = \pi$, with $r_a = a(1 - e^2)/(1 - e) = a(1 + e)$, confirming Eqs. 5 in the Chapter.

(B) Now we can show that elliptical orbits are solutions of $\mathbf{F} = m\mathbf{a}$, if the force is Newton's inverse-square gravitational force $\mathbf{F} = m\mathbf{g} = -\dfrac{GMm}{r^2}\hat{\mathbf{r}}$. Begin with the fact that the force is purely radial, with no tangential (θ) component. Then

$$F_\theta = ma_\theta = m(r\ddot{\theta} + 2\dot{r}\,\dot{\theta}) = 0, \qquad \text{(A5)}$$

using the expression for a_θ given in Eq. 20. Now the time derivative of the product $r^2\dot{\theta}$ is

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = r^2\ddot{\theta} + 2r\,\dot{r}\dot{\theta} = r\left(r\ddot{\theta} + 2\dot{r}\,\dot{\theta}\right) = 0, \qquad \text{(A6)}$$

because of Eq. A5. Therefore the quantity $r^2\dot{\theta}$ is *conserved*, which is not surprising, since the angular momentum of the planet around the Sun is

$$L = mr^2\dot{\theta}, \qquad \text{(A7)}$$

and we know that the planet's angular momentum is conserved because the central gravitational force exerts no torque. The radial component of $\mathbf{F} = m\mathbf{a}$ is $F_r = ma_r$,

$$-\frac{GMm}{r^2} = ma_r = m(\ddot{r} - r\dot{\theta}^2) = m\ddot{r} - mr\left(\frac{L}{mr^2}\right)^2 \qquad \text{(A8)}$$

using Eq. 20 for a_r and eliminating $\dot{\theta}$ using Eq. A7. Solving for $\ddot{r}$,

$$\ddot{r} = \frac{L^2}{m^2r^3} - \frac{GM}{r^2}. \qquad \text{(A9)}$$

Is an elliptical orbit a solution of this equation? Using Eq. A4 for an ellipse, $r = \dfrac{a(1-e^2)}{1+e\cos\theta}$, we can find the time derivative

$$\dot{r} = \frac{a(1-e^2)}{(1+e\cos\theta)^2}\left(e\sin\theta\;\dot{\theta}\right) = \frac{a(1-e^2)\,e\sin\theta}{(1+e\cos\theta)^2}\left(\frac{L}{mr^2}\right) = \frac{L\,e\,\sin\theta}{ma\,(1-e^2)} \qquad \text{(A10)}$$

where we have used Eq. A7 to eliminate $\dot{\theta}$ in the second step, and the shape Eq. A4 in the final step. Differentiating again,

$$\ddot{r} = \frac{L\,e}{ma\,(1-e^2)}\cos\theta\;\dot{\theta} = \frac{L\,e\cos\theta}{ma\,(1-e^2)}\left(\frac{L}{mr^2}\right). \tag{A11}$$

Now we are ready to substitute into Eq. A10, to see if an ellipse solves the equation. That is, is it true that

$$\frac{L^2\,e\cos\theta}{m^2 a\,(1-e^2)r^2} = \frac{L^2}{m^2 r^3} - \frac{GM}{r^2}\quad ? \tag{A12}$$

Multiply through by the denominator on the left-hand side, and substitute the ellipse Eq. A4 for the remaining r in the denominator of the first term on the right. Then the hoped-for equality is

$$L^2\,e\cos\theta = L^2(1+e\cos\theta) - GMm^2a(1-e^2), \tag{A13}$$

or

$$L^2 = GMm^2a(1-e^2). \tag{A14}$$

Therefore an ellipse is indeed a solution of $\mathbf{F} = m\mathbf{a}$ for planetary motion, as long as the planet's angular momentum L on the left is related to the geometrical quantities a and e on the right by Eq. A14. Note that for a given semimajor axis, the larger the eccentricity the smaller the angular momentum.

The reader can demonstrate another relation between the physics and geometry of elliptical orbits. The total energy $E = K + U$ [i.e., $E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r}$] of a planet is related to the semimajor axis of its elliptical orbit by*

$$E = -\frac{GMm}{2a} \tag{A15}$$

* See problem AP35

Finally, we can derive Kepler's *third* law $T^2 \propto a^3$, relating the periods of planets with their semimajor axes, using Newton's laws. In Section A.2(c) we derived the result

$$T^2 = \frac{4\pi^2}{GM} r^3, \tag{14a}$$

showing that KIII is valid for the special case of circular orbits. To show that KIII is also valid for arbitrary *elliptical* orbits, start with Eq. 10 of Section A.2(b),

$$A = \int dA = \int \frac{L}{2m}\, dt = \frac{L}{2m}\, t \tag{10}$$

relating the area swept out by a radial line as a function of time, as illustrated at the right. Here L is the angular momentum of the planet and m is its mass. If we integrate Eq. 10 over one complete period of the orbit, then $t = T$, the orbital period and A is the entire area of the ellipse.

The area of an ellipse is $A = \pi ab$, where a and b are the semimajor and semiminor axes. Also b is related to a by $b = a\sqrt{1-e^2}$, where e is the eccentricity. Therefore

$$A = \pi ab = \pi a^2\sqrt{1-e^2} = \frac{L}{2m}\, T \tag{A16}$$

using Eq. 10. Squaring the last equality and solving for T^2 gives $T^2 = \dfrac{4\pi^2 m^2 a^4 \left(1-e^2\right)}{L^2}$. According to Eq. A14, the angular momentum obeys $L^2 = GMm^2 a(1-e^2)$, so finally

$$T^2 = \frac{4\pi^2}{GM} a^3, \tag{A17}$$

which proves Kepler's third law. Newton's laws not only show that $T^2 \propto a^3$, but they also provide the proportionality constant. Given the mass of the Sun, we can now calculate the planet's orbital period given the semimajor axis, or the semimajor axis given the period.

Special Topic: Hohmann Transfer Orbits

What is the best way to send a spacecraft to Mars? Depending upon what one means by "best", many routes are possible. But the trajectory requiring the *least fuel* is a "minimum-energy" transfer orbit or *Hohmann transfer orbit*. This orbit takes full advantage of the motion of the Earth to help the spacecraft get out to Mars.

Once a spacecraft's engines are turned off and the craft gets far enough from Earth so the Sun's gravitational force dominates its motion, it obeys all the equations we have derived so far, including Kepler's laws. For example, it coasts toward its destination in an elliptical orbit with the Sun at one focus. If it has enough fuel to achieve a certain speed "v" relative to Earth, the craft will have the largest initial speed in the frame of the Sun if it is launched in the direction of Earth's velocity, v_e. Then the initial velocity in the Sun's frame will be $v + v_e$, the largest value it can have for given v. Its subsequent orbit toward Mars is shown. The elliptical path is tangent to Earth's orbit at launch, and tangent to Mars's orbit at arrival, just barely making it.

How long does this transfer orbit take to get to Mars? We can use KIII to find out. Note that the major axis of the craft's orbit is

$$2a_c = r_e + r_m \qquad \text{(S1)}$$

assuming Earth and Mars move in nearly circular orbits. The semimajor axis is therefore

$$a_c = \frac{r_e + r_m}{2} = \frac{1.50 + 2.28}{2} \cdot 10^8 \text{ km} = 1.89 \cdot 10^8 \text{ km}. \qquad \text{(S2)}$$

From Kepler's third law, we find that the period T_c for the craft's elliptical orbit obeys

$$\left(\frac{T_c}{T_e}\right)^2 = \left(\frac{r_e + r_m}{2r_e}\right)^3. \qquad \text{(S3)}$$

The craft traverses only half of this orbit in traveling to Mars, so the travel time is

$$T = T_c / 2 = \frac{1}{2}\left[\frac{r_e + r_m}{2r_e}\right]^{3/2} T_e = \frac{1}{2}\left[\frac{1.89}{1.50}\right]^{3/2} (1 \text{ year}) = 0.707 \text{ years} = 258 \text{ days.*}$$

There are only certain times ("launch windows") when the spacecraft can depart Earth, because Mars has to be at the right spot when the craft arrives at Mars orbit. Specifically, for a Hohmann transfer orbit, Mars must be on the direct opposite side of the Sun from where the Earth was when the spacecraft was launched. The diagrams below show the sequence.

*In his science-fiction novel "Stranger in a Strange Land", Robert Heinlein looks back on the first peopled journeys to Mars: "an interplanetary trip . . . had to be made in free-fall orbits -- from Terra to Mars, 258 Terran days, the same for return, plus 455 days waiting at Mars while the planets crawled back into positions for the return orbit."

Questions and Problems on *Orbits, Chapter A*

AQ1. Stars in the disk of the Milky Way galaxy, like our Sun, move in approximately circular orbits about the galactic nucleus. For a large range of distances from the center, stars all have an orbital velocity of approximately 220 km/s. Is the force on stars in the galaxy inverse r-squared? Is it nevertheless possible that gravity alone is the force keeping the stars in circular orbits?

AQ2. Geosynchronous satellites rotate about Earth in 24 hours. Could one station a geosynchronous satellite permanently above Moscow?

AQ3. Two astronauts are in the same circular orbit of radius R around Earth, 1800 apart, as shown below. Astronaut A has two ham sandwiches, and Astronaut B has none. How can A throw a ham sandwich to B? Find the time to arrive, and the semimajor axis of the sandwich in transit in terms of R. There are many solutions!

AQ4. Circles, ellipses, parabolas, and hyperbolas are the only allowed orbital shapes for a particle experiencing an attractive central inverse-square gravitational force. These shapes are also called *conic sections*, because these figures are the possible intersections of a plane with a cone. Sketch a cone and a plane tilted at various angles, which intersects the cone. How is the plane oriented to generate each of the figures?

AQ5. Suppose the gravitational force exerted by the Sun on a planet were not proportional to the mass of the planet. Show that $KIII$ would not be valid in this case.

AQ6. The planets in a hypothetical solar system all move in circular orbits, and the ratio of the periods of any two orbits is equal to the ratio of their orbital radii squared. What is the dependence of the central force on distance from their sun?

AQ7. An astronaut is marooned in a powerless spaceship in circular orbit around the asteroid *Vesta*. The astronaut reasons that puncturing a small hole through the spacecrafts outer surface into an internal water tank will lead to a jet action of water vapor expanding into space. Which way should the jet be aimed so the spacecraft will descend in the least time to the surface of Vesta? (See Isaac Asimov's first published story, *Marooned off Vesta*. His protagonist punctured a hole in the tank at a spot directly opposite the direction of the surface. Was that the best angle?)

AQ8. A thrown baseball travels in a small piece of an *elliptical orbit* before it hits the ground. What is the semimajor axis of the ellipse, as a fraction of Earths radius?

AQ9. Why don't any planets move in circular orbits or any comets move in parabolic orbits?

AQ10. Assume the period of elliptical orbits depends upon G, M (the Sun's mass), and a (the semimajor axis). Prove Kepler's third law using dimensional arguments alone.

AQ11. Einstein's *general theory of relativity* predicts that light rays from distant stars will bend as they pass by the Sun. The bending angle is directly proportional to G, and also depends upon the Sun's mass M, the speed of light c, and the distance d between the Suns center and the light ray at its point of closest approach to the Sun. If ray b passes by twice as far from the Sun's center as ray a, what is the ratio of their bending angles?

AQ12. Using his *general theory of relativity*, Einstein showed that the perihelion of the planet Mercury's orbit (and the perihelia of other planets as well) should slowly precess; that is, the

perihelion should drift to a slightly different place from one revolution to the next, as though the elliptical orbit were slowly twisting, as illustrated in the exaggerated picture below. The angle of precession is directly proportional to G, and it also depends upon the Sun's mass M, the speed of light c, and the mean (i. e. average) radius r of the planet's orbit. The mean radius of Mercury's orbit is 58×10^6 km, and its perihelion precesses by 43 seconds of arc per century. The mean radius of Earth's orbit is 149×10^6 km. How much should Earth's perihelion precess?

Section A.1 Circular orbits

AP1. A satellite is placed in circular orbit about Earth, a distance of one Earth's radius above the surface. Find the satellite's (a) period, in minutes; (b) speed, in km/s; (c) angular speed, in radians/minute.

AP2. The period T of a particle that moves in a circular orbit just outside the surface of a homogeneous sphere of radius R might depend upon Newton's gravitational constant G, the radius R, and the sphere's mass density ρ. (a) Using dimensional reasoning alone, find by what factor T would change if R were doubled while keeping all else the same. (b) Find a complete formula (including constants) for T in terms of any or all of the given parameters. (c) Find T (in hours) for any sphere having a mean density equal to the density of water (the planet Jupiter is a near-example of such a sphere!) (d) What is the mean density of Earth, given that a hypothetical surface-skimming Earth satellite would have a period of 87 minutes?

AP3. (a) Communication satellites are put into geosynchronous orbit. That is, they are typically put into Earth's equatorial plane, with a period of 24 hours. How many Earth radii is the radius of such an orbit? Then find the satellite's (a) speed (b) acceleration.

AP4. The Moon orbits Earth in 27.3 days. A communications satellite orbits Earth in a single day, so that it stays above the same point on Earths surface. What is the ratio of the Moon's orbital radius to that of the satellite?

AP5. Two satellites of equal mass orbit Earth. Satellite A is in a circular orbit of radius r_A, and satellite B is in a circular orbit of radius $r_B = 2r_A$. What is the ratio of their (a) speeds, v_B/v_A? (b) periods, T_B/T_A? (c) kinetic energies K_B/K_A? (d) potential energies U_B/U_A? (e) total energies E_B/E_A?

AP6. A satellite is to be placed in synchronous orbit around the planet Jupiter to study the famous red spot. That is, the satellite rotates perfectly with the planet, so it hovers over the same point on the planets equator. How many Jupiter radii is the radius of such an orbit? (The rotation period of Jupiter is 9.9 hrs, its mass is about 320 Earth masses, and its radius is about 11 times that of Earth.)

AP7. The Sun moves at a speed v_s 220 km/s in a circular orbit of radius r_s 30,000 light-years around the center of our galaxy. Also, Earth takes time $T_e = 1$ year to describe an almost circular orbit of radius $r_e = 1.5 \times 10^{11}$ m around the Sun. (a) Find a formula for the total mass responsible for keeping the Sun in its orbit, as a multiple of the Sun's mass M_0, in terms of v_s, r_s, T_e, and r_e. (b) Evaluate this mass numerically, as a multiple of M_0.

AP8. Gas and dust particles circle a black hole in what is called an accretion disk. The innermost stable orbit of such particles has radius $r = 6GM/c^2$, where M is the mass of the black hole and c is the speed of light. Find the rotation period (in seconds) of a dust particle with such a minimum radius around a black hole whose mass is that of (a) the Sun (b) 10^6 Suns. (c) The find the speed of such a particle in each case according to Newton's laws, expressed as a multiple of c.

AP9. Imagine a hypothetical galaxy in which all the stars rotate with the same period in circular orbits about the galactic nucleus. They therefore move as though they were points on a rigid wheel rotating with constant angular velocity. If this motion is caused by a force $F = kmr^n$ on stars directed toward the nu-

cleus, where k is a constant, m is the mass of the star, and r is the distance of the star from the nucleus, (a) what must be the value of the exponent n? (b) What is the period of rotation in terms of k? (c) what is the torque exerted on each star by the force F, taking the axis to pass through the nucleus, perpendicular to the plane of rotation?

Section A.2. Kepler's laws and planetary orbits

AP10. A spy satellite designed to peer closely at HMC every day at noon has a 24-hr period, and a perigee of 100 km altitude directly above HMC. What is the altitude of the satellite at apogee?

AP11. The perihelion and aphelion of asteroid *Apollo* in its orbit around the Sun are 0.964×10^8 km and 3.473×10^8 km, respectively. Apollo therefore swings in and out past Earth's orbit. Find (a) the semimajor axis (b) the eccentricity (c) the period of Apollo's orbit in years. Apollo is only one of many "Apollo asteroids" that cross Earth's orbit. Some asteroids have struck Earth in the past, and others will strike in the future.

AP12. A hypothetical Earth-crossing asteroid has its perihelion at 0.900 A. U. and its aphelion at 4.50 A. U. as it orbits the Sun. (a) If its speed is v_a at aphelion, what is its speed at perihelion, in terms of v_a? (b) What is the period of this asteroid in years?

AP13. Some orbital radii and rotation periods are given in the table for the four Galilean satellites of Jupiter. Fill in the missing information, using $KIII$.

moon	orb rad (10^3 km)	period (days)
Io		1.77
Europa	671	3.55
Ganymede	1070	
Callisto		16.69

AP14. The radii of planetary orbits are given in Table 3.1. Find the ratio of the orbital angular momentum per unit mass of each planet to the orbital angular momentum per unit mass of the Earth. That is, find

$$(L_{\text{planet}}/m_{\text{planet}})/(L_{\text{Earth}}/m_{\text{Earth}})$$

for the other eight planets (counting Pluto as a planet). Assume the planets move in circular orbits with radius equal to the observed semi-major axis.

AP15. Verify $KIII$ for the orbits of Mercury and Neptune by comparing their periods and semi-major axes.

AP16. The radius of Earth's orbit varies between $r_p = 1.471 \times 10^8$ km at perihelion, its closest distance to the Sun, and $r_a = 1.521 \times 10^8$ km at aphelion, its farthest distance. Earth's orbital speed at aphelion is $v_a = 29.29$ km/s. (a) What is Earth's orbital speed at perihelion? What area does the orbit sweep out in 24 hours (b) centered on aphelion? (c) centered on perihelion?

AP17. Solve AP16 for Pluto's orbit, with perihelion $r_p = 4.4 \times 10^9$ km, aphelion $r_a = 7.4 \times 10^9$ km, and orbital velocity $v_a = 3.7$ km/s at aphelion.

AP18. A planet of mass m has constant angular momentum L as it orbits the Sun. The area A swept out by its orbit depends upon m, L, and the time interval Δt. Using dimensional reasoning alone, how must A depend upon the three variables?

AP19. In terms of polar coordinates r and θ, the angular momentum and energy of a planet of mass m orbiting the Sun (of mass M) are

$$L = mr^2\dot{\theta}$$

and

$$E = K + U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r}$$

(a) Show that in terms of r and $\dot{r}$ alone, the total energy can be written

$$E = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}}$$

where the effective potential energy

$$U_{\text{eff}} = L^2/2mr^2 - GMm/r.$$

(b) Sketch the form of U_{eff} as a function of r. (c) The minimum of U_{eff} corresponds to the minimum total energy the planet can have for given angular momentum. Only a single radius is possible at the minimum, so the planet

moves in a circular orbit. Find the radius r of this orbit and the corresponding total energy E in terms of L, G, M, and m.

AP20. A vertical pole rises through a frictionless sheet of ice. One end of a light, flexible rope of length D is attached to the pole, and the rope is then laid out straight on the ice sheet, as shown below. A skater of speed v_0, moving perpendicular to the rope, grabs the free end of the rope and holds on. The rope gradually winds up around the pole, drawing the skater inward. When the skater's distance from the pole has been reduced to $D/2$, what is the skater's speed? *Hint*: Show that angular momentum and kinetic energy cannot both be conserved. Which of the two is conserved, and why? Why is the other not conserved?

AP21. A particle of mass m, tied to one end of a light string, sits on a frictionless table. The other end of the string is threaded through a tiny frictionless hole in the top of the table, and then tied to a weight of mass M hanging in uniform gravity g, as shown below. If the particle executes a circular orbit of radius R about the hole, in terms of m, M, R, and g, (a) What is the period of the orbit? (b) What is the magnitude of the particle's orbital angular momentum?

AP22. A particle of mass m, attached to one end of a light string, sits on a frictionless table. The other end of the string is threaded through a tiny frictionless hole in the top of the table, as shown in the figure for AP21, except that instead of being attached to a weight of mass M, the string is held in place by a person underneath the table. (a) The particle is set into circular motion of radius r_0 on the tabletop, with rotational period T_0. What is the angular momentum of the particle? What is its kinetic energy? (b) The person gradually pulls on the string, slowly reducing the radius r of the particles orbit. What is the angular momentum of the particle when its orbital radius is r ? *Hint*: What torque acts on the particle while it is pulled inward? (c) What is $F(r)$, the force on the particle, in terms of m, r, r_0, and T_0 ? Note that $F(r)$ is inward, so should be negative. By the time the orbital radius has been reduced to $r_0/2$, (d) What is the kinetic energy of the particle? (e) What has been the total work $W = \int F(r)dr$ done by the person in reducing r? (f) Using the latter result, verify that the work-energy theorem is consistent with the particles initial and final kinetic energies.

AP23. A block of mass m sits at position A on a horizontal frictionless table, and is connected to a massless unstretched spring of force-constant k and length d, as shown below. The other end of the spring is attached to the origin O, and can freely swing around it. An initial impulse gives the block a velocity v_0, as shown. When the block reaches position B, at which the spring is perpendicular to its original orientation, the length of the spring has become $2d$. (a) What physical quantities are conserved as the block moves from A to B? Why? (b) Let the initial velocity be $v_0 = \sqrt{2kd^2/m}$. Find the magnitude of the block's velocity at B in terms of v_0, and find the angle this velocity makes with the spring direction at B.

Section A.3 Velocity and acceleration in polar coordinates

AP24. The tangential acceleration of a particle in polar coordinates is $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$. For purely central forces, with $\mathbf{F} = F(r)\hat{\mathbf{r}}$, the particle's tangential acceleration is zero (Why?) Prove then that the particle's angular momentum $\ell = mr^2\dot{\theta}$ is conserved.

AP25. Starting at time $t = 0$, a bug walks at constant speed $v = 1.0$ inches/second along a radial scratch from the center of a phonograph record to the edge, 6.0 inches away. The record turns at 33 1/3 revolutions per minute. At $t = 3.0$ s find the bug's (a) radial velocity (b) tangential velocity (both in inches/second), and the bug's (c) radial acceleration (d) tangential acceleration (both in inches/second^2).

AP26. The speed of a race car increases at constant rate from 55 km/hour to 170 km/hr over a distance of 250 m along a circular racetrack of 440 m radius. Find the magnitude of the total acceleration of the car when it speed has completed the 250 m.

AP27. Starting from rest at time $t = 0$, a racecar of mass m moving on a circular track of radius R increases its speed at constant rate according to $v = \alpha t$. Assuming the coefficient of static friction between tires and track is μ_s, at what time t does the car slip off the track, assuming $\alpha < \mu_s g$?

AP28. A comet of mass m moves along the indicated parabola described by the path $r = D/(1+\cos\theta)$ referred to its focus, as shown below, where D is a constant length. (a) In terms of D, find the comet's closest distance to the focus. (b) The angular momentum $\ell = mr^2\dot{\theta}$ of the comet about the focus is conserved. Find an expression for the velocity vector $\mathbf{v}$ of the comet as as function of θ (Your answer will include the constants D and ℓ and the unit vectors in polar coordinates, but not r, $\dot{r}$, or $\dot{\theta}$.) (c) Find the speed v of the particle as a function of θ.

AP29. A wire is bent into the form of a spiral, in which the distance r from one end of the wire is related to the angle θ by $r = \beta\theta$, where β is a constant. A bead of mass m is placed on the wire at $r = 0$, and then starting at time $t = 0$ the bead is caused to move outward on the wire at the constant speed v, as shown below. In terms of v, β, and r, find the radial and tangential components of the bead's (a) velocity; (b) acceleration.

AP30. A radar set on the ground tracks an unidentified flying object. Starting at time $t = 0$, and continuing to time $t = 100$ s, the radar set observes the distance r of the UFO from the radar set and the angle θ of the UFO from the horizontal, both as functions of time. The results show that $r = d[1 + (at - bt^2)^2]^{1/2}$ and $\theta = \tan^{-1}(at - bt^2)$, in terms of constants $d = 3.0$ km, $a = 0.33$ s^{-1}, and $b = 1.65 \times 10^{-3}$ s^{-2}. Find the UFO's (a) radial and tangential velocities (b) vertical and horizontal velocities (c) vertical and horizontal accelerations. (d) Describe the UFO's motion.

AP31. A rocket blasts off from Kennedy Space Center. At time $t = t_0$ a tracking radar finds that the rockets distance is $r = 2.7$ km, that it has angle $\theta = 0.84$ rad measured from the horizontal, and that the first and second time derivatives of these quantities are $\dot{r} = 0.25$ km/s, $\ddot{r} = 0.030 km/s^2$, $\dot{\theta} = 0.16$ rad/s, $\ddot{\theta} = 0.025$ rad/s^2. At this time, find the rocket's (a) radial acceleration, (b) tangential acceleration, (c) vertical velocity, and (d) horizontal velocity.

AP32. A ship searching for underwater treasure follows the path $r = 75\ \theta$ m, as shown below, where the angle θ is in radians, at a constant speed of 2.0 knots. (1 knot is one nautical mile/hr = 0.514 m/s.) When $\theta = 2\pi$, find the radial and tangential components of the ship's (a) velocity (b) acceleration.

AP33. An airplane is tracked by a radar antenna, as shown below. Data from the antenna provides the angle $\theta(t)$, its first and second derivatives $\dot{\theta}$ and $\ddot{\theta}$, the range $r(t)$, and its first and second derivatives $\dot{r}$ and $\ddot{r}$. Find the following about the plane in terms of $\theta(t), r(t)$, and their time derivatives. (a) The radial and tangential components of acceleration, a_r and a_θ. (b) the horizontal and vertical components of velocity $v_{x(t)}$ and $v_{y(t)}$ (c) the speed v (d) the horizontal and vertical components of acceleration, $a_{x(t)}$ and $a_{y(t)}$.

AP34. The Moon has no atmosphere, so one could shoot mined material off the surface with the escape velocity, without burning up the material. One conceivable way to hurl buckets of material into space is to use a horizontal boom of radius R rotating with constant angular velocity ω, as shown, looking down from above.

A filled bucket would be placed on the boom a distance R_0 from the rotation axis, instantaneously at rest relative to the boom. The bucket would then slide (assume frictionlessly) out along the boom, and as it slips off at the end it could have a large enough velocity to escape the Moon. (a) Show that as a function of time, the distance r of the bucket from the rotation axis is given by $r = Ae^{\omega t} + Be^{-\omega t}$, where A and B are constants. (b) Find A and B in terms of R_0, given the initial conditions of the bucket. (c) Find the radial and tangential components of the bucket's velocity when it comes off the end of the boom. (d) Find the speed of the bucket when it comes off the boom. (This speed must equal the Moon's escape speed if the bucket is to escape.) (e) Find the normal force exerted by the boom on a filled bucket of mass m, as a function of its instantaneous position r.

Appendix: Elliptical Orbits and the Laws of Newton

AP35. Show that the energy of a planet of mass m orbiting a star of mass M is given by $E = -GMm/2a$, where a is the semimajor axis of the planet's elliptical orbit.

Special Topic: Hohmann transfer orbits

AP36. A spacecraft is sent from the orbit of Europa to the orbit of Ganymede, both moons of the planet Jupiter, by Hohmann transfer orbit. How long is the trip, in days? (The radii of the orbits of Europa and Ganymede are, respectively, 671×10^3 km and 1070×10^3 km. Europa's period of rotation about Jupiter is 3.55 days.)

AP37. Find the Hohmann transfer-orbit time to reach Venus from Earth. (See Table 1 for the orbital radii of Venus and Earth.)

AP38. In *Stranger in a Strange Land*, Robert Heinlein states that travelers to Mars spent 258 days on the journey out, the same for return, "plus 455 days waiting at Mars while the planets crawled back into positions for the return orbit." Show that travelers *would* have to wait about 455 days, if both journeys were by Hohmann transfer orbits. Note: Solving this problem calls for pictures and careful thought!

AP39. The space shuttle is circling Earth in its equatorial plane in an orbit of radius 6700 km from Earths center. In its cargo bay is a solid-fuel rocket with a communications satellite as payload. (a) The crew wants to launch the satellite up to geosynchronous orbit. (An object in geosynchronous orbit circles Earth in 24 hours, remaining above the same point on the equator.) Find the radius of this geosynchronous orbit. (b) Sketch the Hohmann transfer orbit from the shuttle to geosynchronous orbit. (c) How long does it take the satellite to reach its destination?

B. Simple Harmonic Motion

"Everything should be made as simple as possible, but not simpler." - A. Einstein

B.1 Oscillation

Oscillations take place all around us. Tree limbs sway back and forth in the breeze. A car runs over a bump and bounces up and down. A distant churchbell vibrates when struck; the resulting oscillating air molecules spread out in a sound wave, causing our eardrums to vibrate back and forth. Guitar strings, kettle drums, and electrical circuits all oscillate, as do skyscrapers during earthquakes and Earth's surface itself.

In a mechanical oscillator, the displacement of some object (like a point on a guitar string) moves back and forth in a periodic way. In electrical circuits it is the current that oscillates. How do such oscillations arise? What are their common features?

Consider the energy diagram of an egg. If the egg rolls end-over-end, its gravitational potential energy $U(x)$ looks something like that shown in Fig. 1. The center of mass of the egg is higher when the egg stands on its pointy end, lower when it stands on its flatter end, and lower still when it rests on its side. If the total energy E of the egg is less than either potential energy maximum, as shown, there are then two turning points, and the egg rocks back and forth between them. (If the egg is hardboiled it will rock back and forth for some time, because the dissipation is small: That is, friction with the tabletop and the surrounding air drains away the energy quite slowly. The interior of a *raw* egg sloshes around more while it rocks, so the dissipation is greater; the egg's energy is more quickly converted into heat and so it comes to rest more quickly at its equilibrium point where the potential energy is a minimum.)

Fig. 1. Energy diagram of an egg for the end-over-end direction. Oscillation takes place if the energy is within the potential energy valley, where the egg rocks back and forth between the turning points.

B.2 A spring-mass system

The prime example of an oscillating system is a mass on the end of a spring. The results found in this case can be applied, at least conceptually, to a wide range of other oscillating systems, as we will see in Chapter C.

Attach a block of mass m to one end of a spring of force-constant k. The other end of the spring is held fixed, and the block, which feels the Hooke's-law restoring force $F = -kx$, is free to move along the surface of a horizontal frictionless table.

The potential energy of the system is $U(x) = \frac{1}{2}kx^2$, the parabolic function of position shown in the energy diagram of Fig. 2 (derived in *Day Ten*). The equilibrium position of the block is located where the force $F = -\dfrac{dU(x)}{dx}$ is zero, i.e., at $x = 0$ at the potential energy minimum. For given energy E, the block oscillates back and forth between the two turning points. At either turning point, all the energy is spring potential energy $\frac{1}{2}kA^2$, where A is the amplitude of oscillation, the farthest distance the block reaches from equilibrium. At the equilibrium point all the energy is kinetic energy $\frac{1}{2}mv^2$. At an arbitrary position x, the energy of the system is in general the sum of kinetic and potential energies,

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2. \tag{1}$$

Some questions: What is the *position* $x(t)$ of the block as a function of time, and what is the *velocity* of the block $v(t) = \dfrac{dx(t)}{dt}$ as a function of time?

Fig. 2. The energy diagram for a spring-mass system. For any given energy E the block oscillates back and forth between the two turning points.

Note that if x is positive the spring is stretched and the block is pulled back toward its equilibrium position, while if x is negative the spring is *compressed* and the block is *pushed* back toward equilibrium. If the block is pulled a distance A away from equilbrium and released from rest, one finds that the block oscillates back and forth, as expected; the displacement $x(t)$ of the block from equilibrium is plotted in Fig. 3. The *amplitude* of oscillation A is the maximum displacement to the right (and A is also the maximum displacement to the left.)

Fig. 3. The displacement of the block from equilibrium, as a function of time.

The *period* of oscillation is the time for the block to execute one complete oscillation, say from one end of its motion to the other end and back again. The period T is usually measured in seconds. The *frequency* of oscillation ν is the number of oscillations per unit time,

$$\nu = 1/T. \tag{2}$$

The oscillation frequency ν is typically measured in cycles/second, or *Hertz* (Hz), a unit used *only* for frequency, where 1 Hz = 1 cycle/s. In the example shown in Fig. 3, the period $T = 2$ s and the frequency $\nu = 0.5$ Hz.

Sample Problem 1. Find the period and frequency of the oscillation shown in the graph below.

Solution: The time between successive zeros of $x(t)$ is 3.5 s, so the time for one complete oscillation is the period $T = 7.0$ s. The corresponding frequency is $\nu = 1/T = (1/7)$ Hz.

How does the oscillation period T, the time to complete one full cycle, depend upon the physical parameters in the problem? Dimensional reasoning can help. The only parameters in the problem are the mass m, the spring force-constant k, and the oscillation amplitude A. The dimensions of these parameters, in terms of the fundamental dimensions of mass M, length L, and time T, are

$$[m] = \mathrm{M}, \qquad [k] = \frac{Force}{length} = \frac{\mathrm{ML/T^2}}{\mathrm{L}} = \frac{\mathrm{M}}{\mathrm{T^2}}, \quad \text{and} \qquad [A] = \mathrm{L}. \tag{3}$$

The only way to combine these to obtain the period, a *time*, is to take the inverse square root of the force-constant k (to get the dimensions of time in the numerator), and then multiply by the square root of the mass m (to cancel the dimensions of mass). The amplitude can't be involved at all, because nothing else has dimensions of length! The period of a spring-mass system must therefore have the form

$$T = C\sqrt{\frac{m}{k}}\,, \tag{4}$$

where C is an unknown dimensionless constant. We can't find C using dimensional reasoning alone, but we will find it using $F = ma$, as shown in the next section. Nevertheless, we have already shown that the period of an ideal Hooke's-law spring-mass system is independent of amplitude, proportional to the square root of the mass (so that the period would double if we quadrupled the mass), and inversely proportional to the square root of the spring stiffness (so the period would be cut in half if we quadrupled k).

B.3 Motion using $F = ma$

What is the actual analytic function $x(t)$? That is, what is the function sketched in Fig. 3 or in *Sample Problem* 1? Finding $x(t)$ is our goal.

We use $F = ma$ for a block of mass m on the end of a spring with force-constant k. We will assume for simplicity that the mass of the spring is negligible compared with the mass of the block. A Hooke's-law spring exerts the force $F = -kx$, so the acceleration of the block is

$$a = \frac{d^2x}{dt^2} = \frac{F}{m} = \frac{-kx}{m} = -\left(\frac{k}{m}\right)x\,. \tag{5a}$$

In most of the $F = ma$ problems we have solved before, the force (and therefore the acceleration) depends at most on *time*, not on position. With an equation like (5a), however, we

cannot simply integrate the acceleration a with respect to time to find the velocity v, and then integrate v with respect to time to find the position $x(t)$.*

So how *can* we solve Eq. 5a, in which the acceleration depends upon position? Note from this *simple harmonic oscillator equation*

$$\boxed{\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x} \tag{5b}$$

that the position $x(t)$ of the block has an interesting property: Its second derivative is equal to itself, except for a negative multiplying constant $-\left(\frac{k}{m}\right)$. What function has this property? We start with a simpler question. Is there a function whose second derivative is the negative of itself?

The *sine* and *cosine* functions both have this property. Consider $x(t) = \cos t$, for example. Its first derivative is

$$\frac{d}{dt}\cos t = -\sin t \tag{6a}$$

and so

$$\frac{d^2}{dt^2}\cos t = -\frac{d}{dt}\sin t = -\cos t\ . \tag{6b}$$

The second derivative of $\cos t$ *is* equal to itself, except for a minus sign. Now how do we get the constant k/m, which we need to solve Eq. 5b ?

Take the time derivative of $\cos\omega t$ instead, where ω is a constant.** Using the chain rule,

$$\frac{d}{dt}\cos\omega t = -\omega\sin\omega t. \tag{7a}$$

* Suppose we tried to integrate Eq. 5a with respect to time. We would have $\int a\,dt \equiv \int \frac{d^2x}{dt^2}\,dt = -\left(\frac{k}{m}\right)\int x\,dt$. The integral over the acceleration is easy: It is the velocity $\frac{dx}{dt}$, since the time derivative of the velocity is the acceleration $\frac{d^2x}{dt^2}$. The integral $\int x\,dt$ on the right can't be evaluated, however, because we don't know $x\ (t)$. That is what we are trying to find!

* * We have used the ω symbol in rotation problems as well, as a shorthand for the variable angular velocity $d\theta/dt$. In this chapter ω is simply a constant, however, unrelated to any change in angle.

Differentiating a second time,

$$\frac{d^2}{dt^2}\cos\omega t = -\omega\frac{d}{dt}\sin\omega t = -\omega^2\cos\omega t\ . \tag{7b}$$

The second derivative of the function $x(t) = \cos\omega t$ is equal to the function $x(t)$ itself, apart from the negative constant $-\omega^2$. The solution therefore works if we choose the constant ω to be

$$\boxed{\omega = \sqrt{\frac{k}{m}}\ ,} \tag{8}$$

according to the $F = ma$ equation (5b). The function $x(t) = \cos\omega t$ also makes sense physically, because it oscillates.* Figure 4 illustrates the function $x(t)$, its first derivative $\dot{x}(t)$, the velocity of the block, and its second derivative $\ddot{x}(t)$, the acceleration of the block.

Fig. 4. (a) The function $x(t) = \cos\omega t$, drawn as a function of ωt. (b) The corresponding velocity $\dot{x}(t) = -\omega\sin\omega t$. (c) The corresponding acceleration $\ddot{x}(t) = -\omega^2\cos\omega t$. Note that the acceleration is at all times a constant ($-\omega^2$) times the function $x(t)$ itself.

* The sine function $x(t) = \sin\omega t$ works equally well, for the same value of ω, as one can easily verify. See problem BP22.

How *fast* does the solution $x(t) = \cos\omega t$ oscillate? At time $t = 0$, $x(t) = 1$; also $\cos\omega t = 1$ when $\omega t = 2\pi, 4\pi, 6\pi$, etc., as shown in Fig. 4(a). There is therefore one complete oscillation of $x(t) = \cos\omega t$ when ωt increases from 0 to 2π. So as time evolves from $t = 0$ to $t = 2\pi/\omega$, one period $T = 2\pi/\omega$ of the motion has been completed. The constant ω is therefore

$$\omega = \frac{2\pi}{T} \tag{9}$$

The constant ω is also

$$\omega = 2\pi\nu \tag{10}$$

using Eq. 2. That is, the number of radians/second is 2π times the number of cycles/second, which makes sense since there are 2π radians in each revolution. Inverting Eq. 9 and using Eq. 8, the period of oscillation can be written

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{11}$$

in terms of the physical parameters m and k. By dimensional reasoning alone, we learned that $T = C\sqrt{m/k}$; now we have found that the constant $C = 2\pi$.

In spite of its apparent success, the solution $x(t) = \cos\omega t$ has a serious problem. It is not dimensionally consistent! The left hand side of the equation has the dimensions of length, while the cosine function on the right is dimensionless. Can we modify the solution so that it not only satisfies $F = ma$, but also has the correct dimensions?

This problem is easily solved by multiplying the cosine function by a constant with the dimensions of length. In fact, the function

$$x(t) = A\cos\omega t \tag{12}$$

is a solution of the $F = ma$ equation 5(b) (with $\omega = \sqrt{k/m}$), and has the correct dimensions as well, for *any* constant A, as long as A has dimensions of length. The constant A is of course the *amplitude* of the oscillation. It gives the maximum extension of the mass from its equilbrium position.

Sample Problem 2. (a) A ball of mass m is hung on a spring of force-constant k, where the gravitational field is g. How much has the spring been stretched when the ball hangs in equilibrium? (b) The ball is then pulled down below its equilbrium position a distance A and released from rest at time $t = 0$. Find its position as a function of time.

Solution: (a) In equilibrium there is no net force on the ball, so the gravitational force must be balanced by the spring force kz_0, where z_0 is the spring stretch. Therefore $kz_0 = mg$, giving $z_0 = mg/k$.
(b) In general the ball is at position z, where we will measure z positive downward from the end of the unstretched spring. The net force on the ball is

$$F = mg - kz = kz_0 - kz = -k(z - z_0),$$

using the result of part (a). Newton's second law $F = ma$ then gives

$$F = -k(z - z_0) = m\frac{d^2z}{dt^2} = m\frac{d^2(z - z_0)}{dt^2},$$

where in the last expression we are free to subtract z_0 from z, since z_0 is a constant so its derivatives are zero. This equation is just the simple harmonic oscillator equation in the variable $z - z_0$, so its solution is $z - z_0 = A\cos\omega t$. The position of the ball as a function of time is therefore

$$z = z_0 + A\cos\omega t.$$

Note that the ball oscillates exactly the same as it would in the absence of gravity, except that it now oscillates about a new equilibrium point, a distance z_0 below the unstretched spring. The angular frequency of oscillation is still $\omega = \sqrt{k/m}$.

We can confirm that the SHO in uniform gravity behaves the same as it does without gravity (except for the change in equilibrium position) by finding the potential energy of the system, $U(z) = U_{\text{spring}}(z) + U_{\text{grav}}(z) = \frac{1}{2}kz^2 - mgz$. Here the gravitational potential energy is *negative* since we are measuring z positive *downward*. Using the fact that $z_0 = mg/k$, it follows that

$$U(z) = \tfrac{1}{2}kz^2 - mgz = \tfrac{1}{2}k(z - z_0)^2 - \tfrac{1}{2}kz_0^2,$$

where the first term is the potential energy of a spring alone (with the same force constant k) centered about the new equilibrium point z_0, while the second term is just a constant, so does not influence the motion (recall that the force is the negative derivative of the potential energy.) The potential energy shape is the same as without gravity, so the frequency of oscillation is the same.

Box 1. Simple Harmonic Oscillation and Circular Motion

It is interesting that the relationship $\omega = \dfrac{2\pi}{T}$ between the constant ω and the oscillation period T of the spring is *exactly the same* as the relationship between the angular velocity ω of a uniformly rotating wheel and its period of rotation T. Is this just a coincidence?

Suppose an object moves in a circle of radius R, and that a very distant source of light, in the same plane as the circle, shines down upon the object, as shown. The object then casts a shadow upon the ground, also as shown. As the object orbits at constant speed v (and therefore also has constant angular velocity $\omega = v/R$) the shadow moves back and forth along the ground.

How does the shadow move as a function of time? Place the object at the right-most point of the circle, and make it move counterclockwise around the circle at constant angular velocity ω. The initial displacement of the shadow from the center of its trajectory is R, and since the angle of the object around the circle is $\theta = \omega t$, the position of the shadow at an arbitrary time t is $x = R\cos\omega t$. This is the same as the formula for the displacement of a particle on the end of a spring! That is, the shadow moves exactly as though the object were attached to a spring, free only to oscillate back and forth in a horizontal straight line. The amplitude of the oscillation is $A = R$, the radius of the circle, and the constant $\omega = 2\pi\nu$ of the oscillating shadow is the same as the angular velocity ω of the rotating object. Because of this connection between simple harmonic motion and uniform circular motion, the quantity ω for the oscillator is often called the *angular frequency* or the *circular frequency* of the oscillator, even though the oscillator is not changing its angle nor is the mass on the end of the spring moving in a circle. The quantity ω is given in the units rad/sec, consistent with the fact that there are 2π radians in a circle.

B.4 Phase

We released the block from rest in the experiment shown in Fig. 3. That is, the slope of $x(t)$, which is the velocity of the block, is zero at time $t = 0$. What if we give the block a nonzero initial velocity? Such an initial velocity has no effect on the $F = ma$ equation of motion, but it does affect the solution to the equation. The solution must be such that the initial value of $\dot{x}$ corresponds to the given initial velocity of the block. Figure 5 illustrates what happens if the initial velocity is either away from or toward the equilibrium point. The motion is no longer $x(t) = A\cos\omega t$ in either of the cases illustrated, because $\cos\omega t$ is a maximum at $t = 0$. We need a more general expression for $x(t)$.

Fig. 5. Oscillations of a block given an initial velocity (a) away from (b) toward the equilibrium point. The initial velocity is the initial slope of the graph.

Each of the motions shown has in fact the form

$$x(t) = A\cos(\omega t + \phi_0)\,, \tag{13}$$

where ϕ_0 is a *phase angle*. The effect of the phase angle is to shift the cosine function to the left or to the right, relative to $\cos\omega t$. The motion represented by Eq. 13 is actually the *most general* solution of the $F = ma$ equation

$$\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x \tag{5b}$$

for the simple harmonic oscillator, although we won't prove it here. We can at least see that Eq. 13 *is* a solution of Eq. 5(b), by noting that the second derivative of $x(t) = A\cos(\omega t + \phi_0)$ is equal to $-\omega^2$ times itself. Then if we choose $\omega = \sqrt{k/m}$, Eq. 5(b) is obeyed.

The phase angle ϕ_0 specifies the shift of the cosine function to the right or left. The function $x(t)$ is illustrated in Fig. 6 for the cases $\phi_0 = -\pi/4$, $\phi_0 = 0$, and $\phi_0 = \pi/4$. If $\phi_0 = -\pi/4$, for example, the velocity of the block at time t is

$$v(t) = \frac{dx(t)}{dt} = -A\omega \sin(\omega t - \frac{\pi}{4}). \tag{14}$$

At the initial time $t = 0$, the velocity is therefore

$$v(0) = -A\omega \sin(-\frac{\pi}{4}) = A\omega \sin(\frac{\pi}{4}) = \frac{A}{\sqrt{2}}\omega, \tag{15}$$

which is positive, corresponding to an initial *outward* velocity. The cosine function has been shifted to the *right* by an angle of $\frac{\pi}{4}$ radians, or 45°. If instead $\phi_0 = +\pi/4$, the initial velocity is inward; in this case the cosine function is shifted to the *left* by 45°, as shown.

Fig. 6. The function $x(t) = A \cos(\omega t + \phi_0)$. The amplitude A and the period of oscillation T are shown. Three phase angles are illustrated: (a) $\phi_0 = -\pi/4$, (b) $\phi_0 = 0$, and (c) $\phi_0 = \pi/4$.

The quantity $(\omega t + \phi_0)$, the entire argument of the cosine function, is called the *phase* of the motion. The phase is particularly useful in comparing different oscillations. Two oscillations are said to be *in phase* if their phases are either equal to one another or differ by $2\pi n$, where n is an integer, positive or negative. In all such cases the value of the cosine function for one oscillation is the same as for the other oscillation.

If one mass oscillates according to $x_1(t) = A\cos\omega t$, while a different mass oscillates according to $x_2(t) = A\cos(\omega t + \phi_0)$, there is a *phase difference* ϕ_0 between the two motions.

Suppose for example we hang three identical masses m_1, m_2, m_3 from three identical springs as shown. We displace each mass from equilibrium by the same distance A and release each from rest, but not all at the same time! We release m_2 at time $t = 0$; m_1 a little *later* at time $t = t_0$; and m_3 a little *earlier*, at time $t = -t_0$. Then if y is the elevation away from equilibrium, the three equations of motion are

$$y_1(t) = A\cos\omega(t - t_0) \qquad y_2(t) = A\cos\omega t \qquad y_3(t) = A\cos\omega(t + t_0)$$

In terms of the notation $y(t) = A\cos(\omega t + \phi_0)$, the phase angle ϕ_0 is *zero* for mass m_2, while $\phi_0 = -\omega t_0$ for mass m_1, and $\phi_0 = +\omega t_0$ for mass m_3. The motions are illustrated in Fig. 7. (If the outer masses happen to be released at times $\pm t_0$ such that $t_0 = \pi/4$, then Fig. 6 also represents the motion.)

Fig. 7. The motion of three identical masses on three identical springs, out of phase with one another because they are released at different times.

Sample Problem 3. (a) Find the amplitude A and phase angle ϕ_0 for an oscillator whose initial position is x_0 and whose initial velocity is v_0.

(b) Sketch $x(t)$ for an oscillator with angular frequency ω = 1.0 rad/s in the case $x_0 = 5.0$ cm and $v_0 = 7.5$ cm/s.

Solution: (a) Setting $t = 0$ in $x(t) = A\cos(\omega t + \phi_0)$ and in $\dot{x}(t) = -A\omega\sin(\omega t + \phi_0)$, the initial conditions are related to A and ϕ_0 by

(i) $x_0 = A\cos\phi_0$ and (ii) $v_0 = -A\omega\sin\phi_0$.

Dividing Eq.(ii) by Eq.(i) cancels out A and gives the phase-angle ϕ_0 :

$$\phi_0 = \tan^{-1}\left(-\frac{v_0}{\omega x_0}\right).$$

Dividing Eq.(ii) by ω, and then taking the sum of squares of this equation and of Eq.(i) eliminates ϕ_0, using the trig identity $\sin^2\phi_0 + \cos^2\phi_0 = 1$. We can then solve for A to find

$$A = \sqrt{x_0^2 + v_0^2 / \omega^2}\ .$$

(b) In the special case, $\phi_0 = \tan^{-1}\left(-\dfrac{7.5\text{ cm/s}}{1.0\text{ rad/s}\cdot 5.0\text{ cm}}\right) = -\tan^{-1}(1.5) = -56^\circ$ *

and $A = \sqrt{(5.0\text{ cm})^2 + (7.5\text{ cm/s})^2 / (1.0\text{ rad/s})^2} = 9.0$ cm. Therefore $x(t)$ is as shown.

* Note that $\phi_0 = -\tan^{-1}(1.5) = -56^\circ$ *or* -236°. The first solution is the correct one here, since $\phi_0 = -56^\circ$ makes both x_0 and v_0 positive. The solution $\phi_0 = -236^\circ$ makes both x_0 and v_0 *negative*, so would have been correct had it been true that $x_0 = -5.0$ cm and $v_0 = -7.5$ cm/s.

Sample Problem 4. A toy car of mass $m = 0.5$ kg oscillates on the end of a spring of force-constant $k = 2.0$ N/m. Find the amplitude A and phase angle ϕ_0 of the oscillator if the car (a) is released from rest at time $t = 0$, at $x = 1.0$ (b) is initially located at $x = 0$, with initial velocity $v = 3.0$ m/s.

Solution: The position of the car is $x(t) = A\cos(\omega t + \phi_0)$, so its velocity as a function of time is $v = \dfrac{dx}{dt} = -\omega A \sin(\omega t + \phi_0)$. At time $t = 0$ the position and velocity are therefore

$$x_0 = A\cos\phi_0 \quad \text{and} \quad v_0 = -\omega A \sin\phi_0.$$

The angular frequency of the oscillator is $\omega = \sqrt{k/m} = 2.0$ rad/sec.

(a) Given that $v_0 = 0$, it follows that $\sin\phi_0 = 0$, so we can take $\phi_0 = 0$. Therefore

$$x_0 = A\cos(0) = A.$$

That is, with these initial conditions, the amplitude is $A = x_0 = 1.0$ m, and the phase angle is $\phi_0 = 0$. After time $t = 0$, the position of the car is $x(t) = 1\text{ m} \cdot \cos\omega t$ as shown, where $\omega = 2.0$ rad/s.

(b) If $x_0 = 0$, then $\cos\phi_0 = 0$, so we can take either $\phi_0 = \pi/2$ or $\phi_0 = -\pi/2$. In the first case the initial velocity is

$$v_0 = -A\omega\sin\pi/2 = -A\omega,$$

and in the second case the initial velocity is

$$v_0 = -A\omega\sin(-\pi/2) = +A\omega.$$

The initial velocity is given as a positive quantity, so we make the choice $\phi_0 = -\pi/2$, with $A = v_0/\omega = 3.0\text{ m/s} / 2.0\text{ rad/s} = 1.5$ m. The position of the car is therefore $x(t) = 1.5\text{ m} \cdot \cos(\omega t - \pi/2) = 1.5\text{ m} \cdot \sin\omega t$ as shown, where $\omega = 2.0$ rad/s .

B.5 Oscillator energy

What is the energy of a ball oscillating on the end of a spring? The oscillator generally has both potential energy and kinetic energy. We can find each of these as functions of time, and can also check that the total energy of the system remains constant.

The spring stretch is $x(t) = A\cos(\omega t + \phi_o)$, so the spring potential energy is

$$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t + \phi_o) \tag{16}$$

as a function of time. The potential energy $U(t)$ is maximal when $x = \pm A$, when the spring is maximally stretched or compressed. This maximum potential energy is $U_{max} = \frac{1}{2}kA^2$. Also $U(t) = 0$ at $x = 0$, when the spring is neither stretched nor compressed.

The potential energy $U(t)$ is plotted in Fig. 7, in the case where the ball is released from rest at $x = A$, which corresponds to a phase angle $\phi_o = 0$.

Fig. 7. Potential energy $U(t)$, Kinetic energy $K(t)$, and the total energy E of an oscillator released from rest at $t = 0$ with spring stretch $x = A$.

The kinetic energy of the oscillator as a function of time is

$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2A^2\sin^2(\omega t + \phi_o)\,. \tag{17}$$

The kinetic energy $K(t)$ is maximal when the mass is moving the fastest, at the equilibrium point of the spring. This maximum kinetic energy is $K_{max} = \frac{1}{2}m\omega^2A^2$. Note that $K(t)$ is zero at the endpoints of oscillation, at $x = \pm A$. $K(t)$ is also plotted in Fig. 7, for the case $\phi_o = 0$.

What is the total energy of the spring? We expect it to be *constant*, since the spring force is conservative. The sum of the potential and kinetic energies is

$$E = U(t) + K(t) = \frac{1}{2}kA^2\cos^2(\omega t + \phi_o) + \frac{1}{2}m\,\omega^2A^2\sin^2(\omega t + \phi_o)\,. \tag{18}$$

The angular velocity ω of the oscillator is $\omega = \sqrt{\frac{k}{m}}$, as in Eq. 8, so $m\omega^2 = k$ in the last term. The total energy is therefore

$$E = \frac{1}{2}kA^2\cos^2(\omega t + \phi_o) + \frac{1}{2}kA^2\sin^2(\omega t + \phi_o) = \frac{1}{2}kA^2 \tag{19}$$

a constant, as expected. Figure 7 shows the total energy E for the case $\phi_o = 0$.

Sample Problem 5. The pan of a supermarket vegetable-weighing scale has mass m, causing the spring supporting the pan to stretch a distance $x = x_o$. If a sack of onions of mass m is suddenly placed on the pan at time $t = 0$, the pan oscillates up and down. As functions of time, find the kinetic and potential energies of the system.

Solution: Balancing forces for the unloaded pan, $mg = kx_o$, so the force constant of the spring is $k = mg/x_o$. If the onions were added slowly, the spring would stretch an addditional distance x_o when it comes to equilibrium. When the onions are added suddenly, the spring oscillates about this new equilibrium point with amplitude x_o, as shown in *Sample Problem* 2. We can then neglect gravity and measure spring stretch from the new equilibrium point. Since $x = -x_o$ when the onions are suddenly added, the position of the pan is $x = -x_o\cos\omega t$, measured positive downward from the new equilbrium point. The effective spring potential energy as a function of time is therefore $U_{\text{spring}} = \frac{1}{2}kx^2 = \frac{1}{2}kx_o^2\cos^2\omega t$, where $\omega = \sqrt{\frac{k}{2m}} = \sqrt{\frac{g}{2x_o}}$. The velocity of the pan and onions is $v = \dot{x} = x_o\omega\sin\omega t$, so their kinetic energy is $K = \frac{1}{2}(2m)v^2 = mx_o^2\omega^2\sin^2\omega t$. The total energy is $E = U_{\text{spring}} + K = \frac{1}{2}kx_o^2$, a constant, as shown.

Box 2. Deriving Simple Harmonic Motion from Energy Conservation

In Section B.3 we derived the motion of a block on the end of a spring, starting with $F = ma$. Alternatively, we can find $x(t)$ using *energy conservation*. The energy of the spring-block system is the sum of its kinetic and potential energies, $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$, where m is the mass of the block and k is the force-constant of the spring. Dividing by E and rearranging terms,

$$\frac{k}{2E}x^2 + \frac{m}{2E}\dot{x}^2 = 1. \tag{20}$$

Ignoring for a moment the coefficients of the x^2 and $\dot{x}^2$ terms, we notice that the sum of the squares of x^2 and of $\dot{x}^2$ is unity. What function x has this property?

Eq. 20 is reminiscent of the trig identity $\cos^2\theta + \sin^2\theta = 1$, since we know that $\frac{d}{d\theta}\cos\theta = -\sin\theta$. So let us try $x = A\cos\omega t$. We have to include the constants A and ω to get the correct dimensions, since A (the amplitude) must have dimensions of length (x is a length) and ω (the angular frequency) must have dimensions of inverse time. The corresponding time derivative of x is $\dot{x} = -A\omega\sin\omega t$. Substituting into Eq. 20,

$$\frac{kA^2}{2E}\cos^2\omega t + \frac{mA^2\omega^2}{2E}\sin^2\omega t = 1 \tag{21}$$

To use the trig identity we must choose $A = \sqrt{\frac{2E}{k}}$ to make the coefficient of $\cos^2\omega t$ equal to one, and must then choose $\omega = \sqrt{\frac{k}{m}}$ to make the coefficient of $\sin^2\omega t$ equal to one as well.

More generally we could have added in a phase angle, so

$$x = A\cos(\omega t + \phi) \tag{22}$$

with $A = \sqrt{\frac{2E}{k}}$ and $\omega = \sqrt{\frac{k}{m}}$, which also solves the energy conservation Eq. 21. This is the same result found from solving $F = ma$ with the added benefit that we have found the relation $A = \sqrt{2E/k}$ between the amplitude and energy.

State of the Search

1. Many physical systems oscillate periodically in time. The prime example is the spring-mass system, where an object of mass m oscillates at the end of a Hooke's-law spring of force-constant k.

   The spring potential energy is $U(x) = \frac{1}{2}kx^2$, as shown; for any energy E, the block oscillates back and forth between the two turning points located where $E = U(x)$. In this way the energy of the object varies between all potential (at the turning points) to all kinetic (at the equilibrium point $x = 0$).

2. The $F = ma$ equation for the object becomes

$$\ddot{x} + \omega^2 x = 0$$

   where x is the position of the object measured from equilibrium, $\ddot{x}$ is the acceleration (the second time derivative of x) and ω is the angular frequency of oscillation. The equation is called the *simple harmonic oscillator equation.*

3. The general solution of the simple harmonic equation is

$$x(t) = A\cos(\omega t + \phi_0)$$

   in terms of the amplitude constant A and phase angle ϕ_0. These two constants may be found in terms of the initial position x_0 and velocity v_0 of the object, since $x(0) = x_0$ and $\dot{x}(0) = v_0$.

4. The argument $\omega t + \phi_0$ of the cosine function is called the *phase* of the oscillation. The frequency ν and period T of the oscillator are related to one another and to the angular frequency ω by

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi}.$$

PICTURE QUIZ B

1. The velocity of a block on the end of a spring is given as a function of time in the graph below. Which of the following set of answers is the closest to the correct amplitude A and angular frequency ω of the block's position?

(A) 1.3 m, 1.6 s^{-1}

(B) 2.0 m, 1.3 s^{-1}

(C) 1.3 m, 2.0 s^{-1}

(D) 2.0 m, 1.6 s^{-1}

(E) 1.3 m, 3.2 s^{-1}

2. A ball is hung on a spring and the frequency of oscillation is measured. The spring is then cut in half and the experiment is repeated with the same ball on one of the half-springs. In terms of the original oscillation frequency, the new oscillation frequency is

(A) half as large

(B) smaller by the factor $1/\sqrt{2}$

(C) the same

(D) larger by the factor $\sqrt{2}$

(E) twice as large

3. The current i in an electrical circuit obeys $i = A\cos(\omega t + \phi_0)$, as shown in the graph below.

(a) The oscillation frequency (in Hz) is

(A) 0.50 (B) 0.75 (C) 1.0 (D) 1.5 (E) 2.0

(b) The angular frequency (in s^{-1}) is

(A) 0.21 (B) 0.67 (C) 1.2 (D) 3.1 (E) 5.7

(c) The amplitude (in Amperes) is

(A) 1.0 (B) 2.0 (C) 3.0 (D) 5.0 (E) 6.0

(d) The phase angle ϕ_0 (in radians) is

(A) $-\pi$ (B) $-\pi/2$ (C) 0 (D) $\pi/2$ (E) π

(e) The phase at time $t = 1.0$ s is

(A) 0.75 (B) 1.6 (C) 2.1 (D) 3.0 (E) 3.7

PROBLEM QUIZ B

1. A block is attached to one end of a spring of force-constant k, while the other end of the spring is held in place. At time $t = 0$ the spring is compressed a distance x_0, and the block is released from rest. It then executes simple harmonic motion with period T_0. In terms of any or all of k, x_0, and T_0,

(a) what is the mass of the block?
(b) what is the amplitude of the motion?
(c) what is the fastest speed achieved by the block?

2. Four passengers, each of mass m, step slowly into a car of mass M through its four doors. As a result, the car sags a distance d. Now they suddenly all step out of the car. What is the car's subsequent frequency of oscillation ν, in terms of any or all of M, m, g, and d? (Assume the car has no shock absorbers to damp out the oscillation.)

3. At time $t = 0$ the position of a mechanical oscillator is $x_0 = 49$ cm and its velocity is $v_0 = -32$ cm/s. The mass of the oscillator is $m = 0.071$ kg, and its oscillation period is $T = 6.0$ s. Find the oscillator's
(a) amplitude, in cm
(b) energy, in Joules
(b) maximum speed, in cm/s.

4. A bullet of mass m and velocity v is fired from below into a block of mass M hanging at rest on a spring of force-constant k that is attached to the ceiling. The bullet comes to rest within the block before the block has had time to move appreciably.
(a) How high does the bullet + block system rise before again coming instantaneously to rest? (b) What is the amplitude of the resulting oscillation about the equilibrium point of the block + bullet?
(c) What is the angular frequency of oscillation of the system?

Special Topic: Falling Through an Asteroid

We land upon an airless, spherical asteroid. Gravity is weak, so we find it easy to leap hundreds of meters off the surface, and drift back again several minutes later. Exploring the surface, we encounter a hole in the ground, a few meters in diameter, black as the sky except for a few pinpoints of light seen as we stare directly into its depths. Fearless explorers, we step into the hole and begin to fall. What becomes of us?

First of all, we need to investigate some properties of gravity. According to Newton's law of universal gravitation, the attractive force between two point-masses M and m, a distance r apart, is given by the inverse-square law

$$F = -\frac{GMm}{r^2} \tag{S1}$$

where G is Newton's gravitational constant. The corresponding gravitational potential energy of the masses is

$$U = -\frac{GMm}{r}, \tag{S2}$$

which follows from the fact that F and U are related by

$$F = -\frac{dU}{dr} \tag{S3}$$

together with the convention that $U(r) \to 0$ as $r \to \infty$. To find the force of the asteroid on a small object like an astronaut, we have to sum the forces of all the particles in the asteroid as they pull on the astronaut. This seems to be a tough problem, because the particles pull in many different directions with many different strengths, as shown in Fig. S1.

Fig. S1. A few of the enormous number of particles in the sphere, and their forces on an object of mass m. Each force is proportional to $1/r^2$, where r is the distance between m and the particle.

Easier than summing over all the forces is summing over all the potential energies. Potential energies are *scalars*, with no directionality, so are added like numbers. Forces are vectors, however, so have to be added vectorially. Once we have found the potential energy between the spherical asteroid and the astronaut, it will be easy to find the force on the astronaut using $F = -\dfrac{dU}{dr}$.

We begin with the potential energy $U = -\dfrac{GMm}{r}$ between two point-masses M and m, and add up the potential energies of all the particles in the spherical asteroid with the astronaut, to find the potential energy between the astronaut and the entire asteroid. As an intermediate step we find the potential energy between a (point-mass) astronaut m and a spherical *shell* of radius a, thickness Δa, and mass ΔM, as shown in Fig. S2. Then we can assemble the complete sphere out of spherical shells, and find the potential energy due to the sphere as the sum of potential energies due to the shells.

Fig. S2. A spherical shell and a point-mass outside it.

Not all parts of a shell are at equal distance from the point-mass, so we first have to simplify things even more. If we take two vertical slices through the shell, as shown in Fig. S3(a), with the slices taken very close together, we are left with a thin ring of mass, as shown in Fig. S3(b).

Fig. S3. (a) Two slices through a spherical shell, creating a ring at angle θ, with angular bite $\Delta\theta$. (b) The ring, as viewed from the astronaut m at the right. (c) A triangle with vertices at the sphere center, at a point on the ring, and at m.

We would get such a ring if we sliced a hollow tennis ball with a sharp knife. The particular slice can be characterized by its angle θ measured from the center of the spherical shell, as shown in Fig. S3(a). The radius of the ring is $a\sin\theta$, its thickness along the radial direction of the sphere is Δa, and its thickness perpendicular to the radial direction is the arc-length $a\Delta\theta$, where $\Delta\theta$ is the angular bite removed from the shell by the two slices. The cross-sectional area of the ring is therefore $\Delta a \cdot a\,\Delta\theta$. The pleasant feature of this ring is that all parts of it are essentially the same distance r from the point-mass outside the sphere, as shown in Fig. S3(c), so the potential energy between the point-mass m and the ring is simply

$$U_{\text{ring}} = -\frac{G\Delta M_{\text{ring}} m}{r}. \tag{S4}$$

The mass of the ring ΔM_{ring} is the mass density ρ times the volume of the ring,

$$\begin{aligned}\Delta M_{\text{ring}} = \rho\,\Delta V_{\text{ring}} &= \rho \cdot (\text{circumference of ring}) \cdot (\text{cross-sectional area of ring}) \\ &= \rho \cdot (2\,\pi\, a\sin\theta\,) \cdot (a\,\Delta\theta \cdot \Delta a)\,. \end{aligned} \tag{S5}$$

The potential energy is therefore

$$U_{\text{ring}} = -\frac{Gm\rho \cdot 2\pi a^2 \sin\theta\,\Delta\theta\,\Delta a}{r}. \tag{S6}$$

There are two variables in this expression which change from ring to ring: the angle θ and the distance r. If we use the law of cosines with the triangle shown in Fig. S3(c), we have

$$r^2 = R^2 + a^2 - 2aR\cos\theta\,, \tag{S7}$$

which can be used to eliminate either r or θ in U_{ring}. It turns out to be simpler to eliminate θ, by noting that the differential of Eq. S7 is

$$\Delta(r^2) = 2\,r\,\Delta r = \Delta\,(R^2 + a^2 - 2aR\cos\theta) = 2aR\sin\theta\,\Delta\theta \tag{S8}$$

for fixed R. The potential energy between the ring and the point-mass is then

$$U_{\text{ring}} = -\frac{Gm\rho\,\pi\,a\,\Delta a\,2r\,\Delta r}{r\,R} = -\frac{Gm\rho\,2\pi\,a\,\Delta a\,\Delta r}{R}. \tag{S9}$$

Now that we have found the potential energy between the point-mass and one of the rings, we can reassemble the spherical shell as a sum of rings, so that we will know the potential energy between the point-mass and the spherical shell. The angle θ of the rings varies from 0 to π, so according to Eq. S7, r varies from $R - a$ for the ring closest to m (i.e., with $\theta = 0$) up to $R + a$ for the ring farthest away (with $\theta = \pi$). Therefore

$$U_{\text{shell}} = \Sigma\,U_{\text{rings}} = -\frac{Gm\rho\,2\pi\,a\,\Delta a}{R}\int_{R-a}^{R+a} \mathrm{d}r$$

$$= -\frac{Gm\rho\,2\pi\,a\,\Delta a}{R}\left[(R+a) - (R-a)\right] = \frac{-4\pi\,a^2\,Gm\rho\,\Delta a}{R}. \tag{S10}$$

The volume of the shell is its surface area times its thickness; i.e.,

$$\Delta V_{\text{shell}} = 4\pi\,a^2\,\Delta a, \tag{S11}$$

so the mass of the shell is $\Delta M_{\text{shell}} = \rho\,\Delta V_{\text{shell}}$. Therefore

$$U_{\text{shell}} = -\frac{Gm\,\Delta M_{\text{shell}}}{R}. \tag{S12}$$

We have found that the potential energy of a point-mass outside of the spherical shell is exactly the same as if all the shells's mass were concentrated at a point at its center.

The final step is easy. The solid sphere can be built from concentric shells, as shown in Fig. S4. The total potential energy is therefore

$$U_{\text{sphere}} = \Sigma\,U_{\text{shell}} = -\frac{Gm}{R}\left[\,\Delta M_1 + \Delta M_2 + \Delta M_3 + \cdots\right] = -\frac{GmM}{R} \tag{S13}$$

because R is the same for each shell. Here M is the mass of the entire sphere, the sum of the masses of all the shells. Just as we had hoped, a potential energy of this type between point-masses is consistent with the same potential energy function between a sphere and a small object, if

Fig. S4. The complete sphere assembled from spherical shells

the radius is taken between the center of the sphere and the small object. It is important in our calculation that the mass density inside each shell is constant throughout that shell, i.e. that ρ is spherically symmetric. The density *can* change, however, from one shell to another.

We have found that the potential energy $U = -\dfrac{GMm}{R}$ between a point-mass and a sphere has the same form as it has between two point-masses. The gravitational force $F = -dU/dr$ must therefore also have the same form

$$F = -\frac{GMm}{R^2}. \tag{S14}$$

That is, given that the attractive gravitational force between two point-masses is inverse-square in the distance between them, then the gravitational force between a point-mass and a sphere is inverse-square in the distance between the center of the sphere and the point-mass.

This is an amazing result. Isaac Newton guessed it was true, but for a long time was unable to prove it. One of his primary motivations for inventing the integral calculus was to find a way to add up the individual inverse-square forces of particles within a sphere upon an external body, as an important step in understanding the motion of the planets around the sun, and especially the motion of the earth-moon system. It follows from Eq. S14 that if an astronaut stays on or above the surface of the spherical asteroid, the gravitational force on the astronaut is inverse-squared.

Now what happens to an astronaut who falls into the hole? The formula (S10) we found for the gravitational potential energy due to each shell is still correct, for an astronaut *outside* the shell. But if the astronaut is already *inside* some shell, the limits on the integral of Eq. S10 are wrong. For a shell with radius $a > R$, the distance r varies from $a - R$ up to $a + R$, instead of from $R - a$ up to $R + a$, as shown in Fig. S5. So the potential energy due to such a shell is

Fig. S5 A shell outside m. The distance r varies from $a-R$ up to $a+R$.

$$U_{\text{shell}} = -\frac{2\,G\,m\,\rho\,\pi\,a\,\Delta a}{R}\,[\,(a+R) - (a-R)\,]$$

$$= -4\,G\,m\,\rho\,\pi\,a\,\Delta a, \tag{S15}$$

which is a constant! As long as the shell is *outside* the astronaut, with $a > R$, the potential energy is independent of the astronaut's position R. The force due to this outside shell is therefore

$$F = -\frac{dU_{\text{shell}}}{dr} = 0. \tag{S16}$$

The force is zero because part of the shell pulls one way and part pulls the opposite way, so that all forces exactly cancel, as illustrated in Fig. S6(a). This is an important result. None of the *outside* shells exerts any force at all. Only those shells *inside* an astronaut's radius count, so only their mass M_{inside} should be used in the force-law. Therefore the total force on an astronaut of mass m is

$$F = -\frac{GM_{\text{inside}}m}{R^2}, \tag{S17}$$

as illustrated in Fig. S6(b).

Fig. S6. (a) The net gravitational force on an object within a spherical shell is zero by cancellation of forces. (b) The local gravity acting upon an object falling through a spherically symmetric asteroid depends on how much mass is inside the instantaneous position of the object.

Now we can use $F = ma$ to find the astronaut's motion. The mass inside should be the integral of the density over the volume of the sphere inside. This integral depends of course on how the density of the rock depends on position inside the sphere. For small asteroids, however, the density is close to being *constant* throughout the sphere; the rock in the middle is not compressed appreciably compared with the rock near the surface. In that case

$$M_{\text{inside}}(R) = \int_0^R \rho \, dV = \rho \, V(R) = \rho \, \frac{4}{3} \pi R^3 \tag{S18}$$

where R is the astronaut's distance from the center of the sphere. The force is therefore

$$F = -G \rho \, (4/3) \, \pi R^3 \; m / R^2 = -\frac{4}{3} \pi G \rho \, m \, R \, , \tag{S19}$$

directly proportional to R. So $F = ma$ becomes

$$m \ddot{R} = -k R \tag{S20}$$

where the constant

$$k = \frac{4}{3} \pi G \rho \, m \, . \tag{S21}$$

We recognize Eq. S20 as the simple harmonic oscillator Eq. 5b of Chapter B; the only difference is that now the dependent variable is the radius R instead of the Cartesian coordinate x. Therefore an astronaut of mass m falling through a small tunnel drilled through the center of a spherical asteroid of uniform density ρ falls as though he or she were attached to a spring of force-constant $k = \frac{4}{3} \pi G \rho \, m$, pulling always toward the center.

The equilibrium point of the effective spring is at the center of the asteroid; that is where a falling astronaut's speed is greatest. An astronaut oscillates from one end of the tunnel to the other, with period

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{3}{4\pi G \rho}} = \sqrt{\frac{3\pi}{G\rho}} \, . \tag{S22}$$

It is interesting that this period depends only upon the density of the asteroid, and not upon its size.

Sample Problem S1. A narrow tunnel is drilled through the center of a spherical asteroid from one side to the opposite side, and a package of express mail is dropped into one end of the tunnel. Assume the asteroid has the typical density 2.2 g/cm^3.

(a) How long will it take the package to reach the other end of the tunnel?

(b) What is the speed of the package at the middle of the tunnel, if the asteroid has a radius of 1 km? What if the asteroid's radius is 10 km?

Solution: (a) From Eq.S22, the period of oscillation through the tunnel is

$$T = \sqrt{\frac{3\pi}{G\rho}} = \sqrt{\frac{3\pi}{6.67 \cdot 10^{-11}\ \mathrm{m^3/kg\ s^2} \cdot 2{,}200\ \mathrm{kg/m^3}}} = 8.0 \cdot 10^3\ \mathrm{s}\ .$$

It takes only half the time to go from one end of the tunnel to the other, i.e. $4.0 \cdot 10^3$ s, or about 67 minutes. That is, it takes just over an hour to make the one-way trip, regardless of how large the asteroid is, as long as the asteroid's density is 2.2 g/cm^3.

(b) The package moves according to the solution of the simple harmonic oscillator differential equation ($F = ma$). If we orient coordinates so the x-axis lies along the tunnel and the origin is at the center of the asteroid, the most general solution is $x = A\cos(\omega t + \phi_0)$, where A is the amplitude of the motion and ϕ_0 is the phase angle. The solution for which $x = R$ and $v = 0$ at time $t = 0$ is $x = R\cos\omega t$, corresponding to $A = R$ and $\phi_0 = 0$. The velocity is then $v = dx/dt = -\omega R \sin\omega t$. The speed $|v|$ (the magnitude of the velocity) is largest when $\sin\omega t = \pm 1$ (which happens when $t = \pi/\omega, 3\pi/\omega, 5\pi/\omega, \ldots$, as the package passes through the center of the asteroid.)

That is, $|v| = \omega R = \frac{2\pi}{T}R = \frac{2\pi}{8.0 \cdot 10^3\,\mathrm{s}}\left(10^3\,\mathrm{m}\right) = 0.79$ m/s if $R = 1$ km; if $R = 10$ km the package's speed at the center is ten times as large, $|v| = 7.9$ m/s.

Questions and Problems from Chapter B, *Simple Harmonic Motion*

BQ1. A spring-mass system has oscillation frequency ν_0 on Earth. What would be its oscillation frequency on the Moon, where gravity is only one-sixth as strong? What would be its frequency in outer space, far from any gravitating body?

BQ2. An object oscillates simple-harmonically on the end of a spring. For each of the quantities (a) velocity (b) acceleration (c) force, at which point(s) in the motion of the object is the given quantity equal to zero, and at which point(s) is the magnitude of the quantity a maximum?

BQ3. Suppose the mass of the spring itself is not negligible compared with the mass of the block attached to its end. How would this affect the oscillation period of the system?

BQ4. On Earth one can use a spring scale to find the mass of an object by weighing it. In gravity-free space, where the object is weightless, how could the spring scale be used to find the object's mass?

BQ5. An object oscillates simple harmonically. If the amplitude and period of the motion are both doubled, the object's maximum speed is (a) quadrupled (b) doubled (c) unchanged (d) halved (e) quartered.

BQ6. The diagram below shows the position of a mass on the end of a spring as a function of time. What can you say about the velocity and the force on the mass at the instant indicated by the dashed line? (a) velocity is positive; force is to the right (b) velocity is negative; force is to the right (c) velocity is zero; force is to the right (d) velocity is positive; force is to the left (e) velocity is negative; force is to the left (f) velocity is zero; force is to the left (g) velocity and force are both zero.

BQ7. A mass oscillating on a spring has a period of 3 s. What is the period if (a) the mass is doubled? (b) the spring constant is quadrupled? (c) the oscillation amplitude is doubled?

BQ8. The figure below is a graph of the position of a particle in simple harmonic motion as a function of time. (a) What is the phase angle ϕ? (b) What is the overall phase of the particle at each of the three numbered points on the graph?

BQ9. A particle moves with the potential energy shown below. The horizontal line is the total energy of the particle. (a) What is the spring's equilibrium length? (b) Where are the turning points of the motion? (c) What is the particle's maximum kinetic energy? (d) What will be the turning points if the particle's total energy is doubled?

BQ10. The density of a realistic massive spherical body is larger in the middle of the body than near the surface, so the motion of an object dropped into a tunnel drilled through the body will not be sinusoidal. Qualitatively, how would the motion be different from the uniform-density case? Sketch how you think the position of the object will depend on time. How does it differ from a sinusoidal function?

Section B.1 A spring-mass system

BP1. Find the period T (in seconds) and oscillation frequency ν (in Hertz) of the oscillation

shown in the graph below.

BP2. Find the period T (in seconds) and oscillation frequency ν (in Hertz) of the oscillation shown in the graph below.

BP3. A ball oscillates on the end of a spring, with period T_0. The spring is then cut in half, and the same ball is made to oscillate on the end of one of the half-springs. Find the new oscillation (a) period (b) frequency, in terms of T_0.

BP4. A ball oscillates on the end of a spring with period T_0. (The other end of the spring is held in place on a wall.) The spring is then disconnected from the wall, and that end is attached to a second ball, identical to the first, as shown below. The balls are then pulled apart, stretching the spring, and released. In terms of T_0, find their oscillation period.

BP5. A watermelon is dropped into a supermarket spring scale. The scale and watermelon oscillate up and down several times together with frequency ν_0. When the scale comes to equilibrium, it is a distance y_0 below where it started. The melon is removed, and a different watermelon is dropped on the scale. If the oscillation frequency is now $\nu_0/2$, when the scale comes to equilibrium what is the distance y below where it started? Assume that the mass of the scale is small compared to that of either melon.

Section B.2 Motion using $F = ma$

BP6. A block of mass $m = 1.0$ kg is placed on the end of a spring of force constant $k = 0.50$ N/m. The block is pulled 25 cm from equilibrium, and released from rest at time $t = 0$. Find the block's (a) angular frequency ω; (b) frequency ν; (c) period T; (d) position x at $t = 3.0$ s; (e) velocity v at $t = 3.0$ s.

BP7. A block of mass $m = 0.25$ kg is placed on the end of a spring of force-constant $k = 1.0$ N/m. The block is pulled 15 cm from equilibrium, and released from rest at time $t = 0$. Find the blocks (a) angular frequency ω; (b) frequency ν; (c) period T; (d) position x at $t = 5.0$ s; (e) velocity v at $t = 5.0$ s.

BP8. A ball of mass m is hung on a spring. The ball is pulled down a distance A from its equilibrium point, and released from rest. When the ball returns to its equilibrium point, its speed is v_0. Find the force-constant k of the spring, in terms of m, A, and v_0.

BP9. Inhabitants of Tangshan, China, reported that during the catastrophic earthquake of July 28, 1976, they were thrown 2 m into the air as if by a huge jolt from below. (a) With what speed must a body be thrown upward to reach a height of 2 m? (b) Assume that the vertical motion of the ground was simple harmonic with a frequency of 1 Hz. What amplitude of the vertical motion is required to generate a maximum speed equal to that calculated in part (a)?

BP10. A block of mass m is attached to two springs, each of force constant k. Find the frequency of oscillation ν in each of the following cases: (a) the springs are attached sequentially; (b) the springs are attached side-by-side, as shown below. (c) Suppose that in each case the two springs were replaced by a single spring of force constant k' which gives the same oscillation frequency as with two springs k. Find k' for each of the cases (a) and (b).

BP11. A block executes simple harmonic mo-

tion with amplitude $A = 0.100$ m on a frictionless horizontal surface. When the block is 0.060 m away from equilibrium, the block's speed is 0.360 m/s. (a) What is the period of the motion? (b) What is the displacement of the block when its speed is 0.120 m/s? (c) A small object whose mass is much less than that of the block is placed on the oscillating block. If the small object is just on the verge of slipping at the endpoint of the motion, what is the coefficient of static friction between the small object and the block?

BP12. A block of mass m is placed on top of a block of mass M. The coefficient of static friction between the two blocks is μ_s. Block M, which rests on a frictionless table, is attached to a spring of force-constant k, as shown below. In terms of m, M, μ_s, k , and g, what is the maximum distance Δx the spring can be compressed, so that in the resulting motion block m will not slip?

BP13. A car of mass M is supported equally by four springs, one at each wheel. (a) If vertical oscillations of the car have period T_0, what is the force constant k of each spring, assuming they are equal? (b) Now four passengers, each of mass m, get into the car. What is the new period of vertical oscillations, in terms of M, m, and T_0 ?

BP14. A car of mass $M = 1.50$ tonnes (1 tonne = 1000 kg) is supported by four springs, each of force-constant $k = 22{,}000$ N/m. It has no shock absorbers to damp oscillations. (a) If the car rolls (without passengers) over a pothole, what is the subsequent frequency of oscillation of the car? (b) If the experiment is repeated with four passengers, each of mass 75 kg, what then will be the frequency of oscillation?

BP15. A log of radius R and uniform density lies half submerged in a lake, as shown below. If the log is pushed very slightly down into the water and then released, it will oscillate up and down. Find the frequency of this oscillation in terms of R and g. *Hint*: Archimedes showed that the buoyant (i.e., upward) force of a fluid on an object in the fluid is equal to the weight of fluid displaced by the object. For example, if half the log is submerged, the displaced water has weight $\rho_{water}(V/2)g$ where ρ_{water} is the density of water and V is the total volume of the log. In equilibrium this upward force on the log must be balanced by the downward gravitational force $m_{\text{log}}g = \rho_{\text{log}}Vg$. Therefore $\rho_{\text{log}} = \rho_{\text{water}}/2$ in this example.

BP16. A platform oscillates vertically with simple harmonic motion of amplitude A and frequency ν. (a) A block is placed on the platform when it is at its lowest point. Find the maximum value of ν for which the block will remain in contact with the platform throughout its motion. Call this value ν_{crit}. (b) Suppose $\nu > \nu_{\text{crit}}$. How far above the platform's lowest point will the block leave the platform? (c) How far will the block subsequently rise above the highest point reached by the platform?

BP17. A ball on a particular negligible-mass spring naturally oscillates with the angular frequency 3.00 radians/s. The spring is suspended from the ceiling of an elevator and hangs motionless as the elevator rises at the constant speed 2.00 m/s. The elevator then stops suddenly. With what amplitude does the ball begin to oscillate?

BP18. A spring of force constant k, attached to the ceiling of an elevator, supports a ball of mass m. The elevator is moving downward with constant speed v_0, and the ball remains at constant distance from the ceiling. If the elevator suddenly stops, what is the resulting amplitude of oscillation of the spring-mass system?

BP19. A ball of mass $m = 2.0$ kg hangs on a spring of force constant $k = 0.50$ N/m in gravity $g = 10.0$ m/s^2. The ball oscillates up and down with amplitude $A = 1.50$ m. Just as the ball passes downward through its equilibrium point, it is given a sharp upward impulse $I =$

2.0 kg m/s by a baseball bat. What is its subsequent amplitude of oscillation?

BP20. Astronauts in space cannot weigh themselves in the usual way, so they sometimes find their mass by oscillating on a large spring. Suppose an astronaut attaches one end of the spring to her belt and the other end to a wall in the International Space Station. Someone pulls her away from the wall and releases her; the resulting spring length as a function of time is shown in the figure below. (a) What is her mass if the spring constant is 240 N/m? (b) What is her speed when the spring's length is 1.2 m?

BP21. An ultrasonic transducer used in medical ultrasound imaging is a very thin disk of mass $m = 0.10$ g driven back and forth simple harmonically at 1.0 MHz by an electromagnetic coil. (a) The maximum restoring force that can be applied to the disk without breaking it is 50,000 N. What is the maximum oscillation amplitude that won't break the disk? (b) What is the disk's maximum speed at this amplitude?

Section B.3 Phase

BP22. The figure below shows the position of an oscillator as a function of time. Find the oscillator's (a) amplitude A (b) period T (c) phase angle ϕ_0.

BP23. The figure below shows the position of an oscillator as a function of time. Find the oscillator's (a) amplitude A (b) angular frequency ω (c) phase angle ϕ_0. (d) How much has the phase of the oscillation changed between time $t = 0$ and time $t = \pi/\omega$?

BP24. A block of mass $m = 0.20$ kg is placed on the end of a spring of force constant $k = 5.0$ N/m. The spring is compressed a distance 0.30 m, and the block is released at time $t = 0$ with initial velocity 1.5 m/s directed away from its equilibrium position. Find the amplitude A and phase angle ϕ_0 of the oscillator.

BP25. A block of mass $m = 0.10$ kg is placed on the end of a spring of force-constant $k = 0.50$ N/m. The block is pulled 24 cm from equilibrium, and released at time $t = 0$ with outward velocity $v_0 = 35$ cm/s. Find the block's (a) angular frequency ω (b) frequency ν (c) period T (d) position x at $t = 0.40$ s (e) velocity v at $t=$ 0.40 s.

BP26. (a) Show that $x(t) = A \sin \omega t$ is a solution of the simple harmonic oscillator equation 5(b), for an appropriate value of ω. What is the correct ω? (b) Show that $x(t) = A \sin \omega t + B \cos \omega t$ is also a solution of Eq. 5(b), for any constants A and B, for the same value of ω found in part (a). (c) Another solution of the SHO equation is $x(t) = C \cos(\omega t + \phi_0)$, as given by Eq. 13, with C as proportionality constant. Show that in fact this solution is the same as the solution of part (b), if the constants are chosen appropriately. That is, find the constants A and B in terms of the constants C and ϕ_0.

BP27. A block of mass m hung on a spring stretches the spring a distance ℓ in a gravitational field g, when the block is in equilibrium. The block is then set into oscillation about the equilibrium position. (a) What is the angular frequency of oscillation ω of the block in terms of g and ℓ? (b) The position $x(t)$ of the block (in meters) is $x(t) = 0.20 \cos \omega t - 0.10 \sin \omega t$. Find the initial position $x(0)$ and velocity $v(0)$ of the block, in meters and meters/second, respectively. Assume $g = 9.8$ m/s^2, and $\ell = 3.0$ cm. (c) If the position of the block as a function

of time is represented by $x(t) = A\cos(\omega t + \phi_0)$, find the amplitude A and phase angle ϕ_0 numerically.

BP28. A ball moves in a horizontal circular orbit of radius R with period T. A shadow of the ball displayed on a screen moves back and forth as though the shadow were a mass on the end of a spring. (a) If the shadow were a ball of mass m, what would the force constant k of a spring have to be to make the shadow oscillate as it does? (b) If the shadow moves according to $x(t) = A\cos(\omega t + \phi_0)$, what must be the phase angle ϕ_0 if the shadow is at the center of its motion at time $t = 0$, and heading toward the left?

BP29. A horizontal beam of light is directed parallel to the face of a large clock with an exposed minute hand. A shadow of the hand is produced on a wall on the opposite side of the clock, so that the shadow of the minute hands tip moves up and down sinusoidally as the minute hand turns, as shown below. Measurements show that the position $y(t)$ of the tips shadow is given by $y(t) = (2.5\text{ m})\sin(\omega t + 0.32)$, where t is the time read by our wristwatch. (a) What is the length of the minute hand? (b) What is ω, in units s^{-1}? (c) What time does the minute hand read (between 0 and 60 minutes) when $t = 12$ min?

Section B.4 Oscillator energy

BP30. A block of mass m attached to a spring of force-constant k moves with position $x(t) = A\sin(\omega t + \pi/4)$, where $\omega = \sqrt{k/m}$. As functions of time, find the systems (a) potential energy $U(t)$ (b) kinetic energy $K(t)$ (c) total energy $E(t)$.

BP31. A block of mass m attached to a spring of force-constant k oscillates back and forth on a frictionless surface. If its amplitude is A, (a) at what position(s) will the systems potential energy equal its kinetic energy? (b) at what position(s) will the absolute value of the time derivative of the systems potential energy be equal to the absolute value of the time derivative of its kinetic energy?

BP32. (a) Using energy conservation, show that the instantaneous velocity v of a simple harmonic oscillator in terms of its amplitude A, angular frequency ω, and instantaneous displacement x, is given by $v = \omega\sqrt{A^2 - x^2}$. (b) What fraction of the oscillator's energy is kinetic energy when the displacement of the oscillator is one-quarter its maximum value?

BP33. As described in BP32, the expression $v = \omega\sqrt{A^2 - x^2}$ can be derived from the conservation of energy of a simple harmonic oscillator. (a) Differentiate this expression with respect to time, to show that the acceleration a of the oscillator obeys $a = -\omega^2 x$, which is the SHO Eq. (5a), where $\omega = \sqrt{k/m}$. (b) The velocity $v = dx/dt$, so the expression above is equivalent to $dx/dt = \omega\sqrt{A^2 - x^2}$. Divide this equation by $\sqrt{A^2 - x^2}$ and integrate over time to find x as a function of time. Show that the result is expected for a simple harmonic oscillator.

BP34. A block of mass m, at rest on a tabletop, is attached to a horizontal spring having force constant k. The spring is initially unstretched. Then a constant horizontal force F_0 is applied to the block, causing the spring to stretch. (a) Find the speed of the block after it has moved a distance d from equilibrium if the tabletop is frictionless. (b) Repeat part (a) supposing instead that there is a coefficient of kinetic friction μ_k between the block and tabletop.

BP35. A molecular bond can be modeled as a spring between two atoms that vibrate with simple harmonic motion. The figure below is an approximation to the potential energy of an HCl molecule for small excitation energies. The chlorine atom is much more massive than the hydrogen atom, so assume that the hydrogen atom (with mass $m = 1.67 \times 10^{-27}$ kg) vibrates back and forth while the chlorine atom remains at rest. Using the graph, estimate the

vibrational frequency of the HCl molecule.

BP36. A spring is standing upright on a table with its bottom end attached to the tabletop. A block is dropped from a height 5.0 cm above the top of the spring. The block sticks to the top end of the spring and then oscillates with an amplitude of 20 cm. What is the oscillation frequency?

BP37. The amplitude of a system moving in simple harmonic motion is doubled. Find the change in (a) total energy (b) maximum speed (c) maximum acceleration (d) period.

Problems on oscillation within spherical gravitating bodies

BP38. A future civilization lives within an enormous spherical shell of mass M and radius R_0, as shown below. A visitor, starting from rest at a point $r_0 = 4R_0$ outside the shell, begins to fall. Just as he reaches the shell, a trap door is opened, admitting him to the interior. Plot his speed as a function of r, from $r = r_0$ down to $r = 0$.

BP39. Planet Earth has mass 5.975×10^{24} kg and average radius 6371 km. Pretend that Earth is spherical, has uniform density, is airless, and that a straight tunnel is dug from pole to pole. How long would it take a polar bear who falls in at the North Pole to arrive at the South Pole?

BP40. An astronaut steps into the asteroid tunnel as described in the text, and begins to fall. (a) Suppose that the tunnel bottoms out at some depth $d < R_0$, d measured from the surface. With what speed will the astronaut smash into the bottom? Express the answer in terms of any or all of G, the asteroid mass M, d, and R_0.

BP41. A postal worker on a spherical asteroid wants to deliver mail to the exact opposite point (i.e., the antipode), and can drop it into a direct tunnel through the center of the asteroid, or throw it horizontally so that it goes into a circular orbit just above the surface. Which if either is faster? Explain.

BP42. A straight tunnel is dug between two points on the surface of a uniform-density spherical asteroid, separated by angle θ from the center, as shown below. The tunnel walls are smoothed and polished so one can slide along them without appreciable friction. (a) An astronaut starts from rest at one end of the tunnel. How fast is the traveler moving at the tunnel's midpoint? (b) How long does it take to slide from one end to the other? Compare with the travel time for a tunnel bored straight through the center, i.e. for a tunnel with $\theta = 180^o$.

BP43. Suppose there is a force between two point-masses m and M whose potential energy is $U(r) = kMm/r^2$, where k is a constant. (a) Find the corresponding potential energy between a spherical shell of mass M and radius R_0 and a point-mass m external to the shell, as a function of R, the distance between m and the center of the shell. Is $U = kMm/R^2$? (b) Is the force of a spherical shell on a mass m inside the shell equal to zero? Show that it is, or that it isn't.

BP44. In the *Special Topic* it is shown that there is an inverse-square law of gravity between a spherical shell and an external mass. This follows by finding the potential energy $U(R)$ between them and differentiating to get $F = -dU/dR$. The potential energy $U(R)$ is found by summing the potential energies due to rings sliced from the shell. Find F due to a shell directly by first finding the force on an exter-

nal mass due to a ring, and then summing over rings. [Note that F is actually a vector. From symmetry arguments decide which Cartesian component of force is exerted by each ring, and then sum over rings.]

BP45. A uniform-density spherical asteroid of mass M and radius R_0 rotates with angular velocity Ω about an axis through its center. Visiting astronauts drill a straight highly-polished tunnel through the asteroid's center, from one point on the equator to another point on the exact opposite side. Define a critical angular velocity by $\Omega_c = \sqrt{GM/R_0^2}$. (a) How long does it take an astronaut to fall from one end of the tunnel to the other? Assume $\Omega < \Omega_c$. (b) What happens to the astronaut if $\Omega \geq \Omega_c$? *Hint:* The rotating asteroid is at rest in a non-inertial rotating frame; in such a frame there is a centrifugal pseudoforce $mv^2/R = mR\Omega^2$ directed outward, in addition to real forces.

C. Oscillating Systems

When they asked Michelangelo how he made his statue of David he is reported to have said, "It is easy. You just chip away the stone that doesn't look like David."

The motion of a mass on the end of a spring, as discussed in Chapter B, is interesting in itself. Much more interesting, however, is the fact that the spring-mass system is the prime example of a wide variety of oscillating systems.

C.1 The simple pendulum

One common oscillating system is the *simple pendulum* in which a small object of mass m (called the pendulum "bob") is placed at the end of a light cord of length ℓ and allowed to swing back and forth in a plane. Probably the most-often observed pendulums are those in "grandfather clocks", where they are used to determine tick-rates.* These are not *simple* pendulums, however, because the mass is distributed over the entire pendulum length, and not just at the lower end.

The period of a pendulum is the time it takes to complete one cycle back and forth. What does the period of a simple pendulum depend upon? In addition to the length ℓ of the string and the gravitational field g, the period might depend upon the mass m of the bob and the amplitude of oscillation θ_0, where θ_0 is the initial angle of the pendulum as it is released from rest. That is, the period T might be a function of any or all of these quantities:

$$T = T(\, m,\ \ell,\ g,\ \theta_0\,)\,. \tag{1}$$

Obviously T has the dimensions of time, so the right-hand side, which includes some or all of the four variables m, ℓ, g, θ_0, must also have dimensions of time. The dimensions of these four quantities, in terms of the fundamental dimensions of mass M, length L, and time T, are $[m] = \mathrm{M}$, $[\,\ell\,] = \mathrm{L}$, $[g] = \mathrm{L/T^2}$, $[\theta_0] = 1$ (i.e., angles are dimensionless, since they are defined as the ratio of two lengths.)

* The pendulum clock was invented and patented by the Dutch scientist Christiaan Huyghens in 1657.

Can we use dimensional reasoning to determine the dependence of the period on all the other variables? The only quantity on the right of Eq. 1 which includes T as one of its dimensions is g, so g must appear as $g^{-1/2}$ to get a time to the first power in the numerator. Then to cancel the dimension of $L^{-1/2}$ in g, we have to multiply by $\ell^{1/2}$. The only quantity with the dimensions of mass is the bob mass m; therefore it *cannot appear* in the formula for T, because there is no other variable to cancel it out! Using dimensional reasoning alone we cannot determine how the period depends upon the dimensionless quantity θ_0. Nevertheless, we have learned a lot about the simple pendulum by dimensional reasoning. The period cannot depend upon the mass of the bob and must depend upon the other variables according to the equation

$$T = f(\theta_0)\sqrt{\frac{\ell}{g}}\,, \tag{2}$$

where $f(\theta_0)$ is some unknown function of θ_0.

How can we find the full motion as a function of time? That is, how can we find the angle $\theta(t)$, where θ is the angle of the pendulum from the vertical, as shown in Fig. 1? Also, how can we find the function $f(\theta_0)$ in the formula (Eq. 2) for the period?

We can answer these questions using the fundamental equation of rotational dynamics

$$\tau = \frac{dL}{dt}\,, \tag{3}$$

a natural starting place because the pendulum involves rotation.* Let the origin of coordinates be located at the point of support of the cord, and let θ increase in the counterclockwise sense (i.e., pointed out of the page by the right-hand rule). Therefore the time rate of change of angular momentum $\frac{dL}{dt} = \frac{d}{dt}(I\dot{\theta}) = I\ddot{\theta}$ is positive for counterclockwise angular accelerations.

Fig. 1 (a) the simple pendulum (b) positive angle θ

*Alternatively, we can use $F = ma$ (see problem CP5).

When θ is positive, as shown in Fig. 1(b), the torque on the bob due to gravity is $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \ell\, mg \sin\theta$, directed *into* the page. So with the positive direction *out* of the page, $\tau = \dfrac{dL}{dt} = I\ddot{\theta}$ becomes

$$-\ \ell\, mg \sin\theta = I\ddot{\theta} = m\ell^2\, \ddot{\theta}, \tag{4}$$

where we have used the fact that the moment of inertia of a point mass is $I = m\ell^2$. (The force $\mathbf{T}$ due to cord tension exerts no torque on the bob because $\mathbf{r}$ and $\mathbf{T}$ are antiparallel to one another, and so the torque due to the cord is $\mathbf{r} \times \mathbf{T} = 0$. The only torque on the bob is due to gravity.) Solving Eq. 4 for the angular acceleration,

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta\ , \tag{5}$$

which is called the *pendulum equation.*

Equation 5 looks *somewhat* like the simple harmonic oscillator equation

$$\ddot{x} = -\frac{k}{m}x\ , \tag{6}$$

but it is obviously not the same. The important difference is not that x has become θ, or that the constant $\dfrac{k}{m}$ has become the constant $\dfrac{g}{\ell}$. Those differences amount to only changes in symbols. The important difference is that the right-hand side of Eq. 5 includes $\sin\theta$ rather than θ. That makes the equation much harder to solve.

We can *try* sinusoidal oscillating solutions of Eq. 5, such as $\theta = \theta_0 \cos\omega t$, but they don't work.* And since $\cos\omega t$ doesn't solve the equation, for example, then we can't easily find the period of oscillation of the pendulum in terms of g and ℓ, as we found the period of oscillation $T = 2\pi\sqrt{\dfrac{m}{k}}$ of a spring-mass system in terms of m and k Nevertheless, we have made some progress in understanding the pendulum. We know that the period depends upon amplitude θ_0, cord length ℓ, and gravity g according to $T = f(\theta_0)\sqrt{\dfrac{\ell}{g}}$, where

*Try it!

$f(\theta_0)$ is some unknown function of the amplitude. We have also learned that the detailed motion $\theta(t)$ obeys the pendulum equation $\ddot{\theta} = -\frac{g}{\ell}\sin\theta$.

C.2 Small-amplitude motion

A pendulum oscillates back and forth but the pendulum equation (5) is not the simple harmonic oscillator equation, and does not have any simple solution like $\cos\omega t$. Is there any way to find a solution?

Suppose we release the pendulum bob with a *small* initial angle θ_0, so that $\theta_0 << 1$ (θ_0 expressed in radians.) Then the angle θ will be small throughout the motion. That is a great advantage, because we can make the approximation $\sin\theta \cong \theta$ *. Equation 5 then becomes

$$\ddot{\theta} + (g/\ell)\,\theta = 0, \tag{7}$$

which *is* the simple harmonic oscillator equation! All that is changed from the spring-mass $F = ma$ equation is the substitution of the angle θ for the position x, and the substitution of the constant g/ℓ for the constant k/m. Therefore, the solution of Eq. 7 is the SHO solution

$$\theta(t) = A\cos(\omega t + \phi_0) \qquad (\text{if } \theta << 1) \tag{8}$$

where $\omega = \sqrt{g/\ell}$ and A and ϕ_0 are arbitrary constants, the amplitude and phase angle, to be determined by the initial conditions. If we release the pendulum from rest at $\theta = \theta_0$ at time $t = 0$, for example, then $A = \theta_0$ and $\phi_0 = 0$, so $\theta = \theta_0\cos\omega t$.

The period of the pendulum for small-amplitude motion is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}}, \tag{9}$$

consistent with the result $T = f(\theta_0)\sqrt{\frac{\ell}{g}}$ we found from purely dimensional reasoning, where $f(\theta_0)$ reduces to 2π for small θ_0.

*The Taylor's series expansion of $\sin\theta$ is $\sin\theta = \theta - \theta^3/3! + \theta^5/5! - \theta^7/7! \cdots$. For small θ, each term is much smaller than the one before it. In Eq. 7 we ignore all but the first term.

If the oscillation amplitude θ_0 is *not* small, the period depends upon θ_0. Problems CP6 and CP7 show the first steps in finding this dependence. The result is that the period can be represented as an infinite series in the amplitude θ_0. The first three terms are*

$$T = 2\pi\sqrt{\frac{\ell}{g}}\left[1 + \left(\frac{1}{16}\right)\theta_0^2 + \left(\frac{11}{3072}\right)\theta_0^4 + \ldots\right]. \tag{10}$$

The period increases with amplitude and reduces to $2\pi\sqrt{\frac{\ell}{g}}$ if the amplitude is very small, as shown in Fig. 2.

Fig. 2. The dependence of a pendulum's period on the amplitude θ_0. (A pendulum can have amplitudes θ_0 in excess of 90^0 if the string is replaced by a massless rod.)

Why does the period increase with amplitude? It is because the gravitational torque on the pendulum, which is responsible for driving the motion, increases only as $\sin\theta_0$, which grows more slowly than θ_0 itself. If the torque increased linearly with θ_0, the pendulum equation would be the same as the simple harmonic oscillator equation, and the period would be independent of amplitude. Nonetheless, for small amplitudes the period increases only very slowly, as shown in Fig. 2, so slowly that even Galileo himself believed that the period was independent of amplitude when he observed the swinging of a chandelier in the cathedral at Pisa in 1581. The formula for the small-amplitude period was given by the Dutch scientist Christiaan Huyghens in 1673, and the dependence of period on amplitude was worked out by the Swiss mathematician Leonhard Euler in 1736.

* See, for example, *Classical Dynamics of Particles and Systems* (Fourth Edition) by Marion and Thornton, Saunders College Publishing, Fort Worth 1995

C.3 Other rotational problems

The same equations have the same solutions. A particular equation might arise in very different contexts in physics, chemistry, mathematics, engineering, biology, and economics. The variables would be different, representing different quantities; yet if the equations are the same, the general solutions are the same as well.

The $\tau = I\ddot{\theta}$ equation for a small-amplitude simple pendulum is $\ddot{\theta} + (g/\ell)\theta = 0$, for example, and the $F = ma$ equation for a spring-mass system is $\ddot{x} + \frac{k}{m}x = 0$. The variables are different and the constants are different, but the equations are the same. They are both simple harmonic oscillator (SHO) equations of the form

$$\ddot{x} + \omega^2 x = 0 \tag{11}$$

whose general solution is

$$x(t) = A\cos(\omega t + \phi_0) . \tag{12}$$

We write the SHO equation as in Eq. 11, as a reminder that ω, the angular frequency of oscillation, is equal to the square root of whatever constant appears in front of the undifferentiated term. (We assume that any constant preceding the second-derivative term has already been divided out.) Suppose for example that an electrical engineering problem leads to the equation

$$\ddot{Q} + KQ = 0 \tag{13}$$

for the electric charge $Q(t)$ on a capacitor, where K is a positive constant. Then the general solution of the equation has to be

$$Q(t) = A\cos(\omega t + \phi_0) \tag{14}$$

for arbitrary A and ϕ_0, where $\omega = \sqrt{K}$. The actual values of A and ϕ_0 can be determined from the initial conditions, such as the initial charge and initial current in the capacitor.

The SHO equation arises in a wide variety of situations. The following sample problems illustrate this for cases involving rotational motion.

Sample Problem 1. The physical pendulum

A small hole is drilled through a solid object of mass M, and a thin rod is inserted through the hole. The object is then made to oscillate about the rod as an axle. Find the angular frequency of small oscillations of this "physical pendulum". (The pendulum is "physical" rather than "simple", because the pendulum bob is no longer an idealized point mass.)

Solution: If the object is turned through an angle θ, as shown, the magnitude of the torque exerted on the object about the rotation axis is $\tau = Mgd\sin\theta$, where d is the distance from the rotation axis to the center of mass of the object. As with a simple pendulum, the torque direction is opposite to the positive θ direction, so $\tau = I\ddot{\theta}$ becomes $-Mgd\sin\theta = I\ddot{\theta}$, where I is the moment of inertia of the object *about the chosen axis of rotation.* We know that $\sin\theta \cong \theta$ for small θ, so $\ddot{\theta} + \frac{Mgd}{I}\theta = 0$. The angular frequency of oscillation of a small-amplitude physical pendulum is therefore $\omega = \sqrt{\frac{Mgd}{I}}$.

Sample Problem 2. A thin rod of mass M and length L is pivoted frictionlessly at its midpoint. Two springs, each of force-constant k, are attached to opposite sides of the upper end of the rod, as shown. Find the period of small oscillations of the rod, assuming the springs stay essentially perpendicular to the rod at all times.

Solution: If the rod is turned through a small clockwise angle θ, the top end of the rod moves a small horizontal distance $x = (L/2)\theta$ to the right. This *stretches* the left-hand spring, so it *pulls* to the left with a force $kx = k(L/2)\theta$. The right-hand spring is *compressed*, so it *pushes* to the left with a force $kx = k(L/2)\theta$. Both forces are toward the left, so the total spring force on the top of the rod is $F = kL\theta$, to the left. This force exerts a torque on the rod about its rotation axis as origin; the torque is

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \frac{L}{2}kL\theta$$

directed *out* of the page. (Note that the vectors $\mathbf{r}$ (from the origin to the top of the rod) and $\mathbf{F}$ are essentially perpendicular to one another for small oscillations.) We chose *clockwise* rotations (i.e. rotations into the page, by the right-hand rule) to be positive, so $\tau = \frac{dL}{dt} = I\ddot{\theta}$ becomes

$$-\frac{1}{2}kL^2\theta = \frac{1}{12}ML^2\ddot{\theta},$$

since the moment of inertia of a rod about its midpoint is $I = \frac{1}{12}ML^2$. In standard SHO equation form, the equation is

$$\ddot{\theta} + \frac{6k}{M}\theta = 0,$$

so the angular frequency of oscillation is $\omega = \sqrt{\frac{6k}{M}}$, and the period of oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{M}{6k}} = \pi\sqrt{\frac{2M}{3k}}.$$

Sample Problem 3. A bicycle wheel of mass m and radius r rests on a horizontal surface. A spring of force-constant k is attached to the axle, and the other end of the spring is kept fixed. When the wheel is pushed toward the right, stretching the spring, the wheel subsequently oscillates back and forth, rolling without slipping back and forth along the surface. What is the period T of this oscillation?

Solution: In addition to the force $-kx$ due to the spring, a frictional force is exerted on the wheel by the surface. However, if we choose an origin on the surface, the frictional force exerts no torque on the wheel (since **R** and $\mathbf{F}_{\text{fric}}$ are parallel or antiparallel to one another.) Gravity and the normal force exert equal but opposite torques, so their effects cancel out. The only unbalanced torque about such an origin is due to the spring, with magnitude rF and direction out of the page for positive x. The rolling-without-slipping condition can be written $x = r\theta$, where θ is the rotation angle, positive to the *right*. Therefore the torque is $\tau = kr^2\theta$ out of the page, and the angular acceleration $\ddot{\theta}$ is positive *into* the page. So the equation of motion $\tau = I\alpha$ gives $-kr^2\theta = I\ddot{\theta}$, where $I = mr^2 + mr^2 = 2mr^2$, using the parallel axis theorem. Therefore $\ddot{\theta} + \frac{kr^2}{2mr^2}\theta = 0$, so the angular frequency $\omega = \sqrt{\frac{k}{2m}}$ and the period is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2m}{k}}$.

C.4 Motion with an arbitrary conservative force

Suppose a particle of mass m moves in one dimension under the influence of an arbitrary conservative force $F(x)$. That is, the particle moves with a potential energy $U(x)$ such as the one shown in Fig. 3. The force $F(x)$ on the particle is the negative slope of $U(x)$:

$$F(x) = -\frac{dU(x)}{dx}, \tag{15}$$

Fig. 3. A possible potential energy function for a particle

so in particular the force on the particle is zero when the slope is zero, at points a, b, and c. These three points are *equilibrium points* of the particle, because if the particle is placed at rest at any one of them it remains at rest, because the force is zero. Our interest now is whether these points are *stable* equilibrium points. If the particle is displaced slightly from one of these points, will it return to the point, never straying very far? Or will it depart and not return?

(a) First suppose the particle is near point a, a local *maximum* of $U(x)$. If the particle is slightly to the right of a the force is also to the right, because the slope there is negative. The positive force means that the particle is pushed to the right, away from a. If the particle is slightly to the left of a the force is also to the left, again pushing the particle away from a. Therefore a is a point of *unstable* equilibrium, because if the particle is displaced slightly from a it will accelerate away from that point.

A car at the the top of an icy hill is in unstable equilibrium. A slight shove in either direction sends the car sliding off, faster and faster. Another example of unstable equilibrium is a pencil balanced upright on its tip. Ideally, there is neither a net force nor a net torque on the pencil if it is perfectly vertical; it is therefore in equilibrium. However, the equilibrium position is unstable. If the pencil is not perfectly vertical, it experiences a net torque tending to push it even farther from the vertical orientation, so it falls.

(b) If a particle is near point b instead, where the slope of $U(x)$ is zero over some range of x, then the force on the particle is zero in the neighborhood of b. If the particle is placed slightly to the left or slightly to the right of b, it will be urged neither toward nor away from b. Such a point b is an equilibrium point of *neutral* stability, neither stable nor unstable. A car sitting on a frozen lake has neutral stability. A round pencil lying on its side on a horizontal plane also has neutral stability.

(c) If a particle is near point c, a local minimum in the potential energy, then the particle is in *stable* equilibrium. If it is displaced slightly to the right of c, the force on the particle is to the left, tending to return it toward c. If the particle is displaced slightly to the left, the force on it is toward the right; the particle remains in the vicinity of c. A car in an icy valley is in stable equilibrium, as is a skateboarder at the bottom of a concrete bowl.

If the total energy of the particle is only slightly greater than the potential energy at a minimum such as point c, then the particle has two turning points where it comes instantaneously to rest, before heading in the other direction. That is, the particle oscillates back and forth between the two turning points, as illustrated in Fig. 4.

Fig. 4. Oscillation between the two turning points near a point of stable equilibrium

Most objects we see around us are near a stable equilibrium point. If they weren't, they would long since have gone somewhere else. An object displaced slightly from a stable equilibrium point and then released will oscillate about the point, never wandering far away. The car in the valley rolls back and forth about the bottom of the valley. A car pushed down on one end, compressing its springs, will oscillate up and down for a while, especially if it has no shock absorbers.

What happens to a particle near a potential energy minimum such as point c in Fig. 3? We need to find what force acts on it, and then using $F = ma$ find out how it moves in response to the force.

Let x_0 be the value of x at which the potential energy U is a local minimum. For example, x_0 is the point c in Fig. 3. At that location the potential energy has some value U_0. Using Taylor's series, we can find the potential energy $U(x)$ in the vicinity of x_0. That is,

$$U(x) = U(x_0) + \left.\frac{dU(x)}{dx}\right|_{x_0}(x - x_0) + \frac{1}{2!}\left.\frac{d^2U(x)}{dx^2}\right|_{x_0}(x - x_0)^2 + \frac{1}{3!}\left.\frac{d^3U(x)}{dx^3}\right|_{x_0}(x - x_0)^3 + ., \tag{16}$$

where for example $\left.\frac{dU(x)}{dx}\right|_{x_0}$ means the first derivative of $U(x)$ with respect to x, evaluated at the minimum point x_0. However, since $U(x)$ has a minimum at x_0, the first derivative term is zero. We suppose also that all of the terms in the series after the second derivative term can be neglected. This we can do if

(i) The second derivative of $U(x)$ is *not zero* at x_0;

(ii) The function $U(x)$ is smooth, so that all higher derivatives are finite;

(iii) The particle stays close enough to x_0. [Note that the ratio of the $(N + 1)^{st}$ term in the series to the Nth term is proportional to $(x - x_0)$, so the ratio can be made arbitrarily small if the particle is kept sufficiently near x_0.]

Taylor's series therefore reduces to

$$U(x) \cong U(x_0) + \frac{1}{2}k_{\text{eff}}(x - x_0)^2 \tag{17}$$

for x close to x_0, where the constant $k_{\text{eff}} = \left.\frac{d^2U(x)}{dx^2}\right|_{x_0} \equiv U''(x_0)$, the second derivative of the potential energy with respect to position, evaluated at the minimum point x_0.

The first term $U(x_0)$ in Eq. 17 is unimportant; potential energies are always arbitrary within an additive constant, because such a constant can't affect the force $F(x) = -\frac{dU(x)}{dx}$. The second term is the interesting one. It is the same as the spring potential energy

$$U_{\text{spring}} = \frac{1}{2}kx^2 \tag{18}$$

if we identify $k_{\text{eff}} = k$, except that the equilibrium point is at $x = x_0$ instead of at $x = 0$. It is therefore convenient to redefine the origin of coordinates for the particle so that $x = 0$ at the potential energy minimum. Then $x_0 = 0$ and the potential energy near the minimum is

$$U = \frac{1}{2}k_{\text{eff}}x^2 \tag{19}$$

where $k_{\text{eff}} = U''(0)$, the second derivative of the potential energy evaluated at the minimum point. The particle oscillates back and forth as though it were on the end of a spring of force-constant k_{eff}.

Note that we have reduced the behavior of a particle near a potential energy minimum to that of a particle on the end of a Hooke's-law spring. That is why simple harmonic motion is so important.

As long as the second derivative of the potential energy of a particle near a minimum is not zero, the particle will oscillate simple harmonically about the minimum point, if the amplitude of the oscillation is sufficiently small.

Sample Problem 4. *The simple pendulum revisited*

Find the effective force-constant k_{eff} for a bob of mass m swinging back and forth with small amplitude at the end of a string of length ℓ.

Solution: The gravitational potential energy of the bob is

$$U = mgh = mg\,\ell - mg\,\ell\,\cos\theta = mg\,\ell[1 - \sqrt{1 - (x/\ell)^2}\,],$$

since $\ell\cos\theta = \sqrt{\ell^2 - x^2} = \ell\sqrt{1 - (x/\ell)^2}$, by the law of Pythagoras. The first derivative of $U(x)$ is

$$U'(x) = -mg\ell \cdot \frac{1}{2}\frac{\left(-2x/\ell^2\right)}{\sqrt{1 - (x/\ell)^2}} = \left(\frac{mg}{\ell}\right)\frac{x}{\sqrt{1 - (x/\ell)^2}},$$

zero at $x = 0$, as expected. The second derivative is

$$U''(x) = \left(\frac{mg}{\ell}\right)\left[\frac{1}{\left(1 - (x/\ell)^2\right)^{1/2}} + \frac{x \cdot x/\ell^2}{\left(1 - (x/\ell)^2\right)^{3/2}}\right] = \left(\frac{mg}{\ell}\right)\frac{1}{\left(1 - (x/\ell)^2\right)^{3/2}}.$$

The effective force-constant for the pendulum about $x = 0$ is therefore

$$U''(0) = mg/\ell.$$

For small oscillations, the bob behaves as though it were moving on the end of a spring of force-constant $k_{\text{eff}} = mg/\ell$. The angular frequency is therefore

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{g}{\ell}},$$

and so the period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}}.$$

Sample Problem 5. An atom in a diatomic molecule moves with a molecular potential energy modeled by the function $U(r) = \frac{a}{4r^4} - \frac{b}{3r^3}$, where a and b are positive constants, and r is the distance between the two atomic nuclei.

(a) Sketch $U(r)$. (b) Find the location r_0 of the equilibrium point of the atom, and the dissociation energy of the molecule. (c) Find the frequency of small oscillations about the equilibrium point, if the atom has mass m.

Solution: (a) For very small r the first term in $U(r)$ dominates, so $U(r)$ becomes large and positive as $r \to 0$. For very large r the second term is more important than the first, so $U(r)$ is negative, approaching zero as $r \to \infty$. There must be a minimum in $U(r)$ in between, as shown.

(b) The force on the atom is $F = -\frac{dU}{dr}$, the negative slope of $U(r)$. For large separations $U(r)$ has a small positive slope, corresponding to a weak negative (i.e., attractive) force on the atom. For small separations, $U(r)$ has a large negative slope, corresponding to a strongly positive (i.e., repulsive) force. At the bottom of the potential-energy well the slope is zero so the force is zero; the radius at this point is the equilibrium separation of the two nuclei. The derivative of $U(r)$ is $U'(r) = -F(r) = -\frac{a}{r^5} + \frac{b}{r^4}$, which is zero at $r = r_0 = \frac{a}{b}$. The corresponding potential energy is $U(r_0) = \frac{a}{4(a/b)^4} - \frac{b}{3(a/b)^3} = -\frac{b^4}{12a^3}$. In equilibrium the atom sits at the bottom of the potential energy well, so to escape from the well the atom must be given the *dissociation energy* $D = +\frac{b^4}{12a^3}$, to bring its energy up to zero. Note from the sketch above that if $E \geq 0$ there is no right-hand turning point, so the atom can escape from the molecule.

(c) The bottom of the potential-energy well has a parabolic shape, so the potential mimics the parabolic $(1/2)kx^2$ spring potential. That is, near equilibrium the atom behaves as though it were on a spring of force-constant $k_{\text{eff}} = U''(r_0)$. In this case $U''(r) = \frac{5a}{r^6} - \frac{4b}{r^5}$, so $k_{\text{eff}} = U''(r_0)$ $= \frac{5a}{(a/b)^6} - \frac{4b}{(a/b)^5} = \frac{b^6}{a^5}$. The frequency of small oscillations of this atom about equilibrium is therefore $\nu = \frac{1}{2\pi}\sqrt{\frac{k_{\text{eff}}}{m}} = \frac{1}{2\pi}\sqrt{\frac{b^6}{ma^5}}$.

C.5 Damped oscillators (optional)

Left to themselves, real oscillators eventually wind down. An oscillating spring-mass system loses energy so that the oscillations damp out, as illustrated in Fig. 5. The energy losses are caused by so-called *dissipative* or *damping* forces, non-conservative forces like air resistance and friction.

Fig. 5. Real oscillations typically damp out

Vibrating guitar strings and ringing bells damp out for similar reasons. The oscillations of a car body on the springs of its suspension system are damped out by shock absorbers. Oscillating electrical circuits lose energy to the heat produced in circuit elements called resistors. In this section we will investigate damping in a special case, the case of linear damping in a spring-mass system.

In a undamped oscillator, in which the amplitude of oscillation stays constant, there is only a single characteristic time, the period of oscillation, the time to complete one full cycle. In a damped oscillator there is an additional characteristic time, the time it takes the amplitude to decrease by some standard factor. The two time scales are independent. One can have a short-period, rapidly oscillating system with either slow or rapid damping, or a long-period, slowly-oscillating system with either slow or rapid damping, all as illustrated in Fig. 6.

Fig. 6. Two time scales: the time scale for oscillation and the time scale for damping. (a) A short-period oscillator with a long damping time. (b) A short-period oscillator with a short damping time. (c) A long-period oscillator with a long damping time. (d) A long-period oscillator with a short damping time.

In an undamped spring-mass system the oscillation period is determined by the ratio m/k. We expect that an additional parameter will be needed for a damped oscillator, to help determine the time scale for damping. What physics determines this new parameter?

Typical damping forces are *frictional* forces that usually depend upon the velocity $\dot{x}$ of the oscillating object. The friction might be due to air resistance or drag in a fluid like water, where both the magnitude and the direction of the force depend upon velocity.

When both a spring force $-kx$ and a frictional force $F_f(\dot{x})$ act upon the object, Newton's second law $F = ma$ is

$$m\ddot{x} = -kx + F_f(\dot{x}) , \tag{20}$$

which reduces to the usual simple harmonic oscillator equation $\ddot{x} = -\left(\frac{k}{m}\right)x$ if the damping force $F_f(\dot{x}) = 0$. For most forces $F_f(\dot{x})$, Eq. 20 cannot be solved exactly.

There is an interesting special case in which Eq. 20 *can* be solved. Suppose the frictional force increases *linearly* with velocity,

$$F_f(\dot{x}) = -b\dot{x} , \tag{21}$$

where b is a constant and $\dot{x}$ is the velocity. Viscous drag forces have this form. They are the drag forces on an object moving through a fluid at very low speeds, corresponding to low *Reynolds numbers.** The advantage of using viscous drag for the frictional force has more to do with math than physics, because the $F = ma$ equation then becomes

$$m\ddot{x} = -kx - b\dot{x} , \tag{22}$$

which can be solved exactly!**

If we let $k/m = \omega_0{}^2$ in Eq. 22, so that ω_0 is the angular frequency of the undamped oscillator (as shown in Chapter B), and if we also define a new damping constant β by setting

$$\beta = \frac{b}{2m}, \tag{23}$$

*See the Special Topic in *Day Nine* in the text *Motion I* for a discussion of Reynolds numbers.

**For other drag forces it is usually necessary to solve $F = ma$ numerically, i.e., by using computers.

then Eq. 22 can conveniently be written

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0. \tag{24}$$

Note that the new constant β has the dimensions of inverse time. As we will see, $1/\beta$ is a measure of the typical damping time of the oscillator, the additional parameter we need.

What is the solution of this equation? That is, what is the position $x(t)$? It turns out that there are three different types of solutions, depending upon the relative values of the constants β and ω_0. Physically, the three solution types depend upon how much friction there is. If the friction is so small that $\beta << \omega_0$, which is a common situation, the oscillator is said to be *underdamped.* The solution of Eq. 24 in this case is worked out in Appendix I. The result is

$$x(t) = A\, e^{-\beta t} \cos(\omega_1 t + \phi_0) \tag{25}$$

if $\beta < \omega_0$, where the angular frequency ω_1 is

$$\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}\,. \tag{26}$$

Note that $x(t)$, which is the position of the oscillating mass as a function of time, is the product of an exponential damping factor $e^{-\beta t}$ and an oscillating function $\cos(\omega_1 t + \phi_0)$:

The exponential damping factor provides a natural interpretation of the damping constant β. As time goes on, the oscillator position $x(t)$ has a decreasing amplitude of oscillation, such that at time $\tau \equiv 1/\beta$ the amplitude has decreased to $1/e$ of its initial value, as shown in Fig. 7 below, where $e = 2.718$ .. is the basis of the natural logarithms. So $\beta = 1/\tau$, where τ is the time for the amplitude to decrease by the factor e.

The solution (25) for the linearly damped oscillator differs from the solution of the undamped oscillator $x(t) = A\cos(\omega_0 t + \phi_0)$ in two ways:

(i) The damped oscillator includes the "damping factor" $e^{-\beta t}$, which decreases the amplitude of the oscillator as time goes on.

(ii) The frequency of the oscillator is less than it is without damping, since $\omega_1 = \sqrt{\omega_0^2 - \beta^2} < \omega_0$.

As illustrated in Fig. 7, the position $x(t)$ of the damped oscillator can be visualized as the product of the oscillating function $\cos(\omega_1 t + \phi_0)$ and an "envelope" $Ae^{-\beta t}$ of decreasing amplitude. The solution agrees with our observation that real oscillators decrease in amplitude as time goes on.

Fig. 7. The position of a linearly damped oscillator as a function of time. The position is the product of a sinusoidal function and an exponentially-decreasing envelope, represented by the dotted lines above and below the axis. Note that if t_e is the time when the amplitude has fallen off by a factor e, then $\beta = 1/t_e$.

Sample Problem 6. A block of mass m is attached to a spring of force-constant k. The block is pulled a distance D from equilibrium and released from rest. The block also experiences a linear damping force $F = -b\dot{x}$, where b is a damping constant and $\dot{x}$ is the block's velocity.

(a) Find the amplitude A and phase-angle ϕ_0 of the block.

(b) If the damping is small, estimate the time it takes for the block to lose half its initial energy.

Solution: (a) The position of the block as a function of time is

$$x(t) = A\,e^{-\beta t}\cos(\omega_1 t + \phi_0)\,,$$

where $\beta = b/2m$ and $\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$, with $\omega_0 = \sqrt{k/m}$. The corresponding velocity of the block is

$$\dot{x}(t) = A\,e^{-\beta t}\,[-\beta\cos(\omega_1 t + \phi_0) - \omega_1\sin(\omega_1 t + \phi_0)]$$

found by differentiating $x(t)$. At time $t = 0$, we are given that $x(0) = D$ and $\dot{x}(0) = 0$, so

$$D = A\cos\phi_0 \qquad \text{and} \qquad 0 = -A\,[\,\beta\cos\phi_0 + \omega_1\sin\phi_0\,]\,.$$

The constant A cannot be zero (since $D \neq 0$), so the second equation requires that $\omega_1\sin\phi_0 = -\beta\cos\phi_0$. Therefore $\tan\phi_0 = -\beta/\omega_1$, giving the phase angle $\phi_0 = -\tan^{-1}(\beta/\omega_1)$.

Using the triangle shown at the right, we find that $\cos\phi_0 = \cos(-\phi_0) = \dfrac{\omega_1}{\sqrt{\omega_1^2 + \beta^2}}$, since the cosine of a negative angle is the same as the cosine of the corresponding positive angle. The amplitude constant A is therefore

$$A = \frac{D}{\cos\phi_0} = \frac{D\sqrt{\omega_1^2 + \beta^2}}{\omega_1}\,.$$

We have found both the phase-angle ϕ_0 and the oscillation amplitude A in terms of the initial displacement and the physical parameters of the oscillator.

(b) The energy of the system is the sum of the kinetic energy $\frac{1}{2}m\dot{x}^2$ of the block and the potential energy $\frac{1}{2}kx^2$ stored in the spring. For small damping there are many oscillations before the amplitude is reduced appreciably, so we can simply look at the envelope $Ae^{-\beta t}$ to get the typical amplitude at time t, as illustrated. The position x and velocity $\dot{x}$ are both reduced by the factor $e^{-\beta t}$, so the total energy of the system is reduced by the factor $(e^{-\beta t})^2 = e^{-2\beta t}$. The energy is reduced to one-half its initial value when $e^{-2\beta t} = 1/2$, so $-2\beta t = \ln(1/2) = -\ln 2$. The half-energy time in the small-damping approximation is therefore

$$t = \ln 2 / 2\beta \cong 0.35 / \beta\,.$$

If the damping constant β is cut in half, the half-energy time is doubled, a reasonable result.

Appendix I. Solving the *Damped* Oscillator Equation

The $F = ma$ equation for a harmonic oscillator with an added damping force is

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \tag{I 1}$$

where β is the damping constant and $\omega_0{}^2 = k/m$. Here k is the force constant of the spring and m is the mass on the end. Our goal is to find $x(t)$, the position of the mass as a function of time. Equations of this type can be solved by exponential functions, because exponentials have the property that their derivatives are also exponentials. So we try

$$x(t) = e^{\alpha t} \tag{I 2}$$

where α is some unknown constant. Substitution of (I 2) into (I 1) gives the algebraic equation

$$\alpha^2 + 2\beta\alpha + \omega_0{}^2 = 0 \tag{I 3}$$

since the common factor $e^{\alpha t}$ cancels out. We have reduced the difficult problem of solving Eq. I1 to that of solving a simple quadratic equation in the unknown constant α.

Using the quadratic formula, the two solutions of (I 3) are

$$\alpha = -\beta \pm \sqrt{\beta^2 - \omega_0^2}\,, \tag{I 4}$$

which we will call α_+ and α_-. Then the position of the oscillator $x(t)$ can be either $e^{\alpha_+ t}$ or $e^{\alpha_- t}$.

Assume that $\beta \neq \omega_0$ *. In that case the constants α_+ and α_- are *different*. Then most general solution of Eq. I 1 is

$$x(t) = Ce^{\alpha_+ t} + De^{\alpha_- t} \tag{I 5}$$

*We will consider the unique case $\beta = \omega_0$, called *critical damping*, in problem CP37.

where C and D are arbitrary constants. It is easy to show by substitution that Eq. I 5 is a solution of Eq. I 1, for *any* constants C and D. As for the simple harmonic oscillator without damping, these constants can be found from the initial position and velocity of the oscillator.

There are now two special cases with very different behaviors, depending upon the relative sizes of β and ω_0. If the frictional damping is so large that $\beta > \omega_0$, it follows from Eq. I 4 that the constants α_+ and α_- are *real*, and that each is also *negative*. The position of the oscillator is therefore the sum of two decreasing exponentials, so it turns out that the mass never oscillates more than once before $x(t)$ dies out, approaching zero quickly. This is called the *overdamped* solution. We will explore this case further only in a problem (See problem CP38).

The applicable case in most interesting situations is the *underdamped solution* with $\beta < \omega_0$, corresponding to relatively small friction. We expect the mass to oscillate when the friction is small, although the amplitude of oscillation should decrease with time.

There is a problem with the solution, however; if $\beta < \omega_0$ the quantity $\sqrt{\beta^2 - \omega_0^2}$ becomes an *imaginary number*! What can this mean physically? The position of the mass is necessarily real, so how can a real position $x(t)$ be found from exponentials of complex numbers?

The first step in making sense of this is to write

$$\sqrt{\beta^2 - \omega_0^2} = \sqrt{-(\omega_0^2 - \beta^2)} = i\sqrt{\omega_0^2 - \beta^2} \tag{I 6}$$

where $i \equiv \sqrt{-1}$. The square root on the right is *real*, because $\beta < \omega_0$. The general solution of $F = ma$ for this oscillator is therefore

$$x(t) = e^{-\beta t}\left(C e^{i\sqrt{(\omega_0^2 - \beta^2)}\, t} + D e^{-i\sqrt{(\omega_0^2 - \beta^2)}\, t} \right) \tag{I 7}$$

using Eq. A5, and factoring out $e^{-\beta t}$.

The next step in finding a real solution from complex numbers is to use an amazing identity due to the mathematician L. Euler. The identity is

$$e^{i\theta} = \cos\theta + i \sin\theta \tag{I 8}$$

for any θ. That is, the real and imaginary parts of the complex exponential $e^{i\theta}$ are $\cos\theta$ and $\sin\theta$, respectively. It follows that

$$e^{-i\theta} = \cos\theta - i \sin\theta \tag{I 9}$$

using the trig identities $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$, or simply by taking the complex conjugate of Eq. I 8; i.e., by letting every $i \rightarrow -i$. Substituting these results for the case that $\theta = \sqrt{\omega_0^2 - \beta^2}\, t$ into Eq. I 7 gives

$$x(t) = e^{-\beta t}\,[\,E\cos\omega_1 t + F\sin\omega_1 t\,]\,, \tag{I 10}$$

where the constant $E = (C + D)$, the constant $F = i\,(C - D)$, and the angular frequency

$$\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}\,. \tag{I 11}$$

The solution $x(t)$, corresponding to the position of the mass, must be real; therefore both of the constants E and F must also be real. (This implies that the original constants C and D were complex.)

Note that $x(t)$ is now the product of an exponential damping factor $e^{-\beta t}$ and the oscillating function $E\cos\omega_1 t + F\sin\omega_1 t$. This allows a physical interpretation of the damping constant β : As time goes on, the oscillator position $x(t)$ has a decreasing amplitude of oscillation, such that at the time $\tau \equiv 1/\beta$ the amplitude has decreased to $1/e$ of its initial value. So $\beta = 1/\tau$, where τ is the time for the amplitude to decrease by the factor $e = 2.718\ldots$.

It is usually more convenient to express the solution in terms of an amplitude A and phase-angle ϕ_0 :

$$x(t) = A\,e^{-\beta t}\cos(\,\omega_1 t + \phi_0\,)\,. \tag{I 12}$$

The constants A and ϕ_0 can be found in terms of E and F by expanding the cosine of the sum of two angles.

A few properties of complex numbers: Any complex number z can be written as the sum of a real and an imaginary part: That is, $z = a + ib$, where a and b are real numbers. Any complex number can also be written in the form $z = Re^{i\theta}$, where R and θ are real. Using Euler's identity (I 8) above, it follows that $a = R\cos\theta$ and $b = R\sin\theta$. The complex conjugate of z is denoted by z^*, where $z^* = a - ib$, found simply by changing i to $-i$ in z.

State of the Search

1. Many physical systems oscillate periodically in time. The prime example is the spring-mass system, where an object of mass m oscillates at the end of a Hooke's-law spring of force-constant k. The $F = ma$ equation is

$$\ddot{x} + \omega^2 x = 0$$

where x is the position of the object measured from equilibrium and $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency of oscillation. The equation is called the *simple harmonic oscillator equation.*

2. The general solution of the simple harmonic oscillator equation is

$$x(t) = A\cos(\omega t + \phi_0)$$

in terms of the amplitude A and phase-angle ϕ_0. These two constants may be found in terms of the initial position and velocity of the object. The period T and frequency ν of the oscillator are related to its angular frequency ω by

$$T = \frac{1}{\nu} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}.$$

3. A particle moving without dissipation with potential energy $U(x)$ moves simple harmonically about a potential energy minimum, with effective force constant

$$k = \left.\frac{d^2U}{dx^2}\right|_{x_0} = U''(x)|_{x_0}.$$

(If $U''(x)$ evaluated at the minimum is *zero*, the particle may still oscillate, but not simple-harmonically, i.e., not sinusoidally.)

4. A simple pendulum with a point-mass m on a string of length ℓ in gravity g swings simple harmonically if its amplitude is small, because then the *pendulum equation*

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

reduces to $\ddot{\theta} = -\frac{g}{\ell}\theta$, the simple-harmonic oscillator equation. It follows that the angular frequency ω of a small-amplitude simple pendulum is $\omega = \sqrt{\frac{g}{\ell}}$.

5. *Any* system whose equation of motion is the simple harmonic oscillator equation will oscillate sinusoidally, where the angular frequency of oscillation is the square root of the constant preceding the undifferentiated term (assuming there is no constant preceding the second derivative term.)

6. Real oscillators are damped by frictional forces, which tend to reduce the amplitude and energy content of the oscillators. Linear (viscous) damping leads to the equation of motion

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

where x is the position of the particle of mass m, $\omega_0 = \sqrt{k/m}$ if the spring has force-constant k, and β is proportional to the damping. If $\beta < \omega_0$ the oscillator is "underdamped", and has the solution

$$x(t) = A\,e^{-\beta t}\cos(\omega_1 t + \phi_0)$$

where $\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$, A is the amplitude, and ϕ_0 is the phase angle.

PICTURE QUIZ C

1. Place one ball on the end of a spring, to form a simple harmonic oscillator, and another ball on the end of a cord, to form a simple pendulum. Adjust the cord length so that both systems have the same period. Now take them both to the Moon. The ratio of their periods then satisfies which one of the following?

(A) $T_{\text{pendulum}}/T_{\text{SHO}} < 1$

(B) $T_{\text{pendulum}}/T_{\text{SHO}} > 1$

(C) $T_{\text{pendulum}}/T_{\text{SHO}} = 1$

2. The $F = ma$ equation for a small-amplitude pendulum is $\ddot{\theta} = -\left(\frac{g}{\ell}\right)\theta$. Which of the following is a solution of the equation?

(A) $\theta = A\sin(\omega t)$ for $\omega = \sqrt{\frac{\ell}{g}}$

(B) $\theta = A\tan(\omega t)$ for $\omega = \sqrt{\frac{g}{\ell}}$

(C) $\theta = A\cos(\omega t + \frac{\pi}{4})$ for $\omega = \sqrt{\frac{g}{\ell}}$

(D) $\theta = Ae^{(\omega t + \frac{\pi}{4})}$ for $\omega = \sqrt{\frac{\ell}{g}}$

(E) $\theta = A\sin(\omega t - 1.2)$ for $\omega = \sqrt{\frac{g}{\ell}}$

PROBLEM QUIZ C

1. About what initial angle θ_0 must a simple pendulum have so that its period will be 1% greater than it is for very small amplitudes?

2. A thin uniform rod of mass M and length L is made into a physical pendulum by pivoting it about a small hole through the rod a distance ℓ from its center. (a) Find the period of small-amplitude oscillations in terms of g and given parameters. (b) Find the value of ℓ (expressed as a fraction of L) for which the period is a *minimum*.

3. A hoop of mass M and radius R sits on a horizontal surface. Two springs are attached to the upper rim, and the other ends of the springs are kept fixed, as shown. If the hoop is displaced slightly to the right and released, it oscillates back and forth, rolling without slipping on the surface. Find the period of these small oscillations. Assume the springs remain essentially horizontal.

4. The "effective potential energy" of a planet orbiting the Sun is $U_{\text{eff}} = \dfrac{L^2}{2mr^2} - \dfrac{GMm}{r}$, where L is the (constant) angular momentum of the planet about the Sun, m is the planet's mass, M is the Sun's mass, and r is the distance of the center of the planet from the center of the Sun. The total energy of the planet, which is conserved, is then $E = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}}$, where $\dot{r} = \dfrac{dr}{dt}$. (a) Find the value of r for which U_{eff} is a minimum (this is the radius at which a planet would move in a circular orbit for given parameters L, m, M, and G.) (b) In fact, planets do *not* move in circular orbits: Instead, they *oscillate* about circular orbits. Find the period of such oscillations, and compare with the period of the orbit of the planet around the Sun.

5. (Optional) A ball of mass $m = 0.1$ kg is attached to a spring, and initially displaced and released from rest. The ball is also subject to a viscous damping force of the form $F_{\text{fric}} = -bv$, where v is the velocity of the ball. The ball's position as a function of time is shown in the graph. (a) Find the numerical value of β, in units s^{-1}. (b) Find the force-constant of the spring, in N/m. (c) Find the phase angle ϕ_0 in radians, given that $x(t) = Ae^{-\beta t}\cos(\omega t + \phi_0)$.

Special Topic: Resonance

Aside from frictional damping, it is also interesting to see what happens if an oscillator is driven by some outside force. How does the oscillator respond? For example, suppose we hold one end of a spring up in the air, and hang a ball from the other end. If we wiggle the upper end of the spring up and down, this motion is communicated to the ball, causing it to oscillate in interesting ways. Especially interesting is the important phenomenon of *resonance*, where the ball oscillates with very high amplitudes when the frequency of the driving force matches the natural frequency of the oscillator, the frequency it would have if we pulled the ball away from equilibrium and then let it go.

A child on a swing can achieve large amplitudes of oscillation if someone shoves the swing repeatedly at just the right intervals, as illustrated in Fig. S1. Similarly, a mass hanging on the end

Fig. S1. If an oscillating system is pushed at the right frequency, it can build up large amplitudes.

of a spring can achieve very high amplitudes if the upper end of the spring is moved slightly up and down at the same frequency as the natural frequency of the spring-mass system. These are examples of *resonance*. Resonance also plays an important role in making music, from singing in the shower to designing violins. Amplifier circuits in radios and television sets are built so that they will resonate with the frequencies of broadcasted radio waves. Skyscrapers in Los Angeles and Tokyo are built so they will *not* sway in resonance with typical earthquake frequencies.

A. The driven oscillator

As an example of a driven oscillator, we study the behavior of a spring-mass system under the influence of a sinusoidal driving force. As a first step we neglect damping, and only include the effects of friction later on. We begin by considering a mass m subject to both a spring force and a sinusoidally oscillating driving force $F_0 \sin\omega t$. Newton's second law is then $m\ddot{x} = -kx + F_0 \sin\omega t$, or

$$\ddot{x} + \omega_0^2 x = f_0 \sin\omega t \tag{S1}$$

where $\omega_0 = \sqrt{k/m}$ is the natural frequency of the spring-mass system and $f_0 \equiv F_0/m$. Here ω is the frequency of the driving force, which is independent of the natural frrequency ω_0. We can choose ω to be anything we like, while ω_0 is determined by the spring and mass.

Sample Problem S1. Show that one way to apply a sinusoidal force to an oscillator is to hang a mass on a spring, and then shake the top end of the spring up and down sinusoidally.

Solution: The mass is subject to two forces only, the spring and gravity, as shown in the first diagram below. Suppose that the spring has unstretched length ℓ, and that we place the upper end of the spring at $x = 0$. When the mass hangs in equilibrium the spring is stretched an amount Δx, as shown in the second diagram, where $k\Delta x = mg$, balancing the forces.

Now move the upper end of the spring up and down so that its position as a function of time is $x_0(t) = \alpha \sin\omega t$, where α is a constant. The mass is now located at some position $x(t)$, as shown in the third diagram, so the spring stretch has become $\Delta x + x - x_0$. That is, the spring is stretched a distance $x - x_0$ more than before. The total force on the mass, due to the stretched spring and gravity, is $F_{\text{total}} = -k(\Delta x + x - x_0) + mg = -k(x - x_0)$, where the last step follows by cancelling the equilibrium spring force $-k\Delta x$ with gravity mg. Setting $F_{\text{total}} = m\ddot{x}$, we find

$$m\ddot{x} + kx = kx_0 = k\alpha \sin\omega t,$$

which is equivalent to the driven harmonic oscillator equation (S1) if we choose $f_0 = k\alpha/m$.

Now we need to solve the $F = ma$ equation (Eq. S1)

$$\ddot{x} + \omega_0{}^2 x = f_0 \sin\omega t$$

for the position $x(t)$ of the mass as a function of time. The solution is worked out in Appendix S1. The result is

$$x(t) = A\cos(\omega_0 t + \phi_0) + \frac{f_0}{\omega_0^2 - \omega^2}\sin\omega t \qquad \text{(S2)}$$

where A and ϕ_0 are arbitrary constants. The first term is the usual solution of the undriven oscillator, oscillating at the natural frequency ω_0 of the spring-mass system. The second term also oscillates, but at the frequency ω of the driving force instead. This second term is especially important if ω is close to ω_0. The amplitude of this term, $\frac{f_0}{\omega_0^2 - \omega^2} = \frac{f_0/\omega_0^2}{1 - (\omega/\omega_0)^2}$, is plotted in Fig. S2 as a function of ω/ω_0, the ratio of the driving frequency to the natural frequency of the spring. The graph illustrates the phenomenon of resonance. If the frequency of the driving force is close to the natural frequency of the system, the mass oscillates with large amplitude. Note that the amplitude is *negative* for $\omega/\omega_0 > 1$; this just means that x_p is 180^0 out of phase with the driving force.

Fig. S2. The magnitude of the amplitude of the driving term $\left|\frac{f_0}{\omega_0^2 - \omega^2}\right| = \frac{f_0/\omega_0^2}{1 - (\omega/\omega_0)^2}$ as a function of the driving-force frequency ω, expressed as the ratio ω/ω_0 .

The solution represented by Eq. S2 *does* show the phenomenon of resonance, but it goes too far! The actual amplitude of oscillation cannot be infinite if $\omega = \omega_0$. We can certainly make the driving force oscillate at the natural frequency of the spring-mass system, but this cannot really lead to infinitely-large oscillations. Why does the analysis lead to a physically unrealistic result?

The cure to the disease is in the *physics*, not the mathematics. An important piece of the physics has been left out.

B. *Damped* driven oscillations

The problem is that we have neglected *friction*. Large amplitudes of oscillation imply large velocities, which imply large drag forces. However, we left out drag forces entirely in the preceding section. So to understand resonance better we need to include friction, which we can do as before by introducing a linear (viscous) drag $F_{\text{fric}} = -b\dot{x}$. Including now all three forces, $F = ma$ becomes

$$m\ddot{x} = F_{\text{spring}} + F_{\text{drag}} + F_{\text{driving}} = -kx - b\dot{x} + F_0 \sin\omega t \tag{S3a}$$

or

$$\ddot{x} + 2\beta\,\dot{x} + \omega_0^2 x = f_0 \sin\omega t \tag{S3b}$$

where $\beta = b/2m$ is the damping constant, $\omega_0 = \sqrt{k/m}$ is the natural frequency of the undamped spring-mass system, and $f_0 = F_0/m$.

The solution of the $F = ma$ equation (S3b) is worked out in Appendix S2. The result is

$$x(t) = A\,e^{-\beta t}\cos(\omega_1 t + \phi_0) + C\sin(\omega t - \delta)\,. \tag{S4}$$

The first term is the usual solution for a damped oscillator which is not being driven by an outside force. The amplitude A and phase-angle ϕ_0 are arbitrary constants, which can be determined from the initial position and velocity of the oscillator. The second term is a new term caused by the driving force. The amplitude C of this term is not arbitrary, but depends upon the driving force f_0 and the frequency ω of the driving force,

$$C = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}\,. \tag{S5}$$

The phase-angle δ of the new driving term can be found from

$$\tan\delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}, \tag{S6}$$

which also depends upon ω, the frequency of the driving force.

The new term $C\sin(\omega t - \delta)$ has quite a simple form: In this term the mass oscillates at the same frequency ω as the driving force, but is out of phase with it by the angle δ. This solution is often called the *steady-state* solution because it keeps going indefinitely. When a damped spring-mass system is first excited by a sinusoidal driving force, the motion of the mass is generally quite complicated, because it includes both terms in Eq. S4. But the first term $A\,e^{-\beta t}\cos(\omega_1 t + \phi_0)$ damps out exponentially, while the steady-state term $C\sin(\omega t - \delta)$ is all that remains in the end.

The most interesting part of the new term $C\sin(\omega t - \delta)$ is its amplitude C, which depends upon the driving frequency. The amplitude C is drawn as a function of the ratio ω/ω_0 in Fig. S3, for a variety of damping constants β. Notice that damping has cured the problem of infinite amplitudes found in the preceding section. The $4\beta^2\omega^2$ term in Eq. S5 prevents the denominator from going to zero, and so prevents the amplitude from diverging. The amplitude is high and narrow for small damping, and only blows up as $\beta \to 0$. The curve becomes flatter and broader as the damping is increased.

Fig. S3. The amplitude response of a damped oscillator to a driving force as a function of the driving frequency ω, expressed as the ratio ω/ω_0. The curves show a resonance peak at a frequency near ω_0. The curves are more peaked for small damping than for large damping.

Sample Problem S2. *The resonance frequency and resonance amplitude*

(a) Find the angular frequency for which the steady-state oscillator amplitude is a maximum. This is called the *amplitude resonance frequency* ω_R.

(b) Find also the amplitude of oscillation at resonance C_R.

Solution:

(a) The amplitude $C = \dfrac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$ is a maximum when its denominator D is a minimum; D is a minimum when its square $D^2 = (\omega_0{}^2 - \omega^2)^2 + 4\beta^2\omega^2$ is a minimum. Differentiating D^2 with respect to ω^2, we set

$$\frac{dD^2}{d(\omega^2)} = -2(\omega_0{}^2 - \omega^2) + 4\beta^2 = 0$$

where $\omega = \omega_R$ is the resonance frequency. The result is

$$\omega_R = \sqrt{\omega_0^2 - 2\beta^2}\,.$$

The resonance frequency is somewhat less than the natural frequency of the undamped oscillator. That is, to get the largest response we should drive the oscillator at a slightly lower frequency than the natural frequency ω_0. If the damping is small, however, then $\omega_R \approx \omega_0$.

(b) Substituting $\omega = \omega_R = \sqrt{\omega_0^2 - 2\beta^2}$ into the formula for the amplitude C, it is easy to show that the amplitude of oscillation at resonance is

$$C_R = \frac{F_0}{2m\beta\omega_1}\,,$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ is the frequency of the damped, undriven oscillator. Note that C_R is large if the damping β is small.

C. Power input to a driven, damped oscillator

Suppose that an oscillator is both damped and driven, and that the oscillator is in steady-state, with constant amplitude, as illustrated in Fig. S4. Left by itself the oscillator would die away because of the damping; the driving force is adjusted to just make up for it. The oscillation of a child on a swing naturally dies away, but the swing can be pushed with just the right force to keep it swinging at constant amplitude. Similarly, if a mass is hung on the end of a spring, one can shake

Fig. S4. Constant-amplitude oscillation of a forced, damped oscillator.

the upper end of the spring up and down sinusoidally, with just the right force so that the oscillation amplitude of the mass at the bottom stays constant. This force produces a power input to the oscillator, which is balanced by a power output due to frictional heating. If the input power exceeds the frictional output power, the amplitude increases; if the frictional output power exceeds the input power, the oscillation amplitude decreases.

What is the required average input power needed when a damped oscillator is in steady-state oscillation? By definition, the instantaneous power delivered to an object is $P = Fv$, the applied force times the object's velocity, where the driving force is $F = F_0 \sin\omega t$ and the velocity of the mass is $v = \omega C \cos(\omega t - \delta)$ for steady-state motion. This velocity is found by differentiating the position of the mass, $x(t) = C \sin(\omega t - \delta)$, with respect to time. The delivered power is therefore

$$P = Fv = F_0 \omega C \sin\omega t \cos(\omega t - \delta) = F_0 \omega C \sin\omega t \, [\cos\omega t \cos\delta + \sin\omega t \sin\delta] \tag{S7}$$

where in the last step we used the trig identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$. The average value of the product $\sin\omega t \cos\omega t$ over one or more periods is *zero*, while the average value of $\sin^2\omega t$ is $\langle \sin^2\omega t \rangle = 1/2$ (See problem CP40). Therefore the average power delivered to the oscillator by the driving force is

$$\langle P \rangle = \frac{1}{2} F_0 \omega C \sin\delta . \tag{S8}$$

As shown in Appendix S2 (Eqs. S26 and S27), the amplitude C is related to $\sin\delta$ by

$$C = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} = \frac{F_0}{2mb\omega}\sin\delta \tag{S9}$$

using $f_0 = F_0/m$. These relationships show that the average power delivered can be expressed in either of the forms

$$<P> = \frac{F_0^2}{4m\beta}\sin^2\delta \tag{S10a}$$

in terms of the phase-angle δ, or

$$<P> = \frac{\beta\,\omega^2 F_0^2}{m\left[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2\right]}, \tag{S10b}$$

in terms of the input-frequency ω. Notice from Eq. S10(a) that the maximum power is delivered when $\delta = \pi/2$. The form given in Eq. S10(b) shows also that the needed input power is zero if the damping $\beta = 0$. This makes sense: If there is no damping, no power is required to keep the oscillator going.

Sample Problem S3. A ball of mass m = 0.20 kg is suspended on a spring of force constant k = 0.40 N/m. The ball is dipped in a fluid, so the drag force on it is $F_{\text{drag}} = -bv$, where b = 0.20 kg/s. If the upper end of the spring is moved up and down with amplitude $x_0(t) = \alpha \sin\omega t$, where α = 2.0 cm and the driving frequency is $\omega = 1.0\ \text{s}^{-1}$. Find the average input power required when the system is in steady state oscillation.

Solution: The damping constant is $\beta = \frac{b}{2m} = \frac{0.20\ \text{kg/s}}{0.40} = 0.50\ \text{s}^{-1}$ and the natural frequency is $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.40\ \text{N/m}}{0.20\ \text{kg}}} = \sqrt{2.0}\,s^{-1}$. The phase angle $\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$ $= \tan^{-1}\left(\frac{2(0.50\ \text{s}^{-1})(1.0\ \text{s}^{-1})}{(2.0\ \text{s}^{-2}) - (1.0\ \text{s}^{-2})}\right) = \tan^{-1}(1.0) = 45^0$, and the applied force is $F_0 = k\alpha =$ 0.0080 N, using the discussion in *Sample Problem* S2. Substituting all of this into Eq. 10(a), we find that the steady-state power input is $<P> = \frac{F_0^2}{4m\beta}\sin^2\delta = 8 \cdot 10^{-5}$ Watts.

Appendix S1. Solving the *driven* oscillator equation

If an undamped harmonic oscillator is shaken by a sinusoidal external force, the $F = ma$ equation is

$$\ddot{x} + \omega_0{}^2 x = f_0 \sin\omega t \tag{S11}$$

Its solution is the sum of two parts, $x = x_c + x_p$, where x_c is the general solution of the undriven oscillator equation $\ddot{x} + \omega_0{}^2 x = 0$, and x_p is any solution of the full equation. We already know from Chapter B that

$$x_c = A\cos(\omega_0 t + \phi_0) \tag{S12}$$

with arbitrary constants A and ϕ_0, so that half of the problem has already been solved.

The other half of the problem is to find a solution x_p of the full equation (S11). Note that the function

$$x_p(t) = C \sin\omega t \tag{S13}$$

should work for some value of the constant C, because the second derivative of $\sin\omega t$ is proportional to $\sin\omega t$ itself, so there would be a $\sin\omega t$ in each term, which could then be divided out. Only constants remain, so C can be chosen to satisfy the resulting equation. In fact, if $x_p(t) = C \sin\omega t$, then the second derivative $\ddot{x}_p(t) = -\omega^2 C \sin\omega t$, so Eq. S11 becomes

$$(-\omega^2 + \omega_0{}^2)\, C = f_0\,, \tag{S14}$$

cancelling the $\sin\omega t$. Therefore $x_p(t) = C \sin\omega t$ *is* a solution of $F = ma$, if C is chosen to be

$$C = \frac{f_0}{\omega_0^2 - \omega^2}\,. \tag{S15}$$

The position $x(t)$ of the mass is the complete solution of $F = ma$,

$$x(t) = x_c + x_p = A\cos(\omega_0 t + \phi_0) + \frac{f_0}{\omega_0^2 - \omega^2}\sin\omega t\,, \tag{S16}$$

where A and ϕ_0 are constants depending upon the initial position and velocity of the mass, and f_0 is proportional to the strength of the driving force. The first term, the characteristic solution, oscillates at the natural frequency ω_0 of the spring-mass system, while the second term, the particular solution of the full equation, oscillates at the same frequency ω as the driving force.

Appendix S2. Solving the *damped and driven* oscillator equation

The $F = ma$ equation for a driven, damped harmonic oscillator is

$$\ddot{x} + 2\beta\,\dot{x} + \omega_0{}^2 x = f_0 \sin\omega t\,. \tag{S17}$$

As for the driven, *un*damped oscillator solved in Appendix S1, the solution of this equation is the sum of the general solution $x_c(t)$ of the $F = ma$ equation without the driving term on the right, and a particular solution $x_p(t)$ of the full equation. In Chapter C we have already found the solution $x_c(t)$, as given in Eq. 25:

$$x_c(t) = A\,e^{-\beta t}\cos(\omega_1 t + \phi_0)\,, \tag{S18}$$

where $\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$, and the amplitude A and phase-angle ϕ_0 are arbitrary constants. Notice that this term dies out as time goes on; it is therefore often called the "transient" solution. It is the other solution $x_p(t)$ that prevails in the end.

How do we find this particular solution? We can't use the form $C\sin\omega t$ because the term $2\beta\dot{x}$ in Eq. S17 converts the sine to a cosine, so $\sin\omega t$ can't be divided out of the equation. Trial and error shows that a solution of the form

$$x_p(t) = C\sin(\omega t - \delta) \tag{S19}$$

does work, however. This trial solution oscillates at the same frequency ω as the driving force, but is shifted by some phase angle δ. The function works because expanding the sine of the difference between two angles gives a $\cos\omega t$ term as well as a $\sin\omega t$ term. In fact, substitution of $x_p(t) = C\sin(\omega t - \delta)$ into the $F = ma$ equation (S17) gives

$$C\,[\,(\omega_0{}^2 - \omega^2)\sin(\omega t - \delta) + 2\beta\omega\cos(\omega t - \delta)\,] = f_0 \sin\omega t\,. \tag{S20}$$

The goal is to find constants C and δ which make this equation true for all times.

First, we need to expand the sine and cosine on the left using the trig identities

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B \tag{S21a}$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B\,. \tag{S21b}$$

The result is

$$C\,[\,(\omega_0{}^2 - \omega^2)(\,\sin\omega t\,\cos\delta\, -\, \cos\omega t\,\sin\delta) \,+\, 2\beta\omega\,(\,\cos\omega t\cos\delta + \sin\omega t\,\sin\delta\,)\,]$$

$$= f_0 \sin\omega t\,, \tag{S22}$$

which must hold at all times. In particular, at times t such that $\omega t = 0, \pi, 2\pi,$ etc., the $\sin\omega t$ terms in (S22) are all zero, so the $\cos\omega t$ terms alone must satisfy the equation. That is,

$$C[\,-(\omega_0{}^2 - \omega^2)\sin\delta \,+\, 2\beta\omega\cos\delta\,] \,=\, 0 \tag{S23}$$

at any of the times mentioned above. But all of these quantities are independent of time, so this expression must *always* be zero. Therefore the quantity inside square brackets vanishes; i.e.,

$$\tan\delta \,=\, \frac{2\beta\omega}{\omega_0^2 \,-\, \omega^2}\,. \tag{S24}$$

We can check this result in at least a couple of ways. If the damping $\beta \rightarrow 0$, it follows that $\tan\delta \rightarrow 0$, so the phase angle $\delta \rightarrow 0$ as well. This agrees with our previous solution to the driven, undamped oscillator, as derived in Appendix S1, where the position of the mass moved in phase with the driving force. The same thing happens for very low applied frequencies ω. The phase angle $\delta \rightarrow 0$ as $\omega \rightarrow 0$, which agrees with a simple observation that if the upper end of a spring is very slowly moved up and down, a mass hung on the lower end of the spring moves up and down in phase with it.

Equation S24 must also be true for all times such that $\omega t = \pi/2, 3\pi/2, 5\pi/2,$ etc., that is when each $\cos\omega t$ term is zero. Only the $\sin\omega t$ terms survive, so we must have

$$C\,[\,(\omega_0{}^2 - \omega^2)\cos\delta \,+\, 2\beta\omega\sin\delta\,] \,=\, f_0\,. \tag{S25a}$$

The constant C is therefore

$$C = \frac{f_0}{(\omega_0^2 \,-\, \omega^2)\cos\delta \,+\, 2\beta\omega\sin\delta}\,. \tag{S25b}$$

Fig. S5. A triangle, one of whose angles is the phase angle δ.

With the help of the triangle shown in Fig. S5, constructed from the Pythagorean theorem and our knowledge that $\tan\delta = \dfrac{2\beta\omega}{\omega_0^2 - \omega^2}$, we can read off $\sin\delta$ and $\cos\delta$. In fact,

$$\sin\delta = \frac{2\beta\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \quad \text{and} \quad \cos\delta = \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \tag{S26}$$

Substituting these into Eq. S25(b) gives the final result for the amplitude C:

$$C = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \,. \tag{S27}$$

We have now completed our particular solution $x_p(t)$ of the full $F = ma$ equation of motion for the damped driven oscillator. It is

$$x_p(t) = C\sin(\omega t - \delta) \tag{S28}$$

where the amplitude C is given by Eq. S17 and the phase-angle δ by Eq. S24. The complete solution, adding is $x_c(t)$, is

$$x(t) = A\,e^{-\beta t}\cos(\omega_1 t + \phi_0) + C\sin(\omega t - \delta) \tag{S29}$$

which was our goal.

Questions and Problems from Chapter C, *Oscillation and Waves*

CQ1. A pendulum is made from a string and a mass, and a simple harmonic oscillator is made from a spring and a mass; each has the same period of oscillation. If they are taken to a planet where gravity is four times that on Earth, what is the ratio of the period of the small-amplitude pendulum to that of the simple harmonic oscillator? Explain.

CQ2. How could you use a pendulum to determine the acceleration of gravity on a mountaintop?

CQ3. A boy swings on a swing and finds that the period of motion is 4.0 s. A girl of the same mass joins him; now, assuming the amplitude of the motion is the same as before, the period is (a) 8.0 s (b) 6.0 s (c) 4.0 s (d) 3.0 s (e) 2.0 s (f) one cannot tell without knowing the swing length.

CQ4. Explorers visit a new planet where the value of g is unknown. They set up a pendulum, and find that it oscillates with a period of 1.0 s. What would be the period of this pendulum if (a) its mass were doubled? (b) its length were doubled? (c) its amplitude of oscillation were doubled?

Questions from the Special Topic on Resonance

CQ5. A soprano reportedly can shatter a high-quality crystal glass if she sings at just the right frequency, but she cannot shatter a glass of poor quality. Why might this be true?

CQ6. If you wet a finger and rub it around the rim of a crystal glass, the glass emits a musical sound. How does this work? (Benjamin Franklin invented a musical instrument called the *Glass Armonica* based on this principle.)

CQ7. In 1831 a bridge collapsed in England when soldiers marched across it. Now soldiers are sometimes told to break step while marching across a bridge. Why?

CQ8. A car body is suspended on springs to smooth out road bumps, and shock absorbers are used as a damping force. What would it be like to ride in a car without shock absorbers?

CQ9. Sample Problem S2 shows that the angular frequency of a driven oscillator for which the amplitude of oscillation is maximum is $\omega_R = \sqrt{\omega_0^2 - 2\beta^2}$. As $\beta \to \omega_0/\sqrt{2}, \omega_R \to 0$; for even larger damping, the formula implies that ω_R becomes imaginary. What is going on physically in these cases? Sketch the amplitude C as a function of frequency for $\beta = \omega_0/\sqrt{2}$ and also for $\beta = 2\omega_0/\sqrt{2}$.

CQ10. How can the following complex numbers z be written in the forms (a) $z = a + ib$ (b) $z = Re^{i\theta}$, where a, b, R, and θ are real? (1) $z = 1/i$ (2) $z = 1/(2+3i)$ (3) $z = 3\cos\pi - 4i\sin(\pi/2)$

Section C.1 The simple pendulum

CP1. A simple pendulum consists of a cord of length $\ell = 2.0$ m and a bob of mass $m = 3.0$ kg. When the string is at angle $\theta_0 = 30^o$ to the vertical, the bob is pushed in a direction perpendicular to the string at speed $v_0 = 3.5$ m/s in the uphill direction. (a) What is the maximum angle θ_{max} to the vertical the bob will achieve? (b) What speed will the bob have when it reaches its lowest point?

CP2. Using dimensional reasoning alone, the period T of a pendulum was shown in the text to have the form $T = f(\theta_0)\sqrt{\ell/g}$, in terms of its length ℓ and gravity g, where $f(\theta_0)$ is an unknown function of the pendulum's angle amplitude θ_0. A particular pendulum clock (a "grandfather's clock") is running 5% slow. (a) Should you lengthen or shorten the pendulum to make it tell time correctly? (b) Find the necessary fractional change $(\Delta\ell/\ell)$ in length required.

CP3. A pendulum of mass m and length ℓ is released at rest with amplitude θ_0 in uniform gravity g. (a) At what position of the pendulum bob is the tension a maximum? Why? (b) Find this maximum tension in terms of the given parameters.

CP4. What is the length of a pendulum whose period on the moon matches the period of a 1.0

m long pendulum on Earth?

CP5. Earth's free-fall acceleration varies from 9.78 m/s^2 at the equator to 9.83 m/s^2 at the poles, both because Earth is rotating and because it is not a perfect sphere. A pendulum whose length is precisely 1.000 m can be used to measure g; this is called a *gravimeter*. (a) How long do 100 oscillations take at the equator? (b) How long do 100 oscillations take at the North Pole? (c) Is the difference between these measurable? What kind of instrument could you use to measure the difference? (d) Take the gravimeter to the top of a mountain near the equator. Then 100 oscillations take 201.0 s. What is g on the mountain top?

CP6. With about what percentage accuracy would you have to measure the period of a pendulum of known length, to measure any difference between Earth's gravity at sea level and at the top of Mount Everest (altitude about 29,000 ft above sea level)? Assume that Earth's gravity g falls off as the square of the distance from Earth's center.

CP7. In the text, the pendulum equation $\ddot{\theta} = -(g/\ell)\sin\theta$ was found using $\tau = I\ddot{\theta}$. Starting with an isolation diagram, derive the same equation using $F = ma$ instead.

CP8. (a) Show that the energy of a pendulum is $E = (1/2)m\ell^2\dot{\theta}^2 + mg\ell(1-\cos\theta)$ if its potential energy is zero at the bob's lowest point. (b) By taking the time derivative of this conserved energy, show consistency with the pendulum equation $\ddot{\theta} = -(g/\ell)\sin\theta$. (c) Solve the energy equation for $\dot{\theta} \equiv d\theta/dt$, and then put dt on one side of the equation, and $d\theta$ and all else on the other side. Integrate both sides to show that the period of the pendulum is given by

$$T = 4\sqrt{\ell/2g}\int_0^{\theta_0} d\theta(\cos\theta - \cos\theta_0)^{-1/2},$$

where θ_0 is the angle amplitude of the pendulum. (The integral is not easy to evaluate!)

CP9. The Taylor series expansion of $\cos\theta$ is

$$\cos\theta = 1 - \theta^2/2! + \theta^4/4! - \ldots$$

Use the first two terms only (for both θ and θ_0) in the integral expression for the pendulum period given in part (c) of problem CP6, to show that $T = 2\pi\sqrt{\ell/g}$, independent of θ_0. *Optional extra problem*: Use all three terms (for both θ and θ_0) in the $\cos\theta$ expansion, and insert into the integral expression for T. You can use the binomial expansion $(1+x)^n = 1+nx+..$ (a good approximation if $x << 1$) to simplify the integrand. Show that the resulting integrals can be evaluated, and that they give the improved expression

$$T = 2\pi\sqrt{\frac{\ell}{g}}\left(1 + \frac{\theta_0^2}{16}\right),$$

showing the dependence of the period on the amplitude up to the term of order θ_0^2. Keeping additional terms in the expansions yields terms proportional to θ_0^4, θ_0^6, etc.

Section C.2 Small-amplitude motion

CP10. A small-amplitude simple pendulum of length 1.27 m oscillates back and forth 50 times in 113 s at a specific spot on Earth. What is the acceleration of gravity there?

CP11. Find the length of a small-amplitude simple pendulum whose period (assuming uniform gravity equal to that at the surface of Earth) is the same as the time to circle Earth once in a gravitational orbit just above the surface. Express the answer in terms of Earth's radius R.

CP12. An astronaut rides in a centrifuge on Earth in a horizontal circle of large radius R. The centrifuge turns with angular velocity ω. What is the period of a small-amplitude pendulum suspended within the centrifuge? Does the answer depend upon whether the plane of oscillation is radial or tangential to the circle?

Section C.3 Other rotational problems

CP13. Show that if the quantities I and d for a physical pendulum, as defined in *Sample Problem 1*, are chosen to correspond to the I and d of a simple pendulum, then the period of the motion of a physical pendulum reduces to the period of a simple pendulum.

CP14. A physical pendulum is constructed of a uniform disk of mass M and radius R; the frictionless pivot point is located a distance d above the center of the disk, as shown below. Find the period of small-amplitude swings of the disk. The mass of the pendulum arm is negligible.

CP15. A small hole is punched through a uniform disk of mass M and radius R, at a distance r from the center, as shown below. A nail projecting from a wall is threaded through the hole, so the disk can swing back and forth in a horizontal plane. What is the frequency ν of small oscillations?

CP16. One end of a string of length ℓ and negligible mass is tied to one end of a thin uniform rod of mass m and length L. The other end of the string is attached to a fixed point on the ceiling. The rod and string are pulled aside a small distance, and allowed to swing back and forth, always staying straight relative to one another. Find the period of such small-amplitude swings.

CP17. A ring of mass M and radius R is hung on a nail, as shown below. In terms of any or all of M, R, and gravity g, what is the period of small-amplitude swings of the ring?

CP18. Show that the period of a physical pendulum of mass M and moment of inertia I, pivoted at a point which is a distance d from the center of mass, would be the same as the period of a simple pendulum whose mass is all located at a point a distance $\ell = I/Md$ from the pivot point. This point on the physical pendulum is called the "center of oscillation."

CP19. A torsional pendulum consists of a solid uniform disk of mass m and radius R suspended by a rod, as shown below. The rod has a restoring torque $\tau = -\kappa\theta$, where κ is a constant. The disk can be turned about its axis of symmetry, twisting the rod. (a) Find the period of oscillation of this system. (b) Show, by using the fact that the work done by the torque is $W = \int \tau d\theta$, that the potential energy stored in the pendulum is $U = (1/2)\kappa\theta^2$.

CP20. Two identical thin rods, each of mass m and length L , are joined to form the right-angled object shown below. If the object is balanced on a sharp edge and then deflected slightly, it will oscillate. Find the frequency of oscillation.

CP21. A disk of mass m and radius r is attached to a horizontal massless spring with force-constant k as shown below. The disk has sufficient friction with the surface that it rolls without slipping. The equilibrium position is at $x = 0$. At time $t = 0$, the center of the disk is at $x = 0$, and has velocity v_0 to the right. Find (a) the angular frequency of oscillation of the disk about its equilibrium position; (b) the amplitude of oscillation; (c) the maximum fric-

tional force between the disk and the surface.

CP22. The effective force-constant of a small-amplitude simple pendulum was computed in *Sample Problem 4*. A quicker way to get the result is to begin with the binomial approximation $(1+x)^n \cong 1+nx$ (for $|x| << 1$) to approximate the potential energy function $U = mg\ell[1\sqrt{1-(x/\ell)^2}]$. Find k in this alternative way, and compare with the result found in *Sample Problem 4*.

CP23. A solid ball of radius r rolls without slipping in a cylindrical trough of radius R, as shown below. Show that the period of small-amplitude oscillations of the ball about the bottom of the trough is $T = 2\pi\sqrt{7(R-r)/5g}$. *Hint*: It is necessary to find the turn-angle of the ball about its center in terms of the angle θ of the ball's center relative to the vertical, as measured from point A at the center of the cylinder.

CP24. A uniform rod of mass m and length ℓ is smoothly hinged at point H. It is held in the horizontal position by a spring of force-constant k, which is attached a distance s from H, as shown below. The free end of the rod is pushed up a very short distance, thus compressing the spring slightly, and then released. The angle of maximum displacement θ_0 is small enough so that $\sin\theta_0 \cong \theta_0$ and $\cos\theta_0 \cong 1$. Find the period of motion.

Section C.4 Motion with an arbitrary conservative force

CP25. The 6-12 potential $U(r) = -2a/r^6 + b/r^{12}$, where a and b are positive constants, is sometimes used to approximate the potential energy between two atoms in a diatomic molecule, where the atoms are separated by the distance r. (a) Sketch $U(r)$, and find the location r_0 and potential energy $U(r_0)$ at the equilibrium point of the atoms. (b) Find the effective force constant and the angular frequency of small oscillations of an atom of mass m about the equilibrium point, taking $U(r)$ to be the atom's potential energy.

CP26. An atom of mass m moves in a molecular potential $U(r) = a/3r^3 - b/2r^2$ for $r > 0$, where a and b are positive constants. (a) Sketch the shape of $U(r)$. Then in terms of a and b, find (b) the position r_0 of the potential energy minimum; (c) the dissociation energy D; (d) the effective force-constant k_{eff} near equilibrium; (e) the frequency ν of small oscillations about equilibrium.

CP27. The "Morse potential"

$$U(r) = D_e(1 - e^{a(r-r_o)})^2$$

is a function which fits molecular potential energies quite well. Here r is the separation between two atoms in a diatomic molecule, and D_e, a, and r_0 are constants. In terms of the three constants, (a) Find the radius r and potential energy U at the equilibrium point. (b) Find the effective force constant for an atom near the equilibrium point. (c) Find the period of small oscillations of an atom of mass m near equilibrium, taking $U(r)$ above to be the potential energy of the atom.

CP28. The mutual potential energy of two ions Li^+ and I^- as a function of their separation r is approximately

$$U(r) = -\frac{e^2}{4\pi\epsilon_o r} + \frac{A}{r^{10}}.$$

The equilibrium distance between the centers of these ions in the Li I molecule is $r_0 \cong 0.24$ nm. The electric charge $e = 1.6\times10^{-19}$ C (C stands for *Coulomb*), $1/4\pi\epsilon_o = 9\times10^9$ N m^2/C^2, and

A is a positive constant. (a) Find the dissociation energy of this model Li I molecule in *electron volts*. (b) Find the frequency ν of small oscillations of the Li^+ ion. (*Hint*: assume the I^- ion is fixed, because it is much heavier than the Li^+ ion.)

CP29. Diatomic oxygen (O_2) has a dissociation energy $D = 2.5$ eV and a bond length $r_0 = 0.121$ nm. Suppose we model the potential energy of interaction between the atoms as $U(r) = a/r^4 - b/r^3$, where r is the separation of the two oxygen atoms. (a) Find a and b, including units as well as numerical values. (b) Find the frequency of small vibrations about the equilibrium position r_0. (c) The vibration of this molecule produces electromagnetic radiation whose frequency is comparable to the frequency of small vibrations. In what region of the electromagnetic spectrum (radio, microwave, infrared, visible, ultraviolet, x-ray, γ-ray) is this radiation? (See the CRC Handbook, for example)

CP30. The reduced mass $\mu = m_1 m_2/(m_1+m_2)$ is the proper mass to use in calculating the frequency of oscillation of two bodies m_1 and m_2 bonded together by a spring of force-constant k_{eff}, using $\nu = (1/2\pi)\sqrt{k_{\text{eff}}/\mu}$. (a) Find the reduced mass of an HCl molecule. (b) If the oscillation frequency of the HCl molecule is $\nu = 8.7 \times 10^{13}$ Hz , find the effective force constant between the two atoms. (c) How does k_{eff} compare with force constants you would expect for common macroscopic springs?

other oscillation problems:

CP31. An "L - C circuit" is constructed by connecting an inductor (L) with a capacitor (C) by means of wires, as shown below. In general there is an electric charge $q(t)$ on the capacitor, where q obeys the equation

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0$$

and where L and C are constants. (a) Find the angular frequency ω of oscillations of the circuit, in terms of L and C. (b) If $q = q_0$ and $dq/dt = 0$ at time $t = 0$, find $q(t)$, the charge on the capacitor as a function of time.

CP32. There are thought to be three types of the particles called *neutrinos*: electron-type (ν_e), muon-type (ν_μ), and tau-type (ν_τ). If these neutrinos are all massless, they cannot spontaneously convert from one type into a different type. If there is a mass difference between two types, call the types ν_1 and ν_2, the amplitude A that a neutrino starting out as a ν_1 will have become a ν_2 after moving a distance L is given by $A = \sqrt{S_{12}}\sin^2(L/\lambda)$. The probability P they will convert is the square of the amplitude, so $P = S_{12}\sin^2(L/\lambda)$. Here S_{12} is called the *mixing strength parameter* (which we take to be a constant) and λ is a characteristic length, given in kilometers by

$$\lambda = \frac{E}{1.27\Delta(m)^2},$$

where E is the energy of the neutrino in units of GeV (10^9 eV), and $\Delta(m)^2 = m_1^2 - m_2^2$, the difference in the *squares* of the two masses, in units eV^2. Neutrinos are formed in Earth's atmosphere by the collision of cosmic-ray protons (coming from outer space) with atomic nuclei in the atmosphere. The giant detector called *Super Kamiokande*, located deep underground in a mine west of Tokyo, saw equal numbers of electron-type neutrinos coming (1) from the atmosphere above the detector and (2) from the atmosphere on the other side of Earth, which pass through the planet on their way to the detector. However, Super Kamiokande saw more muon-type neutrinos coming down from above than those coming up from below. This amounts to strong evidence that muon-type neutrinos are oscillating into tau-type neutrinos (which Super Kamiokande cannot detect) as they pass through Earth, since it requires more time to go 13,000 km through Earth than to go 20 km from the atmosphere above the mine. (a) Suppose $(\Delta m)^2 = 0.01$ eV^2 between ν_μ and ν_τ and that the neutrino energy is $E = 5$ GeV. What is λ? How would this explain the

smaller number of muon neutrinos seen from below than from above? (b) The best experimental fit is $(\Delta m)^2 = 0.0022$ eV^2. Again assuming $E = 5$ GeV, what is λ? Taking account of the fact that neutrinos from below travel by many different distances to reach the detector, about what ratio would one expect for the number of muon neutrinos from below and from above?

Section C.5 Damped oscillators

CP33. A block of mass $m = 1.34$ kg is hung from a spring of force-constant $k = 5.27$ N/m. A viscous drag force bv also acts on the block, where $b = 0.143$ kg/s. The block is pulled from its equilibrium position and released from rest. (a) How long does it take for the amplitude of oscillation to fall to 1/4 its initial value? (b) How many cycles are made by the block during this time?

CP34. A block of mass m is attached to a spring of force-constant k. The block also feels a linear damping force with damping constant $\beta = (1/2)\sqrt{k/m}$. The block is started at $t = 0$ with position $x(0) = 0$ and velocity $v(0) = v_0 > 0$. Find an expression for $x(t)$ for this oscillator in terms of ν_0 and ω_0, where $\omega_0 = \sqrt{k/m}$.

CP35. (a) Find the constants E and F in Eq. I 10 (Appendix I) in terms of the amplitude A and phase-angle ϕ_0 in Eq. I 12. (b) Find A and ϕ_0 in terms of E and F.

CP36. Hang a ball of mass $m = 0.235$ kg on a spring of force-constant $k = 1.55$ N/m; then pull the ball 21.5 cm from its equilibrium position and release it from from rest. After eight complete cycles the amplitude of the balls motion falls to 1/2 its original value. Assuming a linear drag force bv, what is the value of b?

CP37.The quality factor Q of a lightly-damped oscillator is defined to be

$$Q = 2\pi \frac{E}{|\Delta E|}$$

where E is the total energy of the oscillator and $|\Delta E|$ is the energy loss during one cycle. If the damping increases, the quality factor decreases. (a) Show that $Q \cong \pi/\beta T$, where β is the damping constant and T is the period of oscillation. *Hint*: The Taylors series expansion of e^x is $e^x = 1+x+x^2/2!+x^3/3!+...$ (b) A simple pendulum loses 1% of its energy during each cycle. What is Q in this case? (c) A mass $m = 0.50$ kg oscillates on a spring of force-constant $k = 2.0$ N/m with damping constant $\beta = 0.02$ s^{-1}. What is Q for this oscillator ?

CP38. Find (a) the potential energy, and (b) the kinetic energy of an underdamped oscillator as a function of time, in terms of its amplitude A, phase-angle ϕ_0, frequency ω_1, and other variables as needed. (c) Show that if the damping is very light, the total energy of the oscillator is approximately $E = (1/2)kA^2e^{2\beta t}$.

CP39. A critically damped oscillator is one for which $\beta = \omega_0$, so $\alpha_+ = \alpha_- = \beta$ in Eq. I 4 (Appendix I), providing only one solution to Eq. I 1. (a) Show that if $\beta = \omega_0$, then $x(t) = Bte^{\gamma t}$ is a second solution of Eq. I 1, where the constant B is arbitrary. Find the constant γ in terms of β . (b) The most general solution of Eq.A1 is $x = Ae^{\beta t} + Bte^{\gamma t}$, with the γ found in part (a). Find A and B if the mass is released from rest at $t = 0$, with initial $x(0) = x_0$. Then sketch $x(t)$ as a function of time in this case. (c) Suppose the mass m starts at some position x_0. Find the required initial velocity v_0, in terms of x_0, β, and the crossing time t_0, so that m will cross through $x = 0$ exactly once. (d) Is it possible for m to cross through $x = 0$ *more* than once, for any initial velocity? Prove that it is or that it isnt. [A critically damped oscillator approaches equilibrium faster than either an underdamped or overdamped oscillator. A pneumatic tube door-closing system should be critically damped, for example.]

CP40. A block of mass $m = 2.5$ kg is attached to a spring of force constant $k = 0.16$ N/m. The damping constant $b = 1.3$ kg/s. (a) Is the oscillator underdamped or overdamped? (b) Find the general solution of $F = ma$ in this case. *Hint*: Try $x = e^{\gamma t}$ and show that two different values of γ solve the equation. The general solution is an arbitrary linear combination of the two solutions.

CP41. Oscillating systems often show an exponential decrease in their stored average energy with time, $< E >=< E_0 > e^{-\gamma t}$. We can define a *quality factor* Q for these systems by $Q \equiv \omega_0/\gamma$, where ω_0 is the natural angular

frequency. (a) When a "middle C" is struck on a piano, with frequency 256 Hz, the energy decreases (say) to half its original value in 1.0 s. What is the Q of the string? (b) The next higher C on the piano, one octave higher, has a frequency of 512 Hz. If the energy takes about the same time to decay, what is the Q of this string? (c) What about the C below middle C, assuming its decay time is also 1.0 s?

CP42. According to classical electromagnetism, an accelerated electron radiates energy at the rate Ke^2a^2/c^3, where K is a known constant, e is the electron charge, a is the instantaneous acceleration, and c is the speed of light. (a) What is the quality factor Q of this oscillation in terms of given quantities, if, during any one cycle, the motion is essentially $x = A\sin\omega t$? (Electrons oscillate back and forth in a linear antenna, for example, so this could be taken as the quality factor of the antenna.) (b) How many complete cycles would occur by the time the energy of motion had dropped to half its initial value?

Special Topic: Resonance

CP43. In Sample Problem S1, the top end of a spring is moved sinusoidally up and down with input amplitude α. (a) In the very low frequency limit ($\omega \to 0$), what is the output amplitude, the amplitude of the sinusoidal motion of the *bottom* end of the spring? Compare the two amplitudes, and explain physically why they are related as they are. (b) Find a value of ω (with $\omega > \omega_0$) for which the output amplitude has the same magnitude as it does for $\omega = 0$.

CP44. Show that the time-average values of the following quantities over one or more complete periods of oscillation are as given. (a) $< \sin\omega t \cos\omega t >= 0$ (b) $< \sin^2\omega t >= 1/2$.

CP45. We showed that the phase-angle δ of the damped driven oscillator goes to zero in the limit of small frequencies, so if the upper end of a hanging spring is moved slowly up and down, a mass hanging on the other end also oscillates up and down in phase with it. (a) Show that as ω is increased to ω_0, the phase-angle δ becomes $\pi/2$. Describe in words what this implies about the motion of the mass if the upper end of the spring is moved up and down at frequency ω_0. (b) Show that as ω is greatly increased so that $\omega >> \omega_0$, the phase-angle δ approaches π. Describe what this implies about the motion of the mass if the upper end of the spring is moved up and down at very high frequency.

CP46. A car body supported on springs has a mass $M = 1500$ kg. When four people, each of mass $m = 75$ kg, get into the car, the car body sags a distance $d = 1.7$ cm. The car is then driven at speed $v = 35$ km/hr over a road with equally-spaced bumps. How far apart must be the bumps in the road, so that the frequency of the cars natural oscillations match the frequency of jolts felt by the car? Assume the shock absorbers don't work, so the damping is small and the amplitude will be large.

CP47. The quality factor Q defined in problem CP41 also describes the sharpness of the resonance curve of a driven damped oscillator. Higher Q corresponds to sharper (higher, narrower) resonances, i.e. to oscillators with smaller damping. For lightly-damped oscillators, show that to a good approximation $Q = \omega/\Delta\omega$, where $\Delta\omega$ is the difference in angular frequencies between the two locations on the average power resonance curve (See Sample Problem 4(b)) for which the power delivered is one-half of the power delivered at ω_0. Thus the narrower the curve, the larger the quality factor of the oscillator. (*Optional*: Show also that $Q = \omega_0/\Delta\omega$ for a very lightly-damped oscillator if $\Delta\omega$ is the angular frequency difference between the two locations on the amplitude resonance curve for which the amplitude is $1/\sqrt{2}$ that at resonance.)

CP48. In the second problem of *Self Quiz 3*, one finds the maximum average power input to a driven, damped oscillator, and the angular frequency ω at which the maximum power input occurs. It is also important to know how *broad* the resonance curve is. One measure of broadness is the *full width at half maximum* (FWHM). Find the difference between the two angular frequencies at which the power resonance curve is half the height it has at maximum.

CP49. A 2.0 g spider dangles at the end of a silk thread. You can make the spider bounce up and down by tapping lightly on it with a pencil. You give the spider the largest amplitude if you tap exactly once every second. What is the spring constant of the silk thread?

D. Waves on a Rope

"When you get to the end of your rope, tie a knot and hang on." - Anonymous

Two people hold opposite ends of a long clothesline. The one at the right holds the rope in place, while the one at the left shakes the rope up and down. The shaking produces a wave, a disturbance that travels down the length of rope. Various wave shapes can be made this way. A single quick up-and-down motion causes a "kink" to form, which keeps its shape as it propagates

Fig. 1. Waves on a rope (a) a kink (b) a sinusoidal wave, each at three different times.

along the rope, as in Fig. 1(a). Smooth, repeated sinusoidal up-and-down shaking causes a sinusoidal wave that propagates along the rope, as shown in Fig. 1(b). It is the *shape* that propagates, not the rope itself. Individual bits of the rope move up and down as the wave passes*, but they do not propagate from one end of the rope to the other.

Waves are similarly set up when a guitar string is plucked, a violin string is bowed, or a piano string is struck. The waves created on these strings determine the pitch and quality of the musical sound produced.

*Actually, individual bits move purely up and down only in the small-amplitude approximation. If the rope is shaken with large amplitudes, bits move back and forth somewhat along the rope as well as up and down.

D.1 Traveling Sinusoidal Waves

Suppose we generate a propagating sinusoidal wave, as shown in Fig. 1(b). Let x be the horizontal coordinate, measured from the left end of the rope, and $y(x)$ be the vertical displacement as a function of x. The displacement $y(x)$ is the "shape" of the rope at a particular time. At the time shown in the final picture of Fig. 1(b), for example, the shape is

$$y(x) = A \sin(\frac{2\pi}{\lambda}x) \tag{1a}$$

in terms of two constants, A and λ, as illustrated in Fig. 2. Here A is the *amplitude* of the wave, which is the maximum displacement from $y = 0$, since the sine function by itself ranges only from -1 to $+1$. The amplitude A has the dimensions of length. The constant λ is the *wavelength* of the wave, the horizontal distance corresponding to one complete cycle. For example, λ is the horizontal distance between two successive crests of the wave. Why is the argument of the sine function equal to $\frac{2\pi}{\lambda}x$? It is because in going (say) from $x = 0$ to $x = \lambda$, the quantity $\frac{2\pi}{\lambda}x$ changes from zero to 2π, so the sine function goes through one complete cycle.

Fig. 2. A snapshot of a sine wave on a rope. The wavelength is the distance between successive crests.

To make the notation more compact, we often write

$$y(x) = A \sin kx\,, \tag{1b}$$

where the constant

$$k = \frac{2\pi}{\lambda} \tag{2}$$

is called the *wave number*; k obviously has dimensions of inverse length. If the wavelength is increased, the wave number is decreased, and vice versa.

Note that the shape of the rope is $y(x) = A\sin(\frac{2\pi}{\lambda}x)$ at some *particular* time, but it cannot be the shape at *all* times, because this $y(x)$ is always zero at $x = 0$. Actually $y(x = 0)$ changes with time as we shake the rope up and down. Just prior to the final picture in Fig. 1(b), for example, our hand has generated a *crest* in the wave by being at $y > 0$; later our hand has moved down so that $y(0) = 0$ in the final picture shown; then in a subsequent picture our hand will have moved to $y < 0$ in order to generate a valley in the wave.

At $x = 0$, the position of our hand, the height of the sinusoidal wave must be

$$y(t) = A\sin\left(-\frac{2\pi}{T}t\right) \tag{3a}$$

where T is the time it takes for us to shake the rope once up and down; here T is the *period* of the oscillation. The minus sign is needed so that $y(t)$ will be greater than zero just before $t = 0$, and less than zero just after $t = 0$, as discussed above. The 2π is needed so that the sine function will oscillate once completely as the time advances from $t = 0$ to $t = T$.

The frequency of oscillation is $\nu = 1/T$, and we recall from simple harmonic oscillation that the angular frequency, measured in radians/second, is $\omega = 2\pi\nu = 2\pi/T$. The height of the rope at $x = 0$ may therefore be written more simply as

$$y(t) = A\sin(-\omega t). \tag{3b}$$

Now if $y(x) = A\sin kx$ is the height of the rope for all x at the particular time $t = 0$, and $y(t) = A\sin(-\omega t)$ is the height of the rope for all times at the particular position $x = 0$, what is the height of the rope at *all* positions and *all* times?

The answer is

$$y(x, t) = A\sin(kx - \omega t). \tag{4}$$

Why is this expression correct? First note that Eq. 4 reduces to the correct functions $y(x) = A\sin kx$ if $t = 0$ and $y(t) = A\sin(-\omega t)$ if $x = 0$. Furthermore, it keeps the same sinusoidal shape for all x at any later time, except that the shape continually moves to the right, in agreement with what we observe when we shake the rope in this way. Finally, as we will show later, Eq. 4 is the solution of the $F = ma$ equation for the rope that is consistent with the conditions given in Eqs. 1b and 3b.

How do we know that the wave expressed by Eq. 4 moves to the *right*? Note that if the quantity $kx - \omega t$ (which is the "argument" of the sine function) remains constant, then the height y of the rope remains constant as well. That is, as $-\omega t$ gets more and more negative, kx must get more and more positive to make up for it. As time increases, the quantity $kx - \omega t$ can clearly only remain constant if x increases as well. Consider a particular point x_0 on the rope at time $t = 0$, for example. At that point the rope has height $y = A \sin kx_0$. Slightly later, at time Δt, which nearby point on the rope will have this same height? It will be the point on the rope located at position $x = x_0 + \Delta x$, where

$$k(x_0 + \Delta x) - \omega \Delta t = kx_0 - \omega \cdot 0 \tag{5}$$

chosen to keep the quantity $kx - \omega t$ constant. Therefore the point at the same height has moved a distance

$$\Delta x = \frac{\omega}{k} \Delta t \tag{6}$$

to the right, as illustrated in Fig.3. This is true for every point along the rope, which allows us to say, using $\Delta x = v\,\Delta t$, that the wave form moves to the right along the rope at the constant velocity

$$v = \frac{\omega}{k} = \frac{2\pi\nu}{2\pi/\lambda} = \lambda\,\nu. \tag{7}$$

The wave velocity is the product of the wavelength and the frequency of the wave.

Fig. 3. The wave $y(x, t) = A \sin(kx - \omega t)$ propagates to the right at speed $v = \dfrac{\omega}{k} = \lambda\,\nu$.

Sample Problem 1. A sinusoidal wave on a string is observed to have wavelength $\lambda = 3.6$ m and to move with velocity $v = 1.2$ m/s to the right. Find its oscillation frequency ν and period T, its wave number k and angular velocity ω.

Solution: From Eq. 7, the frequency is $\nu = v/\lambda = \dfrac{1.2\ m/s}{3.6\ m} = 0.33$ Hz, so the period is $T = 1/\nu = 3.0$ s. The wave number $k = 2\pi/\lambda = 1.7\ \text{m}^{-1}$, and the angular velocity is $\omega = 2\pi\nu = kv = 2.1\ \text{s}^{-1}$.

Reasoning in a way similar to that for rightward-moving waves, it is clear that the alternative sinusoidal wave form

$$y(x, t) = A \sin(kx + \omega t) \tag{8}$$

with a plus sign instead of a minus sign, represents a wave traveling to the *left*, since as time increases x must *decrease* to keep the quantity $kx + \omega t$ constant. Although the propagation direction is different, the speed of the wave is still $v = \dfrac{\omega}{k} = \dfrac{2\pi\nu}{2\pi/\lambda} = \lambda\nu$. We could create such a wave by shaking the *right*-hand end of the rope up and down, so the wave would move from the right toward the left as shown in Fig. 4.

Fig. 4. The wave $y(x, t) = A \sin(kx + \omega t)$ propagates to the left at speed $v = \dfrac{\omega}{k} = \lambda\nu$.

Equations 4 and 8 represent only special cases of sinusoidal traveling waves. In both cases $y(x = 0, y = 0) = 0$, so that the equations can only represent the wave if the left-hand end of the rope (at $x = 0$) happens to be initially undisplaced, with $y = 0$ at $t = 0$. More generally, the waves could have a different *phase*. For example, we might shake the left-hand end so that y is not initially zero, but has some different value. A more general wave moving toward the right would have the form

$$y(x, t) = A\sin(kx - \omega t - \phi)\,, \tag{9}$$

where a phase angle ϕ has been included within the sine function. Now $y(0, 0) = A\sin(-\phi) = -A\sin\phi$, so by a proper choice of ϕ we can let $y(0, 0)$ take on any value between $-A$ and $+A$. Fig. 5 shows the shape of two waves at time $t = 0$. Fig. 5(a) shows a wave with $\phi = 0$, and Fig. 5(b) shows a wave with $\phi = \pi/4$. The sine shape has been shifted to the right in Fig. 5(b), so that at $x = 0$, $y(x) = A\sin(-\pi/4) = -A/\sqrt{2}$.

Fig. 5. (a) A sinusoidal wave at time $t = 0$ with phase angle $\phi = 0$. (b) A similar wave with $\phi = \pi/4$.

Similarly, the most general sinusoidal wave moving to the left is

$$y(x, t) = A\sin(kx + \omega t - \phi). \tag{10}$$

The only effect of ϕ is to shift the wave form to the right or to the left relative to the wave with $\phi = 0$. The added phase angle ϕ does not affect the speed of the wave, which is still $v = \dfrac{\omega}{k} = \dfrac{2\pi\nu}{2\pi/\lambda} = \lambda\nu$.

When we shake a rope sinusoidally up and down, we can determine the frequency ν of the wave just by how quickly we wiggle our hand. The resulting wave will then automatically take on the wavelength $\lambda = v/\nu$, where v is the wave speed. But what determines the speed of a wave on a rope? What physical parameters influence how fast a wave moves? Obviously, the wave speed v must depend upon properties of the rope itself and, perhaps, external influences as well. What are these properties or influences?

A wave in one part of the rope can't "tell" the total length L of the rope, and it also can't "tell" the total mass M of the rope either. But it *can* tell the local mass *density*, the mass per unit length of the rope, which we denote by $\mu = M/L$. The wave may also depend upon the local tension F in the rope.* There are no other relevant parameters in the problem, so v must depend somehow on μ and F.

Dimensional reasoning can tell us how. In terms of the fundamental dimensions of mass M, length L, and time T, the relevant variables have the dimensions

$$[v] = \frac{\mathrm{L}}{\mathrm{T}}, \qquad [\mu] = \frac{\mathrm{M}}{\mathrm{L}}, \qquad [F] = \frac{\mathrm{ML}}{\mathrm{T}^2}. \tag{11}$$

The only way to combine μ and F to get rid of the unneeded dimension M is to divide the two; the dimensions of the ratio are then $\left[\frac{F}{\mu}\right] = \frac{\mathrm{L}^2}{\mathrm{T}^2}$. Therefore the speed v of waves on a rope must be proportional to the square root of the ratio:

$$v \propto \sqrt{\frac{F}{\mu}}. \tag{12}$$

If the tension is doubled, the speed goes up by the factor $\sqrt{2}$; if the mass density is doubled, the speed goes down by the factor $1/\sqrt{2}$.

To find the dimensionless constant of proportionality we have to use more than dimensional reasoning. That is the subject of the next section.

*We will denote tension by "F" instead of "T", so we can use the symbol "T" for the period of sinusoidal waves.

D.2 The Wave Equation

Why do waves propagate? Why do they retain their shape? Suppose the rope has uniform mass per unit length $\mu = M/L$. Start by stretching the rope so that it also has uniform tension F throughout. Assume the effects of gravity are negligible, so the rope is essentially horizontal. What happens if we shake the left end of the rope, introducing a slight transverse displacement of the rope from its normal horizontal position, as shown in Fig. 6(a)?

At time $t = 0$ the rope has the shape $y(x)$, where x is the horizontal direction along the rope and y is the vertical displacement. Can we predict $y(x, t)$, the shape of the rope at later times?

Fig. 6. (a) A transverse (i.e., sideways) displacement of the rope. (b) A small piece of length Δx and mass $\Delta m = \mu \Delta x$.

We can use $F_{\text{net}} = ma$ to study a small piece of the rope, of length Δx and mass $\mu \Delta x$, shown in Fig. 6(b). The small piece moves up and down, transverse (i.e., sideways) to the length of the rope. Its transverse displacement is $y(x, t)$, so its acceleration is the second derivative of $y(x, t)$ with respect to time,

$$a = \frac{\partial^2 y(x,t)}{\partial t^2} . \tag{13}$$

We have to use the *partial* derivative with respect to time, because we are using $F_{\text{net}} = ma$ for a particular piece of the rope, which has fixed position x.

The mass of the piece of rope is $\Delta m = \mu \Delta x$, so its mass-times-acceleration is

$$(\Delta m)a = \mu \Delta x \frac{\partial^2 y(x,t)}{\partial t^2} . \tag{14}$$

The net force F_{net} on the small piece of rope Δm can only be caused by the rope tensions immediately to the left and to the right of the piece (we will neglect gravity here). The net force turns out to be proportional to the tension F in the rope, and also to the curvature of the rope at the point, i.e. to the second derivative of $y(x, t)$ with respect to position x.* If the rope is straight near some particular piece or rope there is no net force on it there; the tensions to the right and left cancel out. Appendix D I shows in detail how to find F_{net}. The result is

$$F_{\text{net}} = F \frac{\partial^2 y}{\partial x^2} \Delta x \ . \tag{15}$$

Equating Eqs. 14 and 15 gives the $F = ma$ equation, the so-called *wave equation* for a string,

$$\frac{\partial^2 y}{\partial x^2} = \left(\frac{\mu}{F}\right) \frac{\partial^2 y}{\partial t^2} \ . \tag{16}$$

The wave equation involves partial derivatives of the shape $y(x, t)$ with respect to x and with respect to t. It is valid for small-amplitude waves on a string or rope with tension F and uniform mass/length μ. As a wave passes by, the piece of string moves up and down perpendicular to the propagation direction of the wave. It moves in response to the net force acting on it due to the string attached to its left and to its right. *Any smooth small-amplitude wave form $y(x, t)$ which can exist on the string has to be a solution of the wave equation.*

* If the rope had no curvature the tension acting to the right would equal the tension acting to the left, so there would be no net force on the piece of rope.

Sample Problem 2. Show that the sinusoidal traveling wave $y(x, t) = A\sin(kx - \omega t - \phi)$ is a solution of the wave equation.

Solution: The partial derivative of $y(x, t)$ with respect to x is $\frac{\partial y(x,t)}{\partial x} = kA\cos(kx - \omega t - \phi)$, and the second partial derivative is $\frac{\partial^2 y(x,t)}{\partial x^2} = -k^2 A\sin(kx - \omega t - \phi)$. The partial derivative of $y(x, t)$ with respect to t is $\frac{\partial y(x,t)}{\partial t} = -\omega A\cos(kx - \omega t - \phi)$, and the second partial

derivative is $\dfrac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 A \sin(kx - \omega t - \phi)$. Substituting the second derivatives into the wave equation (17), is it true that

$$-k^2 [A \sin(kx - \omega t - \phi)] = \left(\frac{\mu}{F}\right)[-\omega^2 A \sin(kx - \omega t - \phi)] \ ?$$

That is, if we cancel the common factor $A \sin(kx - \omega t - \phi)$, is it true that $k^2 = (\mu/F)\omega^2$? We recognize from Eq. 7 that the ratio $\omega/k = v$, the velocity of the wave. So sinusoidal waves *are* solutions of the wave equation *if* the wave speed in the rope is related to F and μ by the equation $v = \sqrt{\dfrac{F}{\mu}}$. (This is exactly the result we had found from dimensional reasoning alone!)

Sample Problem 3. One end of a horizontal rope of mass-density μ is attached to a weight W and hung over a pulley as shown. The other end is shaken up and down sinusoidally at frequency ν, generating rightward-traveling waves on the string. Find the wave's (a) velocity (b) wavelength.

Solution: (a) the wave velocity is $v = \sqrt{\dfrac{F}{\mu}} = \sqrt{\dfrac{W}{\mu}}$, using the result of *Sample Problem 2*, and (b) the wavelength is $\lambda = v / \nu = \dfrac{\sqrt{W/\mu}}{\nu}$.

When we create a kink in a taut string, we observe that the kink propagates down the string at uniform speed, as illustrated in Fig. 1a. Suppose the kink has some shape $f(x)$ at time $t = 0$, as shown at the right, and that the propagation velocity v is to the right. What is the shape at future times $t > 0$, according to the wave equation? And how does the propagation velocity depend upon the rope and the tension in it?

Using the wave equation (17), it is shown in Appendix D II that

$$y(x, t) = f(x - vt) , \tag{17}$$

where $f(x - vt)$ is any smooth function of $x - vt$. (The sine wave $y(x, t) = A\sin k(x - vt)$ is a special case of this sort of function). The wave $y(x, t)$ doesn't change its shape, but is simply shifted along the rope as time goes on. As shown in Appendix D II, the velocity of the wave is related to the tension F and mass/length μ by

$$v = \sqrt{\frac{F}{\mu}} \tag{18}$$

where F and μ are the string's tension and its mass per unit length. The velocity of waves of any shape travel at the same speed as sinusoidal waves.

Sample Problem 4. At time $t = 0$, the shape $y(x, t = 0) = A\ e^{-a(x - x_0)^2}$ is established on a long string of length ℓ and mass m, which is under tension F, by a person who has shaken the left end of the rope. Here A, a, and x_0 are constants, with $x_0 << \ell$. (a) Sketch the wave form $y(x, 0)$ (b) Find the wave $y(x, t)$ at later times.

Solution: (a) The shape is the Gaussian function shown, a "bell-shaped" curve. The peak of the distribution is located at $x = x_0$. The height of the peak is A, and the constant a controls the width of the peak, such that a larger a gives a narrower peak.

(b) To find the shape at later times, we simply substitute $x - vt$ for x in the initial distribution. The wave is therefore

$$y(x,t) = A\, e^{-a\,(x - vt - x_0)^2}$$

Note in particular that the peak of the wave is located at a value of x which makes the exponent vanish, i.e. at $x = x_0 + vt$; the peak (and each point on the wave) moves to the right at speed $v = \sqrt{F/\mu} = \sqrt{F\ell/m}$.

One solution of the wave equation for a string is $y(x, t) = f(x - vt)$, where $f(x - vt)$ is any smooth function of $x - vt$; this solution represents a wave traveling to the right. There are of course also solutions corresponding to waves traveling to the *left*. In fact, if g is any smooth function, then

$$y(x, t) = g(x + vt) \tag{19}$$

is a solution of the wave Eq. 16 representing a wave traveling to the left (See Problem DP17). The velocity of left-moving waves is $v = \sqrt{\dfrac{F}{\mu}}$, the same as for right-moving waves, as can be verified by substituting the form $g(x + vt)$ into Eq. 16.

The most general solution of the wave equation is a linear combination of right-moving and left-moving waves. In fact, this most general solution is

$$y(x, t) = f(x - vt) + g(x + vt) \tag{20}$$

where $f(x - vt)$ is an arbitrary smooth function of $x - vt$, corresponding to a wave traveling to the right, and $g(x + vt)$ is an arbitrary smooth function of $x + vt$, corresponding to a wave traveling to the left. It is easy to introduce both right-moving and left-moving waves on a string, just by simultaneously shaking both ends of the string. Each wave has a distinct identity, so the two waves actually pass right through one another, as illustrated in Fig. 7. This addition of waves is an illustration of the *principle of superposition*, the principle that if two wave forms are separately possible, the sum of them is a possible wave form as well.*

*This summation feature is a property of differential equations (like the wave equation) that are said to be *linear*.

Fig. 7. A right-moving and a left-moving wave pass by each other; neither wave brings about a distortion of the other, as long as the amplitudes are small enough so that the wave equation is valid.

The wave equation $\frac{\partial^2 y}{\partial x^2} = \left(\frac{\mu}{F}\right)\frac{\partial^2 y}{\partial t^2}$ for the shape $y(x, t)$ of a stretched rope is only one of many wave equations, which govern a wide range of physical systems. A two-dimensional generalization of the wave equation gives the behavior of waves on a kettledrum, for example, and a three-dimensional generalization governs waves in solids, such as seismic waves within Earth.

There is a wave equation for pressure waves in liquids and gases, describing the propagation of sound, as we will see in the following chapter. There is a wave equation for waves on the surface of the ocean, which can be used to help understand the behavior of wind-driven waves and the earthquake-caused waves called Tsunamis.

There is a wave equation for electromagnetic fields, which is the propagation equation for light waves. There is also a wave equation for gravity, predicting the existence of waves of gravity, which have not yet been observed directly, but which should be observed within the next five to ten years.

There are even wave equations for quantum-mechanical particles, whose solutions agree very well with the observed behavior of electrons in atoms and molecules, for example, as well as the constituents of atomic nuclei and the motion of fundamental particles. All of these equations have solutions that have much in common with those we have already found for waves on a rope.

D.3 Standing Waves

What happens if there are *two sinusoidal* waves on the same rope, one traveling to the right and the other to the left? If we generate the rightward wave

$$y_R(x\,,\,t) = A\,\sin(kx - \omega t) \tag{21a}$$

and the leftward wave

$$y_L(x\,,\,t) = A\,\sin(kx + \omega t) \tag{21b}$$

on the same string, then the sum

$$y(x\,,\,t) = y_R(x\,,\,t) + y_L(x\,,\,t) \tag{22}$$

is also a possible wave. Note that each of the waves 21(a) and 21(b) has the same amplitude A, wave number k, and angular frequency ω, so each also has the same wavelength λ and frequency ν. The only difference between them is that the first travels to the right and the second travels to the left. It is easy to make such a combined wave: If we shake the left end of a taut rope sinusoidally, a rightward wave will be formed. When the wave hits the right end of the rope, which is simply fixed in place, the wave bounces back to form a leftward-moving wave. So if we continue to shake the left end of the rope leaving the right end fixed, we automatically generate a leftward wave as well as a rightward one.

What does the combination wave look like? From the trig identity

$$\sin(\,\alpha \pm \beta\,) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \tag{23}$$

it follows that

$$\begin{aligned} y(x\,,\,t) &= A[\,\sin(kx - \omega t) + \sin(kx + \omega t)\,] \\ &= A[\,(\sin kx\cos\omega t - \cos kx\sin\omega t\,) + (\,\sin kx\cos\omega t + \cos kx\sin\omega t\,)\,] \\ &= 2A\sin kx\cos\omega t = (\,2A\cos\omega t\,)\sin kx\ . \end{aligned} \tag{24}$$

The combination wave, a superposition of rightward and leftward waves, is a *product* of a function of time, ($2A\cos\omega t$), and a function of position, $\sin kx$. Therefore the locations at which $y(x, t)$ is zero are always at the same place! In particular, $y(x, t)$ must always be zero at an end where the rope is held in place.

In fact, note that $y(x, t) = 0$ at $kx = n\pi$ ($n = 0, 1, 2, \ldots$) for all t. So the *node* positions of the wave (the locations where the displacement $y = 0$) remain always at

$$x = \frac{n\pi}{k} = n\frac{\lambda}{2} \qquad (n = 0, 1, 2, \ldots.) \tag{25}$$

since $k = 2\pi/\lambda$. There is a node at $x = 0$, and others spaced a half-wavelength apart, as illustrated in Fig. 8. These waves don't travel; they stay put. They are therefore called *standing waves*.

Fig. 8. A standing wave, showing nodes and antinodes. The nodes, where the displacement is zero, remain in the same locations. At the nodes the rope never moves. The antinode displacements vary with time. At the antinodes the rope moves up and down a maximum amount.

As shown in Fig. 8, the nodes remain at the same places, at the zeros of the function $\sin kx$. The *antinodes* are halfway between the nodes, at the locations of maximum displacement, where $\sin kx = \pm 1$. The maximum displacement actually varies with time, between $2A$ and $-2A$, because $\sin kx$ is multiplied by $2A\cos\omega t$ in Eq. 24. At those times t when $\omega t = 0, 2\pi, 4\pi, 6\pi$, (i.e., at $t = 0, 2\pi/\omega, 4\pi/\omega$, etc.), the standing wave is as shown in Fig. 9(a). Shortly after, when $\omega t = \pi/2, 3\pi/2, 5\pi/2, \ldots$, then $\cos\omega t = 0$, so the rope is straight everywhere, as shown in Fig. 9(b). Still later, when $\omega t = \pi, 3\pi, 5\pi, \ldots$, the rope is again maximally displaced, but in the opposite direction to its displacement at $t = 0$, as shown in

Fig. 9(c). In other words, the antinodes of the string vibrate up and down harmonically in time with angular frequency ω, between $y = \pm 2A$.

Fig. 9(a) The standing wave $y(x, t) = 2A\cos\omega t \sin kx$ at time (a) 0 ; (b) $\pi/2\omega$; (c) π/ω. The dotted lines represent the position of the string when there is no wave on it.

Now suppose that standing waves are excited on a string of length L, where both ends of the string are fixed so that the displacement is zero. That is, there is a node at both ends of the string, at $x = 0$ and at $x = L$. Then it is clear that only certain wavelengths can exist on the string. In fact, there can only be an integer number of half-wavelengths along the string; i.e., $n\dfrac{\lambda}{2} = L$, so that the possible wavelengths are

$$\lambda_n = \frac{2L}{n} \qquad (n = 1, 2, 3, \ldots\ldots) \ . \tag{26}$$

The corresponding oscillation frequencies are

$$\nu_n = v/\lambda_n = \frac{n}{2L}\sqrt{\frac{F}{\mu}} \ . \tag{27}$$

If $n = 1$, there is only a half-wavelength on the entire string, as shown in Fig. 10(a), and the frequency is

$$\nu_1 = v/\lambda_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}} \ . \tag{28}$$

Fig. 10. The lowest modes of oscillation on a string: (a) the $n = 1$ mode; (b) the $n = 2$ mode .

For $n = 2$, there is a full wavelength on the string, as shown in Fig. 10(b), and the frequency is

$$\nu_2 = v / \lambda_2 = \frac{2}{2L}\sqrt{\frac{F}{\mu}}, \tag{29}$$

twice as large as ν_1 . Each of these is called a *mode* of oscillation. The lowest mode is the $n = 1$ mode, for example; it has the lowest oscillation frequency. There are a countable infinity of such modes (on an idealized continuous string of zero thickness), one for each value of n. As n increases the wavelength gets smaller and the frequency rises.

A guitar player can excite the lowest mode of oscillation on a guitar string; the corresponding frequency is called the *fundamental frequency*. It is the primary frequency we normally hear when the string is plucked. It is also possible to excite the second ($n = 2$) mode by plucking the string while holding down the center of the string, forcing a node at the center. The doubled frequency is heard as one octave higher than the fundamental frequency. For example, if the fundamental frequency for a string is "A" (440 Hz), then excitation of the second mode will give the next higher "A" (880 Hz). The tone of a plucked string is actually a combination of many frequencies, including the fundamental, which is normally most pronounced, along with a number of overtones, each of which is one of the higher-frequency modes of oscillation of the string.

D.4. Standing waves formed by reflection

Tie the right-hand end of a rope to a hook screwed into a wall. Then shake the left-hand end once up and down to form a wave pulse, as shown. The pulse travels to the right at the wave velocity. What happens when it reaches the hook?

When the pulse reaches the hook it pulls *up* on the hook; consequently (according to Newton's third law) the hook pulls *down* on the rope. In pulling down on the rope, the hook creates another pulse, traveling from right to left and inverted from the incoming wave. Fig. 11 shows, in the left-hand column, the incoming pulse traveling from left to right. The upper pictures are at earlier times and the lower pictures are at later times. The second column shows the reflected pulse at the same times. The third column is the sum of the two pulses, which is what the eye sees in looking at the rope. Note that at large times there is only a leftward moving pulse. This pulse has been reflected and inverted from the incoming pulse.

Fig 11. Incoming (left column) and reflected (middle column) pulses. The sum of the pulses is shown at the right. The time intervals between successive pictures are not all the same.

What happens if we generate a sinusoidal wave at the left-hand end, rather than a single pulse? Note that the pictures shown in Fig. 11 already represent something like a half-cycle of a sinusoidal wave. Also, note that the pictures in the right column look exactly like a portion of a standing wave at successive times. This is consistent with our demonstration in Section D.3 that a standing wave can be built from a wave traveling to the right together with a wave traveling to the left. In the case of a rope tied to a hook screwed into the right-hand wall, the wave generated at the left end forms the wave traveling to the right, while the reflected wave generated automatically by the hook forms the wave traveling to the left. Therefore we can generate a standing wave just by shaking one end of the rope while the other end is kept fixed.

At least that is true if we shake the left end with one of a special set of frequencies so that when the original and reflected waves are added together at our hand, the sum matches the actual position of our hand. Since our hand is moving up and down, the wave would actually look as shown in Fig.12. In this case there are not exactly an integer number of peaks and valleys in the wave, because our hand is moving up and down a little so it is not located at a node in the standing

Fig. 12. An approximate standing wave formed by shaking the left end of a rope while keeping the right end fixed.

wave pattern. If we *stopped* moving our hand up and down (to make that point a node), friction would quickly make the wave die out. To generate something that looks like a standing wave for an appreciable period of time, we have to keep shaking the rope up and down to add energy to the wave to make up for the energy lost to air resistance and frictional heating within the rope itself . And we have to shake with one of a discrete set of frequencies so that an approximate standing wave is formed, with a pattern that repeats itself time after time. If we shake the rope with some other frequency the result is a messy-looking wave that does not repeat itself. The situation is reminiscent of the resonance formed in a spring-mass system if we push on it with its natural frequency. In that case the amplitude builds up. If we push with a different frequency the amplitude is much lower and the response appears to be erratic.

Appendix D I: Derivation of the wave equation for a rope

A horizontal rope is under tension F throughout; assume the effects of gravity are negligible. The rope has uniform mass per unit length given by $\mu = M/L$, where M and L are the mass and length of the rope. What happens if we introduce a slight transverse displacement of the rope from its normal horizontal position, as shown in Fig. I 1? That is, suppose at time $t = 0$ we give the rope the shape $y(x)$, where x is the horizontal direction along the rope and y is the vertical displacement. Our goal is to find $y(x, t)$, the shape of the rope at later times.

Fig. I 1. A slight displacement $y(x)$ is made in the rope at time $t = 0$.

Consider a very small piece of the rope of length Δx, as shown in Fig. I 2(a). This piece has mass $\Delta m = \mu \Delta x$. Newton's second law $F = ma$ tells us that Δm will accelerate if there is a net force on it. In the absence of gravity, the only forces acting on Δm are due to the rope tension to the right and to the left on this piece of rope. That is, the molecules in the rope

Fig. I 2(a) A small piece of a displaced rope, of length Δx and mass $\Delta m = \mu \Delta x$. (b) The forces on this piece are the rope-tension forces due to the rope to the right of this piece, and the rope to the left of this piece.

which are immediately to the right and to the left of the piece exert forces on it. If the rope is displaced from equilibrium, the piece Δm will generally be slightly curved, as shown in Fig. I 2(b). The two tension forces on the right and left therefore pull in slightly different directions, as shown. The resulting unbalanced force causes Δm to accelerate and the wave to propagate.

For small transverse displacements, the horizontal component of the tension F remains essentially constant along the rope. The vertical component of tension is therefore $F_y = F \tan\theta$, where θ is the angle of the rope at some point relative to the horizontal, as shown in Fig. I 3. The tangent of this angle, the slope of the rope, can also be written as the *partial derivative* $\dfrac{\partial y(x,\, t)}{\partial x}$. We have to use the partial derivative, because whereas y

Fig. I 3. If F is the horizontal component of the tension, $F\tan\theta$ is the vertical component.

depends upon both the position x along the rope and the time t, the slope at a fixed time t is the derivative of y with respect to x alone. That is what the partial derivative $\dfrac{\partial y(x,\, t)}{\partial t}$ *means*; one differentiates $y(x, t)$ with respect to x, treating t as a constant.

Let the left-hand end of the piece Δm be located at x_0, and the right-hand end at $x_0 + \Delta x$. According to Taylor's series, the slopes of the rope at the left and right are related by

$$\left.\frac{\partial y}{\partial x}\right|_{x_0 + \Delta x} = \left.\frac{\partial y}{\partial x}\right|_{x_0} + \left.\frac{\partial^2 y}{\partial x^2}\right|_{x_0} \Delta x + \frac{1}{2!}\left.\frac{\partial^3 y}{\partial x^3}\right|_{x_0} \Delta x^2 + \cdots \qquad \text{(I 1)}$$

If $y(x)$ is smooth and Δx is sufficiently small, we can neglect all but the first two terms on the right of Eq. I 1. From Fig. I 3 it is clear that there is a positive vertical force $F \tan\theta = F \left.\dfrac{\partial y}{\partial x}\right|_{x_0 + \Delta x}$ on the right-hand end of Δm, and a negative vertical force $-F \left.\dfrac{\partial y}{\partial x}\right|_{x_0}$ on the left-hand end. The *net* vertical force on Δm is therefore

$$F\left[\left.\frac{\partial y}{\partial x}\right|_{x_0+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x_0}\right] \cong F\left.\frac{\partial^2 y}{\partial x^2}\right|_{x_0}\Delta x \tag{I 2}$$

using Eq. I 1.

The acceleration of Δm is $\frac{\partial^2 y}{\partial t^2}$, the second derivative of y with respect to time, keeping the position x fixed. Now the derivatives are with respect to time rather than position. The left end of the piece of rope moves up and down vertically, with position $y(x_0, t)$, velocity $\left.\frac{\partial y}{\partial t}\right|_{x_0}$, and acceleration $\left.\frac{\partial^2 y}{\partial t^2}\right|_{x_0}$. Newton's second law $F_{\text{net}} = ma$ for the piece of rope is

$$F\left.\frac{\partial^2 y}{\partial x^2}\right|_{x_0}\Delta x = \Delta m\left.\frac{\partial^2 y}{\partial t^2}\right|_{x_0} = \mu\,\Delta x\left.\frac{\partial^2 y}{\partial t^2}\right|_{x_0} \tag{I 3}$$

so

$$\frac{\partial^2 y}{\partial x^2} = \left(\frac{\mu}{F}\right)\frac{\partial^2 y}{\partial t^2}\ . \tag{I 4}$$

Eq. I 4 is the wave equation of the rope for small transverse displacements.

Appendix D II:

Proof that $y(x, t) = f(x - vt)$ is a solution of the wave equation for a rope

We want to show that

$$\frac{\partial^2 f(x - v\,t)}{\partial x^2} - \left(\frac{\mu}{F}\right)\frac{\partial^2 f(x - v\,t)}{\partial t^2} = 0\,. \qquad \text{(II 1)}$$

Define an independent variable u by $u = x - \mathrm{v}\,t$, so that $f(x - \mathrm{v}\,t) = f(u)$. Using the chain rule, the partial derivative of $f(u)$ with respect to x is

$$\frac{\partial f(u)}{\partial x} = \frac{df(u)}{du}\frac{\partial u}{\partial x} = \frac{df(u)}{du}\,, \qquad \text{(II 2)}$$

because $\dfrac{\partial u}{\partial x} = \dfrac{\partial (x - vt)}{\partial x} = 1$. That is, v is a constant, and the partial derivative with respect to x treats t as a constant as well. In Eq. II 2 the partial derivative of $f(u)$ with respect to u has been written as the total derivative, because $f(u)$ depends *only* upon u, so the total derivative means the same as the partial derivative in this case. Taking a second derivative, we find in a similar way that

$$\frac{\partial^2 f(u)}{\partial x^2} = \frac{d^2 f(u)}{d^2 u}\,. \qquad \text{(II 3)}$$

The first partial derivative of $f(u)$ with respect to time is

$$\frac{\partial f(u)}{\partial t} = \frac{\partial u}{\partial t}\left(\frac{df(u)}{du}\right) = -\,v\,\frac{df(u)}{du}\,, \qquad \text{(II 4)}$$

using the chain rule and the fact that

$$\frac{\partial u}{\partial t} = \frac{\partial (x - v\,t)}{\partial t} = -\,v \qquad \text{(II 5)}$$

Similarly, the second derivative is

$$\frac{\partial^2 f(u)}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial f(u)}{\partial t}\right) = \frac{\partial}{\partial t}\left(-v\frac{df(u)}{du}\right) = -v\frac{\partial u}{\partial t}\frac{\partial}{\partial u}\left(\frac{df(u)}{du}\right) = v^2\frac{d^2 f(u)}{du^2} \qquad \text{(II 6)}$$

using the chain rule again. The function $f(u) = f(x - v\,t)$ is therefore a solution of the wave equation if

$$\frac{d^2 f(u)}{du^2} - \left(\frac{\mu}{F}\right) v^2 \frac{d^2 f(u)}{du^2}. \qquad \text{(II 7)}$$

This is only possible in general if $v^2 = \dfrac{F}{\mu}$, that is, if the speed of waves on the rope is

$$v = \sqrt{\frac{F}{\mu}} \qquad \text{(II 8)}$$

where F and μ are the rope's tension and its mass per unit length.

State of the Search

1. Traveling waves can be generated on a stretched rope by shaking one end of the rope up and down. These waves are transverse, since the displacement of the rope is perpendicular to the original stretched rope. If x is measured along the rope, the transverse displacement is given by $y(x, t)$, which must be a solution of the wave equation*

$$\frac{\partial^2 y}{\partial x^2} = \left(\frac{\mu}{F}\right)\frac{\partial^2 y}{\partial t^2}.$$

The waves travel along a rope with speed $v = \sqrt{\frac{F}{\mu}}$, where F is the rope tension and μ is the mass/length of the rope.

2. The general solution of the wave equation is $y(x, t) = f(x - vt) + g(x + vt)$, where $f(x - vt)$ is a wave propagating to the right at speed v, and $g(x + vt)$ is a wave propagating to the left at the same speed. Waves keep their shape as they move. A special case is a sinusoidal wave moving to the right, with

$$y(x, t) = A\sin(kx - \omega t - \phi),$$

where A is the amplitude, $k = 2\pi/\lambda$ is the wave number (with λ the wavelength), $\omega = 2\pi\nu = 2\pi/T$ is the angular frequency (ν is the frequency and T is the period), and ϕ is a phase angle.

3. Standing waves have the form $y(x, t) = 2A\cos\omega t \sin kx$; they are particular linear combinations of waves propagating to the right and to the left. The nodes (i.e., the zeros) of standing waves stay put. If $y = 0$ always at both ends of a rope, there can only be an integer number of half-wavelengths present on the rope.

4. If one end of a rope is kept fixed by tying it to a point on a wall, a wave approaching the fixed point will be inverted and reflected back along the rope. Approximate standing waves can be generated by shaking one end of a rope up and down sinusoidally at one of a set of discrete frequencies that makes the wave pattern oscillate with nodes that are nearly fixed in space.

*There are also wave equations for water waves, seismic waves, sound waves, and light waves, among others. For small amplitudes, these all have the form $\frac{\partial^2 y}{\partial x^2} = C\frac{\partial^2 y}{\partial t^2}$ for some constant C. The speed of the wave is necessarily $v = 1/\sqrt{C}$, just as it is for waves on a rope.

PICTURE QUIZ D

1. A wave travels on a rope. The upper figure is a snapshot of the wave at time $t = 0$, and the bottom figure shows the displacement of the rope at point $x = 0$, plotted as a function of time.

(a) The wave's period T is

(A) 9 s (B) 10 s (C) 11 s (D) 12 s (E) 13 s

(b) The wave's angular frequency ω is

(A) 0.43 s^{-1} (B) 0.52 s^{-1} (C) 0.64 s^{-1} (D) 0.71 s^{-1} (E) 0.85 s^{-1}

(c) The wavelength is

(A) 1 m (B) 2 m (C) 3 m (D) 4 m (E) 5 m

(d) The wave number is

(A) 1.2 m^{-1} (B) 1.3 m^{-1} (C) 1.4 m^{-1} (D) 1.5 m^{-1} (E) 1.6 m^{-1}

(e) The speed of the wave is

(A) 0.33 m/s (B) 0.50 m/s (C) 0.67 m/s (D) 0.75 m/s (E) 1.0 m/s

(f) The phase angle in radians is

(A) $\pi/4$ (B) $\pi/3$ (C) $\pi/2$ (D) $2\pi/3$ (E) π

2. When it is under a certain tension, a rope carries traveling waves of speed $v = 6$ m/s. Keeping the tension the same, the standing wave shown is set up on the rope.

(a) The number of antinodes is

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

(b) The standing wave is the sum of two traveling waves, each of wavelength (in meters)

(A) 0.10 (B) 0.20 (C) 0.40 (D) 0.80 (E) 1.00

(c) The oscillation frequency ν of the standing wave is (in Hz)

(A) 5 (B) 10 (C) 15 (D) 30 (E) 60

(d) If the rope has linear mass density $\mu = 0.1$ kg/m, it is under tension (in Newtons)

(A) 1.2 (B) 1.8 (C) 2.4 (D) 3.0 (E) 3.6

PROBLEM QUIZ D

1. Which of the following represent possible small-amplitude waves on a string, over some restricted range of x and t? For any that does, state the speed of the wave and whether it is traveling to the right or to the left. In each case, x is measured in meters and t in seconds.

(a) $y(x, t) = A(x^2 - c^2t^2)$, where A and c are constants.

(b) $y(x, t) = A\,\ell n(2x + 4t)$

(c) $y(x, t) = A\cos(-2t + 2x^2)$

(d) $y(x, t) = A\tan(-7t + 3x)$

2. Two 10 N weights are tied to opposite ends of a string with mass density 4.0 grams/m. The string is hung over two pulleys a distance 0.50 m apart, as shown.

(a) What is the lowest-frequency standing wave that could be set up on the string between the two pulleys? Assume that there is a node at each pulley. (b) How much weight would have to be added to each side of the string so that the frequency found in (a) would be doubled?

3. The instruments in a string quartet include two violins, a viola, and a cello. The four violin strings are tuned to GDAE (ascending), the viola is tuned a musical fifth lower to CGDA (i.e. the frequency of each string's fundamental is 2/3 that of the corresponding string on the violin), and the cello is tuned an octave lower than the viola, also CGDA, so the fundamental frequency of each cello string is half that of the comparable viola string. (a) If the corresponding strings on each instrument were identical except for their length, and the tensions were the same, how much longer would the viola and cello strings have to be than those of the violin? (b) In fact, the relative lengths of the viola and cello strings are 1.17 and 2.13 times the length of the violin strings. Explain how these instruments are able to tune as they do. (c) Sketch a standing wave on the violin's G string and a standing wave on the viola's C string which have the same frequency as one another.

Questions and Problems from Chapter D, *Waves on a Rope*

DQ1. Which of the following are *not* equivalent forms for the traveling wave $y = A\sin(kx - \omega t)$? (a) $y = A\sin k(x - vt)$ (b) $y = -A\sin\omega(t - x/v)$ (c) $y = A\sin 2\pi[(x/\lambda) - (t/T)]$ (d) $y = A\sin[(2\pi/\lambda)(x - \omega t)]$ (e) $y = A\sin 2\pi(kx - vt)$

DQ2. A wave pulse is generated on a horizontal rope; the picture below shows the pulse at time $t = 0$, traveling to the right. At this instant, which parts of the rope are moving up, and which parts are moving down?

DQ3. A standing wave is set up on a rope. At certain times there is no displacement anywhere on the rope, so that it instantaneously appears to be no wave at all. Is there any energy in the wave at these times?

DQ4. Two wave pulses move in opposite directions on a rope, and are shown below at time $t = 0$. Sketch the rope at times $t = 1$ s, 2 s, and 3 s.

DQ5. A wave pulse travels along a stretched string at a speed of 100 cm/s. What will be the speed if (a) the string's tension is doubled? (b) the string's mass is quadrupled (but its length is unchanged?) (c) the string's length is quadrupled (but its mass is unchanged?)

DQ6. The figure below shows the displacement of a particular point on a stretched string as a function of time. Did the displacement at this point reach its maximum of 2 mm *before* or *after* the interval of time when the displacement was a constant 1 mm?

DQ7. The figure below is a snapshot graph of a sinusoidal wave at time $t = 1.0$ s. What is the phase angle of this wave?

DQ8. What is the frequency of the traveling wave shown below?

DQ9. The wave speed on a stretched string is 100 m/s when the tension is 100 N. What tension will give a speed of 200 m/s?

DQ10. Two strings are cut from the same roll and stretched under the same tension. String A has twice the length of string B. Traveling waves are set up in both strings. On which string is the wave speed greater, or are they the same?

Section D.1 Traveling sinusoidal waves

DP1. A traveling sinusoidal wave on a rope is observed to have period $T = 2.0$ s and wavelength $\lambda = 3.5$ cm. Find the waves angular frequency ω, wave number k, and speed v.

DP2. (a) Find the displacement $y(x,t)$ for a sinusoidal wave traveling to the right with amplitude A, period T, and velocity v, for which $y = A/3$ for $x = 0, t = 0$.

DP3. A wave $y(x,t) = A\sin(kx - \omega t)$ is set up on a piano string. (a) Find $y(x,t)$ in terms of wavelength λ and frequency ν, instead of k and ω. (b) Find $y(x,\ 2.0\text{ s})$ if $\lambda = 75$ cm and $\nu = 256$ Hz. (This is a snapshot of the wave shape at time $t = 2.0$ s.) (c) Find $y(35\text{ cm},\ t)$ for $\lambda = 75$ cm and $\nu = 256$ Hz. (This is the displacement of the string at the point $x = 35$ cm as a function of time.)

DP4. Find (a) the wavelength, (b) wave number, (c) frequency, (d) angular frequency, and (e) speed of the wave illustrated below. The diagram at the top shows a snapshot of the wave

at time $t = 0$, and the diagram at the bottom shows the string displacement at position $x = 0$, as a function of time.

DP5. Find (a) the wavelength, (b) wave number, (c) frequency, (d) angular frequency, (e) speed, and (f) phase angle of the wave below, assuming the wave form is $y = A\sin(kx - \omega t - \phi)$. The diagram at the top shows a snapshot of the wave at time $t = 0$, and the diagram at the bottom shows the string displacement at position $x = 0$, as a function of time.

DP6. Two strings of linear mass density μ_1 and μ_2 are knotted together at $x = 0$ and have the same tension F. A wave $y = A\sin k_1(x - v_1 t)$ in string 1 reaches the junction between the two strings, where it is partly reflected and partly transmitted into string 2. Call these waves $B\sin k_1(x + v_1 t))$ and $C\sin k_2(x - v_2 t))$, respectively. (a) Assuming that the frequency ω is the same in each string, and noting that the displacement at one side of the knot must be the same as the displacement at the other side, show that $A = B + C$. (b) Now assuming that both strings near the knot have the same slope (why?), show that $B = A(k_2 - k_1)/(k_2 + k_1) = (v_1 - v_2)/(v_1 + v_2)$. Under what conditions is B negative?

Section D.2 The wave equation

DP7. One end of a cord of mass $m = 25$ g and length $L = 2.5$ m is attached to a wall, and the last length $\ell = 20$ cm of the other end is hung over a small pulley, so a weight of mass $M = 0.50$ kg can be hung from it. Very near the wall a small kink is placed in the cord at time $t = 0$. (a) How long does it take the kink to reach the pulley? (b) If the initial displacement of the cord is $y(x, 0) = f(x)$, what is its displacement just before the kink reaches the pulley?

DP8. A cord of length $L = 10.0$ m has a mass $m = 75$ g. The last 10% of the cord hangs over a small pulley, and a weight of mass $M = 5.0$ kg is hung from it. The free end of the cord is briefly shaken up and down, placing a kink in the cord. How long does it take the kink to reach the pulley?

DP9. At time $t = 0$, the displacement of a stretched rope is $y(x, 0) = a^3/(a^2 + x^2)$ where a is a constant. Subsequently the wave travels to the right at speed v. (a) Sketch $y(x, 0)$ as a function of x. (b) What is $y(x, t)$, the displacement at a later time t? (c) What is the transverse velocity of a piece of the rope located at point x , at time t? (*Hint*: Explain why this transverse velocity should be $\partial y/\partial t$.)

DP10. The displacement on a string is $y(x, t) = 0.05\cos(2.0x - 3.0t)$, where x and y are in meters and t is in seconds. (a) Find the wavelength λ in meters, and the frequency ν in Hz. (b) Find the partial derivatives $\partial y/\partial x$, $\partial y/\partial t$, $\partial^2 y/\partial x^2$, $\partial^2 y/\partial t^2$ for the given displacement. (c) Then show that $y(x, t)$ is a solution of the wave equation for some velocity v. What is this velocity, in meters/sec? (d) Is your velocity from (c) equal to the product of λ and ν ?

DP11. The displacement on a string is $y(x, t) = 0.15\sin(0.3x - 1.2t)$, where x and y are in meters and t is in seconds. (a) Find the wavelength λ in meters, and the frequency ν in Hz. (b) Find the partial derivatives $\partial y/\partial x$, $\partial y/\partial t$, $\partial^2 y/\partial x^2$, $\partial^2 y/\partial t^2$ for the given displacement. (c) Then show that $y(x, t)$ is a solution of the wave equation for some velocity v. What is this velocity, in m/s? (d) Is your velocity from (c) equal to the product of λ and ν?

DP12. Strings No. 1 and No. 2 have linear mass densities μ_1 and μ_2, respectively. They are both under tension because of the hanging weight of mass M, as shown at the top below. (a) Find the wave speed in each string. (b) The weight is now divided into two weights, of masses M_1 and M_2, with $M_1 + M_2 = M$, as shown in the bottom figure. Find M_1 and M_2 in terms of μ_1, μ_2, and M, so the wave speeds in the two strings are equal.

DP13. A wave travels along a stretched string at a speed of 300 m/s. What will be the speed if the string is replaced by one made of the same material and under the same tension, but having twice the diameter?

DP14. String 1 in the figure below has linear density 2.0 g/m and string 2 has linear density 4.0 g/m. Pulses can be sent in both directions by pulling up sharply on the knot, then releasing it. What should be the string lengths L_1 and L_2 so the pulses will reach the ends of the strings simultaneously?

DP15. A uniform flexible rope of mass m and length ℓ hangs between two walls as shown below; each end makes an angle θ with the vertical. (a) Find the speed of a wave at the ends of the rope and at the midpoint of the rope. (b) Show that the ratio of the wave speed at the midpoint and the ends depends only on θ. *Hint:* Consider half the rope, from one end to the midpoint. Draw the isolation diagram for this part and find the tension at the end and at the midpoint.

DP16. A circular loop of rope of mass M and radius R is caused to rotate at constant angular velocity ω . (a) A small kink is introduced into the rope as shown below. Find the speed of the subsequent wave relative to the rope. (b) What is the speed at which the kink moves relative to the laboratory? Consider both possible directions of the wave motion.

DP17. A uniform rope of mass m and length L hangs from a fixed point. (a) Show that the speed of a wave on the rope is $v = \sqrt{gy}$, where y is the distance from the lower end of the rope. (b) Find the time it takes a wave to travel the length of the rope in terms of any or all of m, L, and g. (c) A mass M is now suspended from the bottom of the rope. Show that the time for a wave to travel the length of the rope is now

$$t = 2\sqrt{\frac{L}{g}}\left(\frac{\sqrt{M+m}-\sqrt{M}}{\sqrt{m}}\right)$$

(d) Show that the result in (c) reduces to your result for (b) when $M = 0$. What is the time if $m << M$? *Hint*: Use the binomial expansion.

DP18. A straight uniform rope of mass M and length L in gravity-free space is caused to rotate at constant angular velocity ω about one end of the rope. (a) Find the speed of a wave on the rope as a function of the distance r from the fixed end of the rope, in terms of the param-

eters M, L, and ω . (b) Check the dimensions of your result. (c) How long would it take the wave to propagate from one end of the rope to the other?

DP19. Prove that if v is the speed of waves on a string, then the displacement $y(x,t) = g(x + vt)$ is a solution of the wave equation, where $g(x + vt)$ is an arbitrary smooth (sufficiently differentiable) function of the quantity $x + vt$. What is v in terms of the tension F and mass/length μ of the string? Then explain why this solution represents a wave traveling to the left.

DP20. Prove that if $f(x - vt)$ and $g(x + vt)$ are each solutions of the wave equation on a string, that the sum $f(x - vt) + g(x + vt)$ is also a solution. Physically, this means that if a string can carry rightward waves and it can carry leftward waves, it can carry both at the same time.

DP21. A displacement of a wave traveling on a string is given by $y = 0.20 \sin \pi(0.20x - 10.0t)$, where x and y are in centimeters and t is in seconds. (a) Find the amplitude, wave number, wavelength, frequency, angular frequency, period, and velocity of the wave. In which direction is the wave moving? (b) Find the speed of a particle at position x in the string as a function of time. What is the maximum speed of the particle?

DP22. The displacement of a wave traveling on a string is given by $y = 3.0 \sin(2.0x + 12.0t)$, where x and y are in centimeters and t is in seconds. (a) Find the amplitude, wave number, wavelength, frequency, angular frequency, period, and velocity of the wave. In which direction is the wave moving? (b) Find the speed of a particle at position x in the string as a function of time. What is the maximum speed of the particle?

DP23. (a) Find an equation for the displacement $y(x,t)$ of a sinusoidal wave traveling in the negative x-direction, with amplitude 0.20 cm, period 0.35 s, and speed 45 m/s, if $y(0,0) = 0.10$ cm. (b) Find the speed of a particle at position x in the string as a function of time.

Section D.3 Standing Waves

DP24. The standing wave shown below oscillates so that once every 1.00 s the rope is everywhere horizontal. What is the speed of traveling waves on this rope?

DP25. The standing wave shown below oscillates on a rope under a certain tension. If at that tension traveling waves move on the rope at speed $v = 1.20$ m/s, what is the period of oscillations of the standing wave, the period being the time between the picture shown and the next time the rope looks the same?

DP26. Given a standing wave

$$y(x,t) = 0.50 \sin(1.5x) \cos(2.0t),$$

where x and y are in centimeters and t is in seconds, find (a) the distance between nodes, (b) the maximum speed of a particle of the string situated at an antinode, and (c) the times when the string is straight.

DP27. A violin string has mass/length 0.030 g/cm and is supported at the bridge and nut, 0.30 m apart. (a) For the fundamental mode to have the frequency $\nu = 196$ Hz, what should be the tension in the string? (b) Then what would be the frequency of the standing wave shown below?

DP28. When under a certain tension, a rope of length 2.0 m carries traveling waves of speed $v = 5.3$ m/s. Keeping the tension the same, the standing wave shown below is set up on the rope. (a) What is the wavelength of each

of the traveling waves making up the standing wave? (b) What is the oscillation frequency of the standing wave? (c) What is the tension if the rope has a mass of 0.30 kg?

DP29. The standing wave shown below is set up in a rope of length 1.0 m and mass density 0.15 kg/m when it is under the tension 5.0 N. (a) What is the oscillation frequency of the standing wave? (b) What tension would double the oscillation frequency, assuming that the standing wave picture shown below stays the same?

DP30. A standing wave on a rope of mass/length μ has the displacement $y(x,t) = A \sin kx \cos \omega t$. Find the total kinetic energy in the portion of the string between two successive nodes, at an instant when the string is straight.

DP31. A weight can be hung from a string with linear mass density $\mu = 2.0$ g/m slung over a light pulley, as shown below. The string is connected to a vibrator of frequency ν_0; the length of the string between the vibrator and pulley is $L = 2.0$ m. When the weight of mass m is either 16 kg or 25 kg, standing waves are observed, but no standing waves are observed for any mass between these values. (a) What is v_0? (b) What is the largest mass for which standing waves could be observed?

DP32. A 2.0 m long string is fixed at both ends and tightened until the wave speed is 40 m/s. What is the frequency of the standing wave shown in the figure below?

DP33. The figure below shows a standing wave that is oscillating at frequency ν_0. (a) How many antinodes will there be if the frequency is doubled to $2\nu_0$? (b) If the tension is increased by a factor of four, for what frequency, in terms of ν_0, will the string continue to oscillate as a standing wave with three antinodes?

DP34. The frequency of a standing wave on a stretched string is ν when the string's tension is T. If the tension is changed by the *small* amount ΔT, without changing the length, find the fractional frequency change $\Delta\nu/\nu$ in terms of the fractional tension change $\Delta T/T$.

E. Sound, Seismic, and Surface-Water Waves; Power Flow

"I'm told that Wagner's music is better than it sounds." - Mark Twain

E.1 Transverse Waves and Longitudinal Waves

In Chapter D we dealt only with waves on a string, including transverse traveling and standing waves. String waves are *transverse* because the wave disturbance is *perpendicular* to the direction of wave propagation. There are also waves called *longitudinal* waves, in which the disturbance is *parallel* to the direction of wave propagation. Take a long spring, for example, held horizontally. We can shake one end up and down, and establish propagating transverse waves similar to those on a rope. However, we can also generate longitudinal waves in the spring by squeezing the spring (i.e., compressing it) at one end, and then letting go. The compression propagates along the spring, followed perhaps by a "rarefaction", as illustrated in Fig. 1.

Fig. 1. A longitudinal wave, consisting of a compression followed by a rarefaction, carried along a spring from left to right. The compression contains an unusually large number of spring windings per unit length, while the rarefaction contains an unusually small number.

Sound is a longitudinal wave carried through a medium like air or water. Air can be compressed by vocal chords, a stereo speaker, or an explosion; a compressional wave then travels out through the air. Without a medium there can be no sound, because sound is a compression or rarefaction of the medium itself. The air molecules move back and forth parallel to the propagation direction of the wave, consistent with the fact that sound waves are longitudinal. In compressed portions the density of air molecules is larger than normal; in rarefied portions the density is smaller than normal.

If an underground Earth fault slips, both transverse and longitudinal *seismic* waves are generated. In the transverse wave, material within Earth moves perpendicular to the propagation direction of the wave; in the longitudinal wave material moves

parallel to the propagation direction so that rock is alternately compressed and decompressed. The two kinds of waves generally travel at different speeds, so that if we measure arrival times of the two waves on a seismograph, we can find the distance of the fault from the seismograph as long as we know the speed of each wave.

Water waves are neither strictly transverse nor longitudinal. If a cork (or floating leaf) is placed on the surface of a pond, the cork moves in a circle as the wave passes by. This shows that any particular cupful of water on the surface also moves in a circle, because it is the motion of the water that causes the motion of the cork.

E.2 Sound

Sound is a longitudinal compressional wave in air, water, metal, or other material. If the material is compressed in some region, that material subsequently expands, compressing a neighboring region, and so on, traveling out from the source as a wave.

Suppose that air is confined inside a long cylindrical tube. A piston at one end of the tube is suddenly and briefly pushed into the column of air, as shown in Fig. 2. A layer of air adjacent to

Fig. 2. A layer at the left of a cylinder of air is suddenly compressed by a piston. The compressed layer travels to the right, constituting a pulse of sound.

the piston is compressed, raising its density and pressure relative to the undisturbed air to its right. The excess pressure causes the compressed layer to expand, compressing a layer immediately to its right, while the original compressed layer relaxes back to its original density and pressure. The compression keeps propagating to the right as a pulse of sound. An individual air molecule moves back and forth as the pulse passes, but it doesn't travel very far. It is the compressional *pulse* that keeps traveling. It may eventually push and pull on an eardrum, creating the sensation of sound

through neural impulses to the brain. Over many millions of years, ears have evolved to detect wide ranges of both frequencies and intensities in these compressional waves.

Figure 3 shows what happens when we begin to move a piston in and out with a sinusoidal motion. Now a sinusoidal traveling wave is generated within the cylinder.*

Fig. 3. The sound wave generated within a cylinder by a piston moving sinusoidally in and out.

* The actual motion of air inside the cylinder is somewhat more complicated than shown in Figs. 2 and 3, in that the viscosity of air would reduce its ability to move back and forth near the cylinder walls.

How fast does a sound-pulse move? Using $F = ma$, Appendix E I shows that the speed of sound waves is

$$v = \sqrt{\frac{dp}{d\rho}}, \tag{1}$$

the square-root of the derivative of the pressure of the medium with respect to its density.* If it does not take much pressure to increase the density substantially, i.e., if the medium is "squishy" like air or marshmallows, the velocity of sound is relatively small.** If a great deal of pressure is needed to cause an appreciable density change, the medium is "stiff" and the velocity of sound is large, as in water or in metals.

The speed of sound at 25° C is about 1500 m/s in pure water and about 1530 m/s in seawater. The speed of sound in steel (depending upon the type) is 5800 - 6000 m/s. In an ideal gas (which is a pretty good approximation for air at standard temperature and pressure) the speed of sound is

$$v = \sqrt{\frac{dp}{d\rho}} = \sqrt{\gamma \frac{RT}{M}} \tag{2}$$

where R is the universal gas constant, T is the temperature, M is the molecular weight, and γ is called the ratio of specific heats (Eq. 2 is derived in Appendix E I). In dry air the sound velocity is 331 m/s at 0° C, increasing with the square root of T as the temperature rises. Hydrogen gas H_2 has a lower molecular weight than air, so the speed of sound in hydrogen is correspondingly higher, 1284 m/s at 0° C.

In Chapter D we showed that the equivalent of $F = ma$ for small-amplitude waves $y(x, t)$ on a string is the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \tag{3}$$

* It is easy to see that the equation is at least dimensionally correct; pressure is force per unit area, with dimensions $(ML/T^2)/L^2 = M/LT^2$ and density has dimensions M/L^3, so their ratio has the dimensions L^2/T^2, the same as the dimensions of velocity squared.

** If there is lots of friction within a medium the wave will die out quickly because its energy is converted quickly into heat. In marshmallows we expect the velocity of sound to be low and to be quickly damped out.

where the speed of the waves is $v = \sqrt{\frac{F}{\mu}}$ on a string with tension F and mass/length μ. Small-amplitude *sound* waves are also solutions of a wave equation. If we characterize sound waves by the excess pressure Δp (Δp is the pressure above the ambient air pressure) as a function of position and time, then for one-dimensional waves (like those in a long narrow tube of air) the wave equation is

$$\frac{\partial^2(\Delta p)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2(\Delta p)}{\partial t^2} \tag{4}$$

where the velocity of the waves is $v = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{B}{\rho}}$. Here B is the *bulk modulus* of the fluid and ρ is the density of the fluid, as described in Appendix E I. Alternatively, sound waves can be characterized by the variation in density $\Delta\rho$ as a function of position. The wave equation is then the same as before, except that $\Delta\rho$ is the dependent variable instead of Δp. We won't show it here, but each of these equations can be derived by applying Newton's second law $F = ma$ to small regions inside the material. Sound speeds in various materials are given in Table 1. (In solids transverse waves can be propagated as well; Table 1 gives only the speed of longitudinal sound waves.)

Substance	v(m/s)	Substance	v(m/s)
Gold (hard-drawn)	3240	Ethanol	1207
Silver	3650	Kerosene	1324
Lead (annealed)	2160	Water (distilled)	1498
Steel (stainless)	5790	Water (sea)	1531
Glass (Pyrex)	5640	Air (dry)	331
Lucite	2680	Hydrogen	1284
Polyethylene	1950	Oxygen	316

Table 1. Sound speeds (longitudinal waves) in various substances. For the liquids sound speeds are given at temperature 25° C, and for the gases sound speeds are given at sea-level pressure and temperature 0° C.

E.3 Seismic Waves

Waves within our planet are usually generated by sudden shifts in Earth's outermost layer, the crust. The source may be near the surface or deep underground. Much of our planet is *elastic*. (An elastic medium, such as stiffened jello, can be distorted by a force, but springs back to its original configuration once the force has been removed.) In an elastic medium a small local distortion subsequently spreads out as a wave. The wave speed may depend upon position, so that the waves can travel in interesting paths.* They can travel through large parts of Earth and then be easily detected far from the source by seismographs, instruments that record the shaking of an arriving wave.

Seismic waves generally have both longitudinal and transverse components. The longitudinal waves are called P waves (P for primary, or pressure). The transverse waves are called S waves (S for secondary, or shear). Typically P waves travel faster than S waves, so the seismograph receives the P waves first and the S waves a bit later. If the speeds of these waves are known, the distance of the source from the seismograph can be calculated from the time interval between the receptions of the two wave types.

The seismic waves formed during earthquakes are our main source of information about conditions far below the surface. We have discovered, for example, that Earth contains a liquid core, because S waves do not propagate through Earth's center. (S waves cannot propagate in fluids, because there is little or no restoring force in a fluid for sideways displacements.) On the other hand, P waves *do* propagate all the way through the planet, including the center, as one would expect: They are much like sound waves, and sound waves move just fine through fluids.

*One can show that when a wave is traveling along the boundary between faster and slower regions, it tends to bend away from the faster into the slower region.

Sample Problem 1. A magnitude 4.1 aftershock of the 1994 Northridge earthquake was recorded at Claremont, CA on Harvey Mudd College seismographs. The record shows the arrival of a particular P wave followed by the arrival of the S wave from the same event. From the record, how far from Claremont was the location of the seismic event? (P waves typically travel at 5.62 km/s and S waves at 3.33 km/s in the upper crust of Earth in Southern California. The record was obtained by G. A. Lyzenga.)

Solution: If D is the distance of the event from the seismograph, the arrival times of the two waves are $t_S = D/v_S$ and $t_P = D/v_P$, respectively. The difference in arrival times is

$$t_S - t_P = D/v_S - D/v_P,$$

so the distance of the event is

$$D = \frac{t_S - t_P}{\left[\dfrac{1}{v_S} - \dfrac{1}{v_P}\right]}.$$

The difference in arrival times is $t_S - t_P \cong (27 - 15)$ s $= 12$ s according to the record, so (substituting in the velocities of the two wave types), we find $D \cong 98$ km. Assembling a number of such seismograph records, by triangulation seismologists concluded that the aftershock epicenter was located about 5 miles north of Simi Valley, CA.

E.4 Surface Water Waves

Waves that travel on the surface of lakes or the ocean are sometimes called gravity waves, because gravity is the force that causes them to propagate.* They are neither pure transverse nor pure longitudinal waves; a cork set on the lake executes circular or elliptical motion, indicating that in addition to the obvious transverse component, water moves back and forth as well in the direction of propagation of the wave, as illustrated in Fig. 4. The velocity of the wave depends upon the local gravity g, and also generally upon both the wavelength λ and the depth h of the water. If the water is shallow, i.e. if the depth $h << \lambda$, then the velocity of the waves is

Fig. 4. A surface water wave as time passes. The wave is traveling to the right, but a molecule of water moves in a circle as shown.

$$v = \sqrt{g\,h} \quad , \tag{4}$$

which is independent of the wavelength. All wavelengths of shallow-water waves travel at the same speed in water of constant depth. In a large shallow pool of water of one-meter depth, the speed of waves whose wavelength is longer than a few meters is a little over 3 m/s, or about 7 mi/hr. Tsunamis, waves formed in the ocean by earthquakes, often have wavelengths hundreds of miles long. They are therefore *shallow*-water waves, since the oceans are typically only a mile or two deep! Tsunami waves all move at the same speed if the ocean depth is constant; if $h = 1$ km, for example, the speed of the waves is 100 m/s , or about 220 mi/ hr.

In the other limit, the deep-water limit $h >> \lambda$, the speed of water waves depends upon wavelength but not depth. After all, why should a wave in the deep ocean with a few-meter wavelength "care" how deep the ocean is? Divers verify that in fact the ocean is usually quite calm beneath the surface a few tens of meters. The velocity of such deep-water waves is

$$v = \sqrt{\frac{g\lambda}{2\pi}} \quad , \tag{5}$$

*If gravity were turned off, a localized bump of water would just stay put, not traveling anywhere.

which exhibits the property of *dispersion*. A wave whose velocity depends upon its wavelength, i.e., $v = v(\lambda)$, is called a dispersive wave. According to Eq. 5, deep-water waves are dispersive, with longer wavelengths traveling faster than shorter wavelengths. If you are sitting in a canoe on a deep lake when a speedboat passes by, expect to receive the long-wavelength swells before the shorter wavelengths arrive. Typical waves in mid-ocean (excepting Tsunamis) are deep-water waves. A wave of wavelength 10 m, for example, has a speed $v \cong 4$ m/s (9 mi/hr).

When waves are nondispersive, the shape of a wave pulse remains the same with time. But if waves are dispersive, the shape can change. Small-amplitude waves on a rope and small-amplitude sound waves are both nondispersive; all wavelengths travel at the same speed. A human speaking voice and a symphony orchestra contain many wavelengths. If different frequencies traveled at different speeds, the human voice would be distorted and we might hear the higher frequency piccolos in the orchestra before the lower frequency bass viols in a chord where they both played together. Light waves in vacuum are nondispersive; all wavelengths travel at the same speed. We see all wavelengths in a distant supernova explosion essentially simultaneously, for example. If light from a supernova were dispersive, we might see the explosion in red light before we saw it in blue light, because these colors correspond to different wavelengths. Light in glass or water *is* dispersive. Commonly blue light travels faster than red light in glass, which causes prisms to disperse white light (consisting of all visible wavelengths) into its constituent colors, by the process known as *refraction*.

Sample Problem 2. A researcher in a small boat in the middle of the ocean on a calm day sees a meteor fall into the ocean some distance away. Forty-five minutes later she observes that the waves reaching her boat have a wavelength of 9 meters. (a) How far away was the meteor impact? (b) How much longer will she have to wait before she observes that the passing waves have a wavelength of 1 meter?

Solution: (a) The waves are deep-water waves, so their velocities are given by $v = \sqrt{\dfrac{g\lambda}{2\pi}}$.

The distance of the splash is therefore $D = v\,t = \sqrt{\dfrac{9.8\ \text{m/s}^2\ (9\ \text{m})}{2\pi}}\,(45 \cdot 60\ \text{s}) = 10\ \text{km}$ away. (b) The 1 - meter wavelength waves travel only one-third as fast, so will take three times as long to arrive. She will therefore have to wait another hour and a half for them.

E.5 Power Flow in Waves

There is no doubt that waves carry energy. Sound waves cause eardrums to vibrate. Seismic waves can shake buildings. Water waves can increase the energy of a body surfer or even destroy a city. How much energy is carried by such waves?

Begin by considering waves traveling to the right on a rope, for example, characterized by the shape $y(x, t)$. The power flow is the energy per unit time passing any point on the rope.

We can find this power flow by first finding the work ΔW done on a small piece of rope Δm due to the rope immediately preceding it (which is the part of the rope transmitting energy to Δm), as Δm moves a small transverse distance Δy, as shown in Fig. 5(a). By definition, the work is $\Delta W = F_T \, \Delta y$, the transverse force on the piece of rope times its tranverse displacement. The transverse force is the transverse component of tension, which is the total tension F times the slope of the rope $\frac{\partial y}{\partial x}$, as shown in Fig. 5(b). This transverse component is *opposite* to the displacement Δy, so the work done is $\Delta W = -F\frac{\partial y}{\partial x}\Delta y$.

Fig. 5. (a) A piece of rope moves a transverse distance Δy. (b) The transverse component of tension, $-F\frac{\partial y}{\partial x}$. The work done by rope to the left is the product of these, $\Delta W = -F\frac{\partial y}{\partial x}\Delta y$.

The power input to the piece of rope due to the rope at its left is the time rate of change of the work done. For the small time interval Δt,

$$P = \frac{\Delta W}{\Delta t} = -F\frac{\partial y}{\partial x}\frac{\Delta y}{\Delta t} . \tag{6a}$$

In the limit $\Delta t \rightarrow 0$,

$$P = -(F \frac{\partial y}{\partial x}) \frac{\partial y}{\partial t} . \tag{6b}$$

In the particular case of the sinusoidal wave $y(x, t) = A \sin(kx - \omega t)$, we can take the two partial derivatives, the derivative with respect to x and the derivative with respect to t, to find the power input

$$P = -F kA \cos(kx - \omega t) \cdot [-A \omega \cos(kx - \omega t)] = F A^2 k\omega \cos^2(kx - \omega t) . \tag{7}$$

The velocity of the wave is $v = \sqrt{\frac{F}{\mu}} = \frac{\omega}{k}$, which we can use to eliminate F and k in Eq. 7. The power can then be written

$$P = A^2 \mu v \omega^2 \cos^2(kx - \omega t) . \tag{8}$$

The power changes periodically with time. The reason for this is that the rate at which work is done on the rope by the person shaking the left-hand end depends periodically on time as well.

We can also find the *average* power carried along the rope. The time-averaged power carried on a rope of mass-density μ by a sinusoidal wave of amplitude A, velocity v, and angular frequency ω, is

$$< P > = A^2 \mu v \omega^2 < \cos^2(kx - \omega t) > = \frac{1}{2} A^2 \mu v \omega^2 , \tag{9}$$

because the time average of $\cos^2(kx - \omega t)$ is 1/2 .* For waves of given speed on a rope of given mass density, the power transferred down the rope is proportional to the *square* of the amplitude of the wave, and to the *square* of the wave frequency.

*See problem EP15.

Sample Problem 3. A rope of length $L = 3.0$ m has a mass $m = 0.10$ kg, and is under a tension $F = 75$ N. A sinusoidal wave of amplitude $A = 2.0$ cm and period $T = 0.05$ s is sent down the rope by a source at one end. If all the power arriving at the other end is extracted, how long must one wait until the energy collected amounts to $E = 100$ J ?

Solution: The mass/length of the rope is $\mu = \dfrac{0.10 \text{ kg}}{3.0 \text{ m}} = 0.033$ kg/m, so the wave velocity is

$$v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{75 \text{ N}}{.033 \text{ kg/m}}} = 47.4 \text{ m/s} .$$

The angular frequency of the wave is $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{.05 \text{ s}} = 126 \text{ s}^{-1}$. The average power propagated along the rope is therefore

$$< P > = \frac{1}{2} A^2 \mu v \omega^2 = \frac{1}{2} (0.02 \text{ m})^2 \cdot 0.033 \text{ kg/m} \cdot 47.4 \text{ m/s} \cdot (126 \text{ s}^{-1})^2 = 5.0 \text{ Watts.}$$

The power is the time rate of change of the energy, so if $<P> = 5.0$ W, it takes 20 seconds to collect 100 J of energy.

The power carried by a traveling wave on a rope is proportional to its amplitude *squared.* This is true of other waves as well, including sound waves. If we characterize sound by the excess pressure Δp, then the intensity of sound is proportional to Δp^2.

An ideal plane wave propagates in a single direction, say the x direction, and extends to infinity in the other (y and z) directions, as illustrated in Fig. 6.

Fig. 6. A schematic snapshot of a sinusoidal plane wave propagating in the x-direction. The planes shown correspond to wave maxima, which move to the right as time goes on.

A sinusoidal plane wave of sound traveling in the x direction has an excess pressure given by

$$\Delta p = \Delta p_0 \sin(kx - \omega t)$$

where Δp_0 is the amplitude. The total power carried by such an idealized wave is infinite, since the wave extends to infinity in the y and z directions. It therefore makes more sense to talk about the power per unit area perpendicular to the propagation direction. This quantity is called I, the intensity in the wave. The intensity is the propagating power/area or the energy per unit time per unit area, which is finite and proportional to $\Delta p^2 = \Delta p_0{}^2 \sin^2(kx - \omega t)$ for a plane wave.

Most often the wave from a localized sound source spreads out in three dimensions. As the wave spreads out, the *intensity* of the sound decreases, even if the total power in the wave stays constant. The intensity I is

$$I = \frac{Power}{Area}, \qquad (10a)$$

the power per unit area in the wave, also the energy per unit time and per unit area. Suppose for example a horn emits sound waves in all directions. The power (the energy/second) of sound passing through a sphere centered on the horn is the same for all spheres, regardless of their radius (neglecting any absorption of energy by the air.) A human ear of fixed size receives less intensity far away from the sound source than it does nearby, however, because the power is spread over a larger and larger area as the wave expands. In fact, for a spherical sound wave the area of the spherical wavefront is $4\pi r^2$, growing quadratically with distance r from the source. The intensity I therefore falls off with the square of r:

$$I = \frac{\alpha}{r^2}, \qquad (10b)$$

where α is a constant. The intensity drops by a factor of four if we are twice as far away.

A typical human ear can hear a range of intensities of something like twelve orders of magnitude, even before the "threshold of pain" is achieved at the upper end. Therefore intensities are often expressed on a logarithmic scale. In fact the *sound level* SL is defined as

$$\mathrm{SL} = 10 \log\left(\frac{I}{I_0}\right) \quad \text{(in decibels, or } db\text{)} \tag{11}$$

where the logarithm is to the base 10. Here $I_0 = 10^{-12}$ W/m^2, an intensity which is near the threshold of typical human hearing. Note that if $I = I_0$, then SL = 0 db;

if $I = 10\, I_0$, then SL = 10 db , about the sound level of nearby breathing;

if $I = 100\, I_0$, then SL = 20 db, about the sound level of a whisper.

The threshold of pain, observed near jackhammers and at some rock concerts, corresponds to

$$I = 1 \text{ W/m}^2 \text{, for which } \mathrm{SL} = 120 \text{ db}.$$

The sound of a nearby jet engine can have an SL = 150 db, and if the sound level near you reaches 165 db, it can light your hair on fire. (In that case you will probably have no need to worry about sound levels.)

Adult humans typically can hear frequencies from about 20 Hz - 15,000 Hz; young children can hear up to about 20,000 Hz. Fig. 5 is a map of typical frequency and intensity ranges audible to people. The intensity at which sound becomes inaudible, and the intensity at which it begins to cause pain, depends upon the frequency as shown. Recall that "middle C" on a piano is 256 Hz, that every octave above that doubles the frequency, and that every octave below that halves the frequency.

Fig. 5. A map of the typical frequency and intensity ranges of sound audible to people.

Sample Problem 4. A loudspeaker broadcasts Tchaikovski's *1812 Overture*, which approaches the threshold of pain to a listener 2m away. If the listener retreats to a distance 20 m from the loudspeaker, estimate the new intensity and sound level. Assume the sound spreads out istropically (i.e., equally in all directions) in three dimensions.

Solution: The distance has increased by a factor of 10, so the intensity has fallen by a factor of 100. Instead of the intensity $I = 1\ \mathrm{W/m^2}$ (with sound level SL = 120 db), the new intensity is $I = 10^{-2}\ \mathrm{W/m^2}$, and the new sound level is

$$\mathrm{SL} = 10 \log\left(\frac{10^{-2}\,\mathrm{W/m^2}}{10^{-12}\,\mathrm{W/m^2}}\right)\mathrm{db} = 10 \log 10^{10}\ \mathrm{db} = 100 \log 10\ \mathrm{db} = 100\ \mathrm{db}.$$

Note: In a typical concert hall the sound does *not* spread out evenly in three dimensions. There are of course the walls, the floor, the ceiling, and the audience which partially absorb and partially reflect sound that strikes them; there are also various other reflective and absorptive surfaces introduced to improve the sound quality.

E.6 The Doppler Effect

The pitch of a train whistle is higher as the train approaches than as it recedes. The same is true of police or ambulance sirens - the pitch of the siren is higher when the vehicle approaches and lower when it recedes. In each of these cases the sound emitter is moving relative to the air, while we as observers are stationary. It is equally true, although less frequently observed, that frequency shifts are heard if the emitter is stationary, while the observer moves through the air. If we drive a car past a stationary whistle, for example, the sound is high-pitched as we approach and low-pitched as we drive away. This change in pitch is known as the *Doppler Effect.*

First suppose that both emitter and observer are at rest relative to the air, as shown in Fig. 6.

Fig. 6. Wave maxima with both emitter and observer at rest relative to the air.

The emitter is at the left, the observer is at the right, and the circular arcs shown are crests of a sinusoidal sound wave, spaced one wavelength apart. The time Δt between the arrival of two successive crests is λ, the distance the wave moves, divided by its speed, the sound-speed v_S relative to air. The time interval Δt is the period of the wave, and also its inverse frequency: That is, if both emitter and observer are at rest relative to the air,

$$\Delta t = \frac{\lambda}{v_S} = \frac{1}{\nu}, \tag{12}$$

confirming the relationship $v_S = \lambda\nu$ valid for any sinusoidal sound wave.

Now suppose the observer remains at rest relative to the air, while the emitter moves toward it at speed v_{em}, as illustrated in Fig. 7. At time $t = 0$ the source emits the crest of a wave when the observer is a distance d away, as shown in Fig. 7(a). At a slightly later time $t = T$, the source emits the *next* wave crest, as in Fig. 7(b), and has itself moved a distance $v_{em}T$, so is now only a distance $d - v_{em}T$ away from the observer. At time $t = t_1 = d/v_S$, the first crest finally arrives at the observer, as in Fig. 7(c), and at the still later time $t = t_2 = T + \frac{\text{distance}}{\text{sound speed}} = T + \frac{d - v_{em}T}{v_S}$, the second crest arrives. The time interval between the arrival of the two crests is

$$\Delta t = t_2 - t_1 = T + \frac{d - v_{em}T}{v_S} - \frac{d}{v_S} = T\left(1 - \frac{v_{em}}{v_S}\right) \tag{13}$$

Fig. 7. The emitter moves toward the observer. (a) At $t = 0$ the emitter sends out a wave crest. (b) At $t = T$ it sends out a second crest. (c) The first crest arrives at the observer. (d) the second crest arrives.

so the observed frequency is

$$\nu_{ob} = \frac{1}{\Delta t} = \frac{1}{T}\frac{1}{\left(1 - \frac{v_{em}}{v_s}\right)} = \frac{\nu_{em}}{\left(1 - \frac{v_{em}}{v_s}\right)}. \tag{14}$$

Note that the velocity of sound v_s is the sound velocity relative to the air in which it moves, and it is independent of the velocities of the emitter and observer. If the emitter is moving *away* from the observer (*i.e.*, toward the *left* instead of toward the right in Fig. 7) it is easy to show that Eq.14 still holds, except we have to let $v_{em} \rightarrow -v_{em}$. So covering both cases,

$$\nu_{ob} = \frac{1}{\Delta t} = \frac{1}{T}\frac{1}{\left(1 \mp \frac{v_{em}}{v_s}\right)} = \frac{\nu_{em}}{\left(1 \mp \frac{v_{em}}{v_s}\right)} \tag{15}$$

for the emitter moving relative to the air, where the upper (minus) sign applies if the emitter is moving toward the observer, which gives a shift toward higher frequency, and the lower (plus) sign applies if the emitter is moving away from the observer, which gives a shift toward lower frequency.

Now suppose the sound *emitter* remains at rest relative to the air, while the observer moves toward it at speed v_{ob}, as illustrated in Fig. 8. At time $t = 0$, when the observer is a distance d away, the emitter sends out a wave crest, as shown in 8(a). At time T it emits a second crest, as in

Fig. 8. The observer moves toward the emitter. (a) At $t = 0$ the emitter sends out a wave crest. (b) At $t = T$ it sends out a second crest. (c) The first crest arrives at the observer. (d) the second crest arrives.

8(b), when the observer is now only a distance $d - v_{ob}T$ away from the emitter. The first crest reaches the observer at time t_1, when the distances covered by the sound wave and the observer sum to d, *i.e.*, when $v_s t_1 + v_{ob} t_1 = d$, or $t_1 = d/(v_s + v_{ob})$, as illustrated in 8(c). The second crest finally reaches the observer at t_2, when the distances covered by the sound wave and the observer sum to $d - v_{ob}T$, except we have to add in the starting time for this crest, which is T. That is, $t_2 = T + (d - v_{ob}T)/(v_s + v_{ob})$. The difference in arrival times is

$$\Delta t = t_2 - t_1 = T + \frac{d - v_{ob}T}{(v_s + v_{ob})} - \frac{d}{(v_s + v_{ob})} = T - \frac{v_{ob}T}{(v_s + v_{ob})} = T\left(\frac{v_s}{v_s + v_{ob}}\right), \tag{16}$$

so the observed frequency is $\nu_{ob} = \dfrac{1}{\Delta t} = \dfrac{1}{T}\left(\dfrac{v_s + v_{ob}}{v_s}\right) = \nu_{em}\left(1 + \dfrac{v_{ob}}{v_s}\right)$. If the observer is moving *away* from the emitter (*i.e.*, toward the *right* instead of toward the left in Fig. 8) it is easy to show that we just have to change the sign of v_{ob}. So covering both cases,

$$\nu_{ob} = \left(1 \pm \frac{v_{ob}}{v_s}\right)\nu_{em} \tag{17}$$

for the observer moving relative to the air, where the upper (plus) sign applies if the observer is moving toward the emitter, which gives a shift toward higher frequency, and the lower (minus) sign applies if the observer is moving away from the emitter, which gives a shift toward lower frequency. In dry air at 20° C, the velocity of sound is $v_s = 331$ m/s $= 740$ mi/hr, so if you drive toward a sound emitter at a speed of 37 mi/hr, which is 5% the speed of sound, the frequency you hear is 5% higher than the frequency emitted by the source, a difference easy for most people to distinguish when the frequencies are compared.

As emitter and observer approach one another, there is always a shift toward higher frequency, and as they recede there is always a shift toward lower frequency. But the magnitude of the shift depends on which is moving relative to the air, since formulas (15) and (17) are different. It can be shown (see problem EP24) that if *both* the emitter and observer are moving relative to the air, one direction or the other along a straight line (i.e., directly toward or away from one another), the observed frequency is

$$\nu_{ob} = \left(\frac{1 \pm v_{ob}/v_s}{1 \mp v_{em}/v_s}\right)\nu_{em}. \tag{18}$$

Upper signs are used if the observer is moving toward the emitter (in the numerator) or if the emitter is moving toward the observer (in the denominator). If either is moving away from the other, the appropriate lower sign is used.

Sample Problem 5. On a windless day, an ambulance with blaring siren, traveling at 50 mi/hr (22 m/s), overtakes and passes a car moving (illegally) at 25 mi/hr (11 m/s). If the average siren frequency heard by the car driver is 1250 Hz as the ambulance approaches, what average frequency does the driver hear after the ambulance has passed?

Solution: When both emitter and observer are moving relative to the air, we can use Eq. 17 to relate emitted and observed frequencies. While the ambulance is catching up with the car, the emitter is moving toward the observer, so we use the minus sign in the denominator of Eq. 17, since that increases v_{ob}. During this time the car is moving away from the emitter, so we also use the minus sign in the numerator, since that *decreases* v_{ob}. Taking the speed of sound to be $v_s = 330$ m/s, the ratio $v_{ob}/v_s = 11/330 = 0.0333$, and the ratio $v_{em}/v_s = 22/330 = 0.0667$. So while the ambulance is catching up with the car,

$$\nu_{ob} = 1250 \text{ Hz} = \left(\frac{1 - 0.0333}{1 - 0.0667}\right) \nu_{em} = 1.036\ \nu_{em}.$$

While the ambulance is catching up with the car, the observed frequency is higher than if both vehicles were at rest relative to the air. After the ambulance passes, the emitter is receding from the observer, so we use the plus sign in the denominator, to decrease ν_{ob}. The observer is moving toward the emitter, so we also use a plus sign in the numerator, to increase ν_{ob}. After the ambulance has passed by the car,

$$\nu_{ob} = \left(\frac{1 + 0.0333}{1 + 0.0667}\right)\nu_{em} = 0.969\ \nu_{em} = 0.969 \left(\frac{1250}{1.036}\right)\text{Hz} = 1170 \text{ Hz}.$$

The frequency falls by over 6% as the ambulance passes by.

State of the Search

1. In addition to transverse waves, such as waves on a rope, in which particles in the rope move perpendicular to the wave propagation direction, there are longitudinal waves, in which particles move parallel (and antiparallel) to the wave propagation direction. Sound is a longitudinal wave. There are both transverse (S) and longitudinal (P) components of seismic waves, waves moving through Earth caused usually by sudden shifts in Earth's crust.

2. Surface water waves are neither transverse nor longitudinal; a molecule of water in the wave moves horizontally as well as vertically, in a circular or elliptical path. Shallow water waves (with wavelength $\lambda >>$ depth h) have velocity $v = \sqrt{g\,h}$, while deep water waves (with $\lambda << h$) have velocity $v = \sqrt{\frac{g\lambda}{2\pi}}$. Dispersive waves are those for which the velocity depends upon the wavelength. Deep-water waves are dispersive, while shallow-water waves are not.

3. The rate at which energy is carried along a traveling wave on a string is proportional to the square of the string's amplitude. The time-average power carried by a sinusoidal wave is

$$< P > = \frac{1}{2} A^2 \mu v \omega^2$$

where A is the amplitude, μ is the string's mass/length, v is the wave velocity, and ω is the angular frequency.

4. The intensity of a sound wave (the power/area) spreading into three dimensions falls off as the square of the distance from the source. The so-called *sound level* SL is related to the intensity I by

$$\mathrm{SL} = 10 \log\left(\frac{I}{I_o}\right) \quad \text{(in decibels, or } db\text{)}.$$

5. The frequency ν_{ob} of observed sound waves generally differs from the frequency ν_{em} emitted by the emitter, if the emitter and/or observer moves relative to the medium. The pitch is higher if the emitter and observer are moving toward one another, and lower if they are moving apart. In fact,

$$\nu_{ob} = \left(\frac{1 \pm v_{ob} / v_s}{1 \mp v_{em} / v_s}\right)\nu_{em}$$

where v_s is the speed of sound, and v_{ob} and v_{em} are the velocities of the observer and emitter relative to the medium. Upper signs are used if the observer and emitter are moving toward one another, and lower signs are used if they are moving apart.

Appendix E I: The Speed of Sound Waves

Consider a long cylindrical tube, as shown in Fig. 2 in Chapter E, and as shown in Fig. E1 as well. A piston at the left end is pushed toward the right, compressing the air within, raising its pressure and density relative to the undisturbed air to its right. The compression travels to a right as a pulse wave. How fast does the sound-pulse move?

Fig. E1. A layer of air is compressed by a piston. The compressed layer travels to the right as a pulse of sound.

Consider a thin layer L of undisturbed air. The width of the layer is Δx, and its cross-sectional area is A. At time $t = t_0$ the compressional pulse just arrives at the left edge of L, as shown in Fig. E2(a) . The pulse then begins compressing L, increasing its density as shown in Fig. E2(b). At time $t = t_0 + \Delta t$ the entire layer has been compressed, as shown in Fig. E2(c). Note that if the pulse moves with velocity v, the layer-width Δx and time Δt are related by $\Delta x = v\,\Delta t$. We assume that the compression of L is small compared with Δx .

Fig. E2. (a) A compressional pulse arrives at the left edge of a layer of undisturbed air at time t_0. (b) The pulse compresses a portion of the layer for times t such that $t_0 < t < t_0 + \Delta t$. (c) The pulse reaches the right edge of the layer at time $t_0 + \Delta t$.

Now we can explore the dynamics of the layer L using $F = ma$. While the pulse moves through L, the pressure (i.e., the force/area) on the right face of L is p, the ambient pressure of the air in the column, while the pressure on the left face is $p + \Delta p$, where Δp is the extra pressure in the compressional pulse. The net force on L is therefore

$$F_{\text{net}} = (p + \Delta p)A - pA = \Delta p A , \tag{E1}$$

Fig. E3. The forces acting on layer L. The net force to the right is $\Delta p A$.

as shown in Fig. E3. The mass of air in L is $m = \rho A \Delta x$, the product of the mass-density ρ and the volume $A \Delta x$. Newton's second law therefore gives

$$F_{\text{net}} = \Delta p A = m a = (\rho A \Delta x) \frac{\Delta v}{\Delta t} \tag{E2}$$

where Δv is the velocity gained by the left-hand edge of the layer as the pulse arrives.*

* The rest of the layer eventually gains this velocity Δv as well.

Cancelling the A and using $\Delta x = v\,\Delta t$ results in

$$\Delta p = \rho\, v\, \Delta \mathrm{v}\,, \tag{E3}$$

showing how the extra pressure Δp in the pulse is related to the velocity gain Δv of the layer. During the time Δt it takes the pulse to move through the layer, the left-hand edge moves the distance $\Delta v\,\Delta t$, while the right-hand edge doesn't move at all, because it only gains speed Δv at the very end of the time interval. The layer has therefore been compressed by $\Delta v\,\Delta t$. That is, the volume change ΔV of the layer has been

$$\Delta V = -\,\Delta v\,\Delta t\;A\,, \tag{E4}$$

negative because the volume has *decreased*. The mass in the layer has not changed, however, so the density of the layer must have *increased*. To see how much, we take the differential of the conserved mass $M = \rho\, V$ (using small finite differences); that is,

$$\Delta M = \rho\,\Delta V + \Delta\rho\, V = 0\,, \tag{E5}$$

so the density change is

$$\Delta\rho = -\,\rho\,\frac{\Delta V}{V} \tag{E6a}$$

$$= -\,\rho\,\frac{-\,\Delta v \Delta t\, A}{\Delta x\, A} = \rho\,\frac{\Delta v}{v} \tag{E6b}$$

since $\Delta x = v\,\Delta t$. The velocity change is

$$\Delta v = \frac{\Delta p}{\rho\, v} \tag{E7}$$

according to Eq. E3, so Eq. E6(b) becomes

$$\Delta\rho = \frac{\Delta p}{v^2}\,. \tag{E8}$$

The speed of the sound pulse is therefore

$$v = \sqrt{\frac{\Delta p}{\Delta \rho}} = \sqrt{\frac{dp}{d\rho}}, \tag{E9}$$

taking the limit as the pressure and density changes are continuous and small. This equation shows that the velocity of sound in a medium is related to how much the density of the medium changes when its pressure is increased.

There is *another* way to express the sound velocity which allows a useful comparison with the speed of waves on a string. The *bulk modulus* of a material is defined to be

$$B = -\frac{\Delta p}{\Delta V / V} \tag{E10}$$

where Δp is a pressure change imposed on the material, and $\Delta V / V$ is its resulting fractional change in volume . The minus sign is inserted in the definition to make B a positive quantity; we expect that ΔV will be negative (the material will shrink) as a result of a positive pressure increase. Stiffer materials have larger bulk moduli, because their shrinkage is less for a given pressure increase.

The fractional changes in volume and density of a material are related by $\frac{\Delta V}{V} = -\frac{\Delta \rho}{\rho}$ because of mass conservation (see Eq. E6(a)), so the bulk modulus can also be written

$$B = \frac{\Delta p}{\Delta \rho / \rho}. \tag{E11}$$

The velocity of the sound in the medium is therefore

$$v = \sqrt{\frac{\Delta p}{\Delta \rho}} = \sqrt{\frac{B}{\rho}} \tag{E12}$$

in terms of the bulk modulus and density. This last expression is reminiscent of the formula $v = \sqrt{\frac{F}{\mu}}$ for the velocity of waves on a string, in terms of the string's tension F and mass/length μ . The tension F is a measure of the stiffness of a string, just as the bulk modulus B is a measure of the stiffness of a medium; the mass/length μ is a measure of the inertia or "sluggishness" of the one-dimensional string, just as the mass/volume ρ is a measure of the inertia of a three-dimensional medium.

Sample Problem E1. The bulk modulus of water is $B = 2.21 \times 10^9$ N/m^2 at 1 atmosphere of pressure and at temperature $T = 20^\circ$ C.

(a) A volume of water at the surface of the ocean has temperature 20° C. What would be the fractional compression in its volume if it were transported to the bottom of an ocean 1 km deep, if its temperature remained constant?

(b) What is the speed of sound in water at $T = 20^\circ$ C?

Solution: (a) The bulk modulus of water is large, meaning that water is "stiff": It does not compress much, even under very high pressures. The pressure 1 km beneath the surface of the ocean can be determined by finding how much force is needed to support a column of water 1 km high. A 1 m^2 area on the bottom of a 1 km deep ocean has above it a column of water of volume 1 m^2 x 1 km = 1000 m^3. The density of water is 1000 kg/m^3 (= 1 g/cm^3), so the mass of water in the column is 10^6 kg. The weight of this much water is

$$W = mg = 10^6 \text{ kg} \times 9.8 \text{ m/s}^2 = 9.8 \times 10^6 \text{ N},$$

so the pressure at this depth is $p = F/A = 9.8 \times 10^6$ N/m^2, equivalent to 0.7 tons per square inch! The bulk modulus is related to the change in pressure and the fractional volume change by $B = -\dfrac{\Delta p}{\Delta V / V}$; the fractional volume change is therefore

$$\frac{\Delta V}{V} = -\frac{\Delta p}{B} = -\frac{9.8 \times 10^6 \text{ N/m}^2}{2.21 \times 10^9 \text{ N/m}^2} = 4.4 \times 10^{-3},$$

a bit less than one half of one percent.

(b) The velocity of sound in water is

$$v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{2.21 \times 10^9 \text{ N/m}^2}{1000 \text{ kg/m}^3}} = 1{,}500 \text{ m/s}$$

to two significant figures. Even though water has a much higher density than air, water is so much "stiffer" than air that the speed of sound in water is about $4\frac{1}{2}$ times as large as it is in air at standard temperature and pressure.

The speed of sound in air

In an ideal gas the molecules behave like particles of negligible size which have no significant long-range attraction or repulsion for one another. Dilute gases like air at typical pressures and temperatures on Earth are close enough to ideal for most purposes. Even the gas in the center of the Sun, which at a density many times that of water is certainly not dilute, is close to being ideal because of its very high (15×10^6 K) temperature. To find the speed of sound $v = \sqrt{\frac{dp}{d\rho}}$ for an ideal gas we need to relate the pressure of a gas to its density.

The pressure p of an ideal gas is proportional to both the number of molecules per unit volume $\frac{N}{V}$ and to the temperature T; the *ideal gas law* is consequently

$$p = \frac{N}{V} kT$$

where k is a constant of proportionality called *Boltzmann's constant.* The constant has dimensions of energy/temperature, and has the value $k = 1.38066 \cdot 10^{-23}$ J/K, where K stands for degrees Kelvin. The ideal gas law is also often used in the form

$$pV = nRT,$$

where n is the number of *moles* of gas, $n = N / N_0$, where $N_0 = 6.02 \times 10^{23}$ mol^{-1} is *Avogadro's number*, the number of molecules per mole, the same for any species of gas. Here $R = 8.31$ J / mol · K is called the *universal gas constant.* Comparison of the two forms of the ideal gas law shows that R is related to Boltzmann's constant by $R = N_0 k$.

To find the speed of sound we need to relate some of the parameters in the ideal gas law to the mass density ρ. The mass of the gas m in *grams* is proportional to the number of moles n by $m = Mn$, where M is called the *molar mass* of the substance (also sometimes called the "molecular weight"). For example, $M = 2 \times 14 = 28$ g/mol for nitrogen gas (N_2), and $M = 2 \times 16 = 32$ g/mol for oxygen gas (O_2). That is, one mole of N_2 gas has a mass of 28 g, while one mole of O_2 gas has a mass of 32 g. The mass density of a gas is $\rho = \frac{m}{V} = M \frac{n}{V}$, so the ideal gas law can also be written in the useful form

$$p = \rho \cdot \frac{RT}{M}.$$

First assume that the temperature of the gas remains constant as the sound wave propagates.

Under this assumption the sound velocity should be $v = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{RT}{M}}$, found by differentiating the gas law above keeping T fixed. We can then predict the sound velocity in air and compare the result with experiment. Dry air is about 78% N_2 and 20% O_2 (the rest is mostly Argon and CO_2), so we will take the average molar mass to be $M = 29$ g/mol. The velocity of sound in air at 0° C (273 K) is then predicted to be

$$v = \sqrt{\frac{RT}{M}} = \sqrt{\frac{(8.31 \text{ J/mol} \cdot \text{K})\,(273 \text{ K})}{29 \text{ g/mol}}}$$

$$= \sqrt{\frac{(8.31 \text{ J/mol} \cdot \text{K})\,(273 \text{ K})}{29 \cdot 10^{-3} \text{ kg/mol}}} = 280 \text{ m/s}.$$

Although this result has the right order of magnitude, it does not agree with the observed velocity $v = 331$ m/s. Something is wrong!

What is wrong is the assumption that air remains isothermal (i.e., that the temperature T stays fixed) as the sound wave travels. A considerably better assumption is that the air remains *adiabatic* (i.e., that no heat leaks in or out of the air) as the wave travels. The reason is that the wave travels so fast that there is no time for an appreciable amount of heat-energy to enter or leave any part of the air. For such adiabatic processes, the ideal gas law takes the form $pV^\gamma = constant$, where $\gamma = C_p/C_v$ is the *ratio of specific heats*, the ratio of the specific heat at constant pressure to the specific heat at constant volume. The gas density ρ is proportional to $1/V$, so the adiabatic law can also be written

$$p = constant \times \rho^\gamma.$$

For an ideal gas $C_p = C_v + R$, where R is the gas constant; also $C_v = \frac{3}{2}R$ for a monatomic gas, $C_v = \frac{5}{2}R$ for a diatomic gas, etc. Air is predominantly diatomic, so we take $\gamma = \frac{7/2\,R}{5/2\,R} = \frac{7}{5}$. Therefore $\frac{dp}{d\rho} = constant \cdot \gamma\rho^{\gamma-1} = \left(\frac{p}{\rho^\gamma}\right)\gamma\rho^{\gamma-1} = \gamma\frac{p}{\rho} = \gamma\frac{RT}{M}$.
The velocity of sound is therefore predicted to be $\sqrt{7/5}$ greater than before; that is,

$$v = \sqrt{\gamma\frac{RT}{M}} = \sqrt{\frac{7}{5}} \cdot 280 \text{ m/s} = 331 \text{ m/s},$$

in excellent agreement with experiment.

PICTURE QUIZ E

1. Someone is yelling at you from far away, but you can't make out what they are saying. So you walk toward them so you are only one-tenth as far away as before. Assuming the sound spreads out is three dimensions without reflections,

(a) The intensity of the sound is thereby increased by about a factor of

(A) $\sqrt{10}$ (B) 10 (C) $\sqrt{1000}$ (D) 100 (E) 1000

(b) and the sound level is increased by about

(A) 2 (B) 10 (C) 20 (D) a factor of 10 (E) a factor of 100

2. A rope is under a certain tension. Sinusoidal waves of a particular wavelength and amplitude travel along the rope.

(a) Suppose the amplitude of the wave were tripled, keeping the wavelength the same. The power carried by the wave would increase by a factor

(A) 1/9 (B) 1/3 (C) 1 (D) 3 (E) 9

(b) Suppose the wavelength of the wave were cut in half, keeping the amplitude the same. The power carried by the wave would increase by a factor

(A) 1/4 (B) 1/2 (C) 1 (D) 2 (E) 4

3. A child waiting on the sidewalk hears the ice-cream truck approaching. A note with frequency 440 Hz in the melody broadcast by the truck is heard as frequency 462 Hz by the child . How fast is the truck moving, assuming it is moving directly toward the child? (The speed of sound that afternoon is 360 m/s.)

(A) 9 m/s (20 mph) (B) 13 m/s (30 mph) (C)18 m/s (40 mph) (D) 22 m/s (50 mph)

PROBLEM QUIZ E

1 . The velocity of a wave of wavelength λ and frequency ν is $v = \lambda\nu = \frac{\omega}{k}$ in terms of its angular frequency ω and wave number k. This is also often called the *phase velocity* v_p of the wave. The *group velocity* v_g of a continuous distribution of waves of different wavelength and frequency is defined to be $v_g = \frac{\partial\omega}{\partial k}$. The group velocity is the speed of a localized wave packet composed of many different sinusoidal waves whose wave numbers k are close to one another, as illustrated. To find it, we must first find ω as a function of k, and then differentiate. Find the group velocity of surface water waves in the cases of (a) shallow water waves (b) deep water waves, expressed as a multiple of the phase velocity. (The phase velocity of shallow-water waves is $v_p = \sqrt{gh}$ and the phase velocity of deep-water waves is $v_p = \sqrt{\frac{g\lambda}{2\pi}}$.)

wave packet

2. The intensity of a sound wave is the average power in the wave per unit area, $I = \frac{<P>}{A}$. The average power in a sound wave is proportional to its amplitude squared (just as for a wave on a string). For a sound wave, the amplitude is the maximum pressure difference Δp in the wave relative to the ambient pressure of the fluid. The intensity depends also on the mass density ρ of the fluid and the velocity v of the wave. Fluid A has density ρ_A, and fluid B has density $\rho_B = 2\rho_A$. Sound waves are propagated in each, with the same speed and the same maximum pressure difference. Find the ratio $\frac{I_B}{I_A}$ of the sound intensities in the two fluids.

3. The noon factory whistle goes off on Tuesday, a still day. A worker standing a mile away hears the pitch as 950 Hz.

(a) At noon on Wednesday, the worker is driving toward the factory at 15 m/s on a still day. What pitch will the worker hear?

(b) At noon on Thursday, the worker is standing a mile away. A steady wind is blowing at 15 m/s, directly from the factory toward the worker. What pitch will the worker hear?

Questions 4 and 5 are dependent on Appendix E I

4. Helium is a monatomic gas at 0° C; its molecular weight is 4.0 g/mol. What is the speed of sound in helium gas at this temperature? Compare with the speed of sound in air at the same temperature. (Assume helium is an ideal gas, a good assumption at atmospheric pressure and 0° C.)

5. The speed of sound in ethanol is 1207 m/s at 25° C, and the density of ethanol at that temperature is 0.79 g/cm^3. Ethanol is placed inside a cylinder of length 1.0 m and radius 3.0 cm, with a piston at one end. What force would you have to exert on the piston to compress the ethanol by 1.0 mm?

Special Topic: Vocalization in Dolphins, Porpoises, and Whales

It is hard to see very far in seawater, but it is easy to hear. Sounds, especially low-frequency sounds, can travel great distances underwater. It is also true that the speed of sound is over four times as great in water as in air, so quite rapid underwater communication is possible. While humans are the beneficiary of a highly evolved visual system, cetaceans (mammals including dolphins, porpoises, and whales) have evolved highly discriminating auditory systems. And they have gone us one better: While human bodies are equipped only for passive reception of visual signals (we haven't yet evolved headlights in our forehead to light up the surroundings at night), dolphins, porpoises, and whales not only receive and make brain "pictures" of the sounds around them; they actively broadcast a wide variety of sounds and listen for echoes to navigate or detect prey.

Two thousand years ago Aristotle and Pliny the Elder both wrote about the squeaks and clicks of captured dolphins. They did not understand their purpose. Researchers have since recorded and analyzed dolphins' steady stream of clicks. Each click is short, on the order of milliseconds; the clicks often vary in intensity and they contain a wide range of frequencies. They can be focussed into highly directional beams, which bounce off fish or seafloor features, producing complex echoes.

How much can dolphins accomplish without using their eyes? It turns out that they can easily navigate at night and avoid complicated objects while blindfolded. They can read sound echoes to distinguish fine details, assembline a "sound picture" in their brains. They can identify types of fish and where the fish are located.

Researchers on Long Island, New York, studied a 25-foot juvenile sperm whale they named "Physty", who was stranded in a boat basin.*

> In the evenings he would come to life; actively swimming and click-training the pier posts and everything in the basin with his "sonar".
>
> His clicks were just unbelievable. When you were standing on this solid wooden dock, or bulkhead filled with dirt, there was no air space between you and the water. When Physty clicked in your direction, you could actually feel the vibration of his clicks coming up through the soles of your shoes.

* "Whale Tales" (Vol 1) collected by Peter J. Fromm, Whale Tales Press, Friday Harbor, Washington.

Each species emits distinct sounds. Aside from the clicks, humpback whales "sing" complex tunes and fin whales deliver basso 20 Hz blasts that may be heard for hundreds of miles. Whales probably use these sounds to communicate or just keep in touch.

How do cetaceans vocalize? The picture below illustrates how orcas ("killer whales") produce their sounds. They force air through nasal sacs and cavities that can be rapidly opened and closed. The resulting sounds strike air sacs within the skull that behave somewhat like parabolic reflectors, reflecting the sounds toward a fatty area in the forehead called the "melon". The melon is an acoustic lens that focusses sounds into a directional beam as they leave the head, as shown in the diagram on the left.*

How do orcas hear the resulting echoes? They receive sounds through a very thin flattened section of the lower jaw called an "acoustic window", as shown on the right. The sounds are picked up by the fat body in the hollow jawbone and carried to the middle and inner ears. The ear bones are encased in a dense ivory-like bone called the bulla, which floats in a bubbly foam. The foam reduces interference from other sounds, including the orca's own vocalizations, by isolating the inner ear from the skull.

Orcas (and other species) are excellent mimics. A researcher who had listened to their sound patterns used an electronic synthesizer to try to mimic their sounds. A pod of orcas swam nearby.

> Then several whales, louder than before, directed a series of terse vocalizations toward the hydrophone suspended under the boat. With no notion of what to expect, I flipped on the tape recorder and turned up the volume pots on the synthesizer. I pressed the keys in the pattern I had devised, monitoring the imitation whale phrase as it passed out the underwater speaker. I held my breath. Two seconds went by. And then it came: A chorus of whales - three, maybe four - sang out a clear, perfect imitation of what I had just played to them - in harmony! They did not repeat their own sound; rather, they duplicated my human accent!

* "Orca: The Whale Called Killer", by Erich Hoyt; A Firefly Book, printed and bound in Canada by Friesens, Altona, Manitoba (1990). (The drawings above are taken from Hoyt's book.) See also "Whales, Dolphins and Porpoises", by Adrienne Mason; Altitude Publishing The Canadian Rockies, Vancouver, and Denver (1999), and "Dolphins Whistle a Signature Tune", by Peter Tyack, *Science* 289, p. 1310 (2000)

Questions and Problems from Oscillations and Waves, Chapter E

EQ1.Why is there a lower limit to the wavelength sound waves can have if they are propagated through air of some given density?

EQ2. In the evening, air near the ground sometimes cools faster than air higher up. At such times a hiker can sometimes easily hear the conversation of other hikers many tens of meters away. Explain why sound waves *bend* in conditions like this, and why this bending amplifies the sound.

EQ3. When tearing a piece of paper, there is a characteristic sound. If you tear the piece of paper *faster*, the tearing sound has a higher pitch. Why?

EQ4. There is a rule of thumb for figuring out how far you are from a lightning stroke. After seeing the flash, count seconds until the thunder is heard; then divide by five. The result is the distance of the stroke in miles. Why does this work? How accurate is it?

EQ5. What are the dimensions of B, the bulk modulus, found from its definition? If the velocity v of a sound wave depends upon B and the density ρ of the material, how must v depend upon B and ρ, using dimensional reasoning alone? (See Appendix E I)

EQ6. In an active sonar system, a submarine sends out sound pulses and detects echos from other objects. How can it be used to detect both the distance of objects and their velocities?

EQ7. High-frequency sound ("ultrasound") can be directed into the human body as a diagnostic tool. Why is ultrasound used (rather than audible sound waves), how might such a device work, and what could physicians learn?

EQ8. In his book *The Theory of Sound*, the British physicist Lord Rayleigh stated that if an observer traveled away from a musical performance at twice the speed of sound, the observer "would hear a musical piece in the correct time and tune, but backwards." Do you agree?

EQ9. Is there a Doppler shift if the source and observer are both moving with the same velocity relative to the air, with the observer chasing the source? What if the source is chasing the observer?

EQ10. Sound wave A delivers 2 J of energy in 2 s. Sound wave B delivers 10 J of energy in 5 s. Sound wave C delivers 2 mJ of energy in 1 ms. Rank in order, from largest to smallest, the sound powers P_A, P_B, P_C of these three waves.

Section E.2 Sound

EP1. At the temperature 0 C the speed of sound in air is 331 m/s. (a) What is the speed of sound in air at room temperature, 20 C? (b) The temperature of air usual drops as one gains altitude away from Earth's surface. What happens to the speed of sound?

EP2. Medical technicians use ultrasound to measure the flow of blood in the carotid artery. If the ultrasound frequency is 41 MHz, find the wavelength of this sound wave in (a) air at 200 C; (b) human tissue, assuming the speed of sound is that in water, 1500 m/s.

EP3. Neutron stars are often formed in supernova explosions. They are extremely dense stars comprised primarily of neutrons. (a) For a certain density range (around 10^{12} g/cm^3) the pressure P within the star is given by an equation of the form $P = k\hbar^2 M^m \rho^n$, where k is a dimensionless constant, $\hbar$ is Planck's constant (with the dimensions of energy $\times$ time), M is the mass of a single neutron, and ρ is the mass density (mass/volume) of the star. Find the exponents m and n using dimensional reasoning. (b) Find an expression for the speed of sound waves within the star, in terms of $\hbar, M, k$, and ρ. (c) At still higher densities (of order 10^{16} g/cm^3) the pressure within the star is proportional to the four-thirds power of density ($P \sim \rho^{4/3}$), and P depends also upon $\hbar, M$,and c, the speed of light. How does P depend upon these three quantities? (d) Find an expression for the speed of sound within the star in this density range, in terms of h, M, c, ρ, and a dimensionless constant.

EP4. One way to monitor global warming is to measure the average temperature of the ocean. Researchers are doing this by measuring the time it takes sound pulses to travel underwater over large distances. At a depth of 1000 m, where ocean temperatures hold steady near 4 C, the average sound speed is 1480 m/s. It is known from laboratory measurements that the sound speed increases 4.0 m/s for every 1.0 C increase in temperature. In one experiment, where sounds generated near California are detected in the South Pacific, the sound waves travel 8000 km. If the smallest time change that can be reliably detected is 1.0 s, what is the smallest change in average temperature that can be measured?

EP5. An ultrasound unit sends a 2.4 MHz sound wave into a 25-cm long tube filled with an unknown liquid. A small microphone right next to the ultrasound generator detects both the transmitted wave and the sound wave that has reflected off the far end of the tube. The two sound pulses are 4.4 divisions apart on an oscilloscope for which the horizontal time sweep is set to 100 μs/division. What is the speed of sound in the liquid?

Section E.3 Seismic Waves

EP6. HMC seismographs registered seismic information from an October 20, 1996 aftershock of the Landers earthquake, including the arrival of the P and S waves as shown below. Typically, P waves travel at 5.62 km/s and S waves at 3.33 km/s in Earth's upper crust in Southern California. How far from Claremont was the location of the seismic event? [Data obtained by G.A. Lyzenga]

EP7. In a torsional wave, a twist of some sort is propagated through a solid medium. Consider for example a horizontal wire supported at several points along its length, so the wire is forced to remain straight, although it is free to twist lengthwise (i.e., about an axis that runs along the center of the wire). A number of thin rods are attached at their centers to regularly-spaced points along the wire, as shown.

The rods are also horizontal, but perpendicular to the wire. If the entire assembly is sitting in equilibrium and one of the end rods is suddenly turned away from its initial horizontal orientation by some angle, a pulse propagates along the system, so that different rods turn back and forth out of the horizontal plane, as illustrated. The parameters that define the system are the moment of inertia I of each rod, the spacing Δx between successive rods along the wire, and the torsion constant κ of each section of wire. The torsion constant is defined as follows: Let a wire section be the piece of wire between two neighboring rods. Fix one end of the wire section so it cannot twist, and then apply a torque τ to the other end of the section, where τ is directed along an axis running along the center of the wire. The end of the wire to which τ has been applied then twists by an angle θ, where the magnitudes of torque and angle are related by $\tau = \kappa\theta$. (a) Using dimensional reasoning, find how the velocity v of torsional waves through the system depends upon the three parameters. (b) Derive the wave equation

$$\frac{\partial^2\theta}{\partial x^2} = \left(\frac{1}{v^2}\right)\frac{\partial^2\theta}{\partial t^2}$$

for the torsional waves in the system, where v is the speed of the waves, which depends in some definite way on I, κ, and Δx. Here θ is the angle of a rod relative to the horizontal at time t, if the rod is located at x . *Hint*: Consider

three successive rods, No. 1, No. 2, and No. 3, tilted by angles $\theta_1, \theta_2, \theta_3$ at time t. The torque on No. 2 due to No. 1 is then

$$\tau_1 = \kappa(\theta_1 - \theta_2) = -\kappa(\theta_2 - \theta_1) \cong -\kappa \frac{\partial \theta}{\partial x}|_{1,2} \Delta x$$

by first-order Taylor series. Here $\partial\theta/\partial x|_{1,2}$ is the rate at which θ increases with x in the wire between rods No. 1 and No. 2. Find likewise the torque on No. 2 due to No. 3, and then find the total torque on No. 2. Use first-order Taylor series again to show that this total torque on No. 2 is approximately

$$\tau_2 = \kappa \left(\frac{\partial^2 \theta}{\partial x^2} \Delta x \right) \Delta x.$$

Section E.4 Surface water waves

EP8. What is the speed of a tsunami wave of wavelength 53 km if the ocean is 2.6 km deep? How long would it take the wave to travel 100 km if the depth remains constant?

EP9. What is the speed v of a deep-water wave in terms of its frequency ν_0 and g? If a fisherman counts 2.0 s between the times his boat is in a trough and a crest, what is the wave speed v? What is the wavelength λ of the wave in that case?

EP10. The speed v of surface water waves depends in general upon gravity g, the depth of water h, and the wavelength λ. Using physical arguments, including dimensional reasoning, what should be the ratio of the speeds of two water waves on two planets whose gravities differ by a factor of two, if h and λ are the same for both waves?

EP11. It can be shown that the velocity of surface water waves is

$$v = \sqrt{\frac{g\lambda}{2\pi} \tanh\left(\frac{2\pi h}{\lambda}\right)},$$

where g is the gravitational field, λ is the wavelength, and h is the water depth. The hyperbolic tangent of a quantity x is defined to be $\tanh x = (e^x - e^{-x})/(e^x + e^{-x})$. (a) For deep-water waves, $h >> \lambda$. Show in this case that the equation above reduces approximately to $v = \sqrt{g\lambda/2\pi}$. (b) What is the fractional error in using the deep-water formula for v if $h/\lambda = 1$? (c) For shallow-water waves, $h << \lambda$. Show in this case that to a good approximation $v = \sqrt{gh}$. (d) What is the fractional error in using the shallow-water formula for v it $h/\lambda = 1$? [Two useful approximations are $(1+x)^n \cong 1 + nx$ if $n << 1$ (binomial series), and $e^x \cong 1 + x$ if $x << 1$.]

EP12. A rule of thumb used by ship designers is that the speed of a ship through the water cannot exceed the speed of an ocean wave of wavelength equal to the length of the ship's hull. (This rule does not apply to boats like hydrofoils that skim along the surface of the water, relying on hydrodynamic lift to keep most of their hull above the water.) This limiting speed is called the *hull speed* of the ship. The reason for this rule is that a ship sailing through the water makes waves; when the speed of the ship reaches the hull speed, the crest of one of these waves will be at the bow and the crest of the next will be at the stem. The ship is then permanently trapped in a wave trough – any increase of engine or sail power merely pushes the ship against the wave crest at its bow; this feeds more energy into the wave, making it higher, but produces only an insignificant increase in the ship's speed. (a) Show that the hull speed (in m/s) is given by $v = 1.2\sqrt{L}$,where L is the length of the ship in meters. Ship designers often measure speed in knots (1 knot = 1 nautical mile/hr = 1.15 mile/hr) and length in feet. Show that in these units the formula becomes $v = 1.3\sqrt{L}$. (b) What is the hull speed for a sailboat of length 30 ft? A clipper ship of length 200 ft? The Queen Elizabeth II ocean liner of length 1013 ft?

EP13. The surface tension of liquids arises from molecular forces. If a wire frame is dipped into a bowl of water, a thin film of water can be drawn upward by a horizontal sliding wire as shown.

It is the surface tension σ of water that supports the film against gravity. The upward force is $F = 2\sigma d$, where d is the horizontal width of the film; the factor of two is due to the fact that the film has two surfaces. Short-wavelength ($<$ 1 cm) waves on the surface of water are related to surface tension. The velocity v of such waves depends upon the density of water ρ and the wave number k as well as σ. (a) Use dimensional reasoning to find how the velocity v depends upon all three parameters. (b) Consider two such waves, the second with twice the wavelength of the first. What is the velocity of the second relative to that of the first?

Section E.5 Power flow in waves

EP14. A rope of length $L = 22$ m has mass $m = 5.5$ kg and tension $F = 250$ N. We want to send a sinusoidal wave of frequency $\nu = 0.50$ Hz along the rope, so it will transmit power $P = 35$ W as a time average. Assuming no energy losses, what must be the amplitude of the wave?

EP15. A rope with mass/length $\mu = 30$ g/m carries the wave $y(x,t) = 0.10\sin(0.75x - 12t)$, where x and y are in meters and t is in seconds. Find (a) the velocity of the wave, including both its magnitude and direction (b) the tension of the rope (c) the wavelength λ and frequency ν of the wave (d) the time-averaged power transmitted by the wave.

EP16. A string has mass/length μ and is under tension F . A motor can deliver the maximum average power P to the string. If the wave amplitude is A, (a) what is the maximum frequency at which the motor can shake the string? (b) what is the minimum wavelength the wave can attain?

EP17. Prove that the time averages $< \cos^2 \omega t >$ and $< \sin^2 \omega t >$ are each 1/2. *Hint*: The time average of any repeating function $f(t)$ is

$$< f(t) >= \frac{1}{T}\int_0^T f(t)dt,$$

the time integral of the function over one period, divided by the period itself.

EP18. A spherical sound wave travels out isotropically (i.e., the same in all directions) from a small source at $r = 0$. (a) Explain why the displacement of the medium $y(r,t)$ could be given by the expression

$$y = \frac{A}{r^n}\sin k(r - vt),$$

where A and k are constants, r is the distance from the source and v is the wave speed. What value of the exponent n is appropriate in this case? (b) What are the dimensions of A? (c) Find a similar expression representing an incoming wave.

EP19. A cylindrically-symmetric sound wave is emitted by a very long line source. The wave travels outward, so the front of the wave covers a cylindrical surface of ever-increasing radius. (a) What is the correct expression for $y(r,t)$ in this case, where y is the displacement and r is the perpendicular distance from the line source? What is the correct value of the exponent n in this case, using the expression given in EP18? (b) Write a similar expression representing an ingoing wave, moving inward toward the central line.

EP20. In the open sea, where the depth h of the water is large, the amplitude of an earthquake-caused *tsunami* is small. When the tsunami enters coastal shallows, where h is small, the amplitude becomes large. (a) Show that the amplitude varies as $A \sim 1/h^{1/4}$. Note that the wavelength of a tsunami is a few hundred kilometers, much larger than the depth of the ocean floor. (*Hint:* As in the case of a wave on a string, the power carried by a water wave is proportional to $v\omega^2 A^2$, if the wave is a "linear" (small-amplitude) wave. Assume that the power in the wave remains constant as it enters the coastal shallows; also note that the frequency remains constant.) (b) Suppose that a tsunami has an amplitude of 4.0 m in the open sea, where $h = 5.0$ km. If this wave reaches a coastal region where $h = 20$ m, what will be its amplitude? [Strictly speaking, one should not use the linear approximation for a tsunami near the shore. In the open ocean the wavelength is much larger than the amplitude, so the linear approximation is good. Near the shore the wavelength becomes much shorter and the amplitude much larger, so the linear approximation is not very good.]

EP21. The sound level of an orchestra passage to a listener in the third balcony is SL= 40 db. If the listener moves up to an empty seat only a third as far away from the orchestra, what would be the sound level of the same passage? (Assume that the sound spreads out isotropically in three dimensions.)

EP22. Two people are conversing at a restaurant table, producing an average sound level of 50 db in the room. How many couples in the same room would bring the average sound level up to 65 db?

EP23. The velocity of sound in the ocean depends upon depth. About 1500 m below the surface there is a minimum in the sound velocity, so that sound waves a bit above this depth are deflected downward, and waves a bit below this depth are deflected upward. The result is the SOFAR channel, in which sound waves spread out from a source throughout the two-dimensional channel rather than spreading out in three dimensions. Consider a sound source at the SOFAR depth, whose intensity is I_0 at a point 100 m from the source. (a) In terms of I_0, what is its intensity 1 km from the source? (b) What would be its intensity at 1 km if the wave spread out uniformly in three dimensions? (The sound from small charges exploded at the SOFAR depths has been detected thousands of kilometers from the source.) (c) Why might Navy submarine captains be interested in the SOFAR channel?

Section E.6 The Doppler effect

EP24. A sound source which is stationary relative to the air emits sound of frequency ν_{em}. Draw a figure analogous to Fig. 6, except that the observer is moving with speed v_{ob} *away* from the emitter. Then find the frequency ν_{ob} detected by such an observer in terms of ν_{em} in a manner similar to that used in the text following Fig. 6.

EP25. An observer is at rest relative to the air. An emitter moves away from the observer at speed v_{em}, emitting sound of frequency ν_{em}. Draw a figure analogous to Fig. 7, except that the emitter is moving *away* rather than toward the observer. Find the frequency v_{ob} detected by the observer in terms of ν_{em} using an analysis similar to that above and below Fig. 7.

EP26. An ambulance traveling at 80 km/hr sounds its siren at 1600 Hz while passing a pedestrian. (a) What is the net change in frequency heard by the pedestrian? (b) After passing by, the ambulance approaches the side of a hospital at the end of the street. What does the pedestrian hear when the reflected sound off the hospital wall is superimposed on the direct sound from the ambulance?

EP27. The text derives the frequency heard by an observer when the observer is at rest relative to the air and the emitter is moving at speed v toward the observer, and also the frequency heard by an observer when the emitter is at rest and the observer is moving at speed v toward the emitter. Calculate the approximate difference between these two frequencies, valid to second order in the ratio v/v_s, where v_s is the speed of sound. [In the case of *light*, Einstein postulated in 1905 that there is no medium that carries light analogous to the air that carries sound, so that only the *relative* velocity between the emitter and observer can matter; therefore for light the correct Doppler formulas must not depend upon which is "moving."]

EP28. With the help of careful pictures, derive Eq. 17, the general expression for the frequency ν_{ob} heard by an observer in terms of the frequency ν_{em} emitted by an emitter, for the case where both are moving relative to the air toward one another, the observer at velocity v_{ob} and the emitter at velocity ν_{em}. The speed of sound relative to air is v_s. This is the case where the upper signs are appropriate in Eq. 17.

EP29. Bats navigate in the dark with the help of ultrasonic chirps, short emissions of high-frequency sound lasting a millisecond or less and repeated several times a second. Assuming the sound made by the bats has a frequency ν = 40 kHz, if a bat is flying toward a wall with a speed v_b = 10 m/s, what is the frequency of the sound reflected off the wall as heard by the bat (velocity of sound v_s = 330 m/s)?

EP30. A "hunter" submarine, moving at speed v_h through a calm ocean, is tailing a second, missile-carrying sub moving away at speed v_m.

The speed of sound in water is v_s. The chasing sub sends out a sonar signal of frequency ν_e. (a) What is the sonar frequency ν_a detected by the missile sub? (b) If the signal reflects off the missile sub, what frequency ν_b is received by the hunter sub's sonar detector? (c) Suppose v_h = 35 km/hr, v_m = 28km/hr, v_s = 1500 m/s. Find the frequencies ν_a and ν_b numerically, if ν_e = 1660 Hz.

EP31. Doppler-effect sound frequencies depend upon whether the observer is moving toward the source or the emitter is moving toward the observer. (a) If the relative speed between the emitter and observer is 10% the speed of sound, find the fractional Doppler shift $\Delta\nu/\nu$ if, relative to the air, (i) only the observer moves (ii) only the emitter moves. (b) In general, the Doppler shifts are nearly the same in the two cases if the relative speed is small compared with the speed of sound. To show this, first derive the binomial expansion

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+...$$

from Taylor's series

$$f(x) = f(x_0)+\frac{df(x)}{dx}|_{x_0}(x-x_0)+\frac{1}{2!}\frac{d^2f(x)}{dx^2}(x-x_0)^2+...$$

(c) Then show, using the binomial expansion, that the moving-emitter frequency, as given by Eq. 16(b), differs from the moving-observer frequency, as given by Eq. 14, by terms of order $(v_{em,ob}/v_s)^2$ or smaller, where $v_{em,ob}$ is the relative speed between the emitter and observer, and v_s is the speed of sound. Thus if $v_{em,ob}$ = 0.1, the difference in the predictions is only about 1%.

EP32. A whistle you use to call your dog has a frequency of 21 kHz, but your dog is ignoring it. You suspect the whistle may not be working, but you cannot hear sounds above 20 kHz. To test it, you ask a friend to blow the whistle, then you hop on your bicycle. Should you drive toward or away from your friend to hear the whistle, and what minimum speed must you have?

EP33. A physics professor demonstrates the Doppler effect by tying a 600 Hz sound generator to a 1.0 m long rope and whirling it around her head in a horizontal circle at 100 rpm. What are the highest and lowest frequencies heard by a student in the classroom?

Problems on sound using Appendix EI

EP34. In the Mariana Trench of the western North Pacific, there is a spot where the ocean is 11,034 m deep. What would be the fractional increase in volume of a sample of water raised from this depth to the surface? Use the bulk modulus $B = 2.2 \times 10^9$ N/m^2.

EP35. (a) Aluminum has a bulk modulus $B = 7.0 \times 10^10$ N/m^2 and a density 2.7 times that of water. What is the speed of sound in aluminum? How does it compare with the speed of sound in water? (b) Repeat the problem for lead, which has $B = 7.7 \times 10^9$ N/m^2 and a density 11 times that of water.

F. The Superposition of Waves

"It ain't over till it's over." - Yogi Berra

Often we need to *superimpose* waves. Superposition is simply the addition of two or more waves. A high-school band might include 76 trombones, for example; the sound wave from all of them is the superposition of the waves due to each individual trombone. Intensity, "richness" of tone, and interference are just three of the many phenomena associated with the superposition of waves.

F.1 The Principle of Superposition

The wave equation on a rope

$$\frac{\partial^2 y}{\partial x^2} = \left(\frac{\mu}{F}\right)\frac{\partial^2 y}{\partial t^2} \tag{1}$$

is a *linear* equation, because the displacement y occurs only to the first power in the equation.* An important mathematical feature of such an equation is that if $y_1(x,t)$ and $y_2(x,t)$ are any two solutions of the equation, then the sum $y(x,t) = y_1(x,t) + y_2(x,t)$ is also a solution. In physics this feature is called the *principle of superposition*. If we create a wave on a rope and then later create a different wave, it is also possible to create a wave which is the sum of those two. We saw for example in Chapter D that the sum of a wave $f(x - vt)$ traveling to the right and a wave $g(x + vt)$ traveling to the left is also a solution of the wave equation for a rope.

Small-amplitude *sound* waves are also solutions of a wave equation. If we characterize sound waves by the excess pressure Δp (Δp is the pressure above the ambient air pressure) as a function of position and time, then for one-dimensional waves (like those moving along a long narrow pipe filled with air) the wave equation is

$$\frac{\partial^2(\Delta p)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2(\Delta p)}{\partial t^2}, \tag{2}$$

where the velocity of the waves is $v = \sqrt{\dfrac{dp}{d\rho}}$ in terms of the fluid pressure p and density ρ, as discussed in Chapter E.

*The wave equation, as introduced in Chapter D, is a linear second order partial differential equation.

The wave equation for sound is also linear, so the sum of two solutions is itself a solution. That is,

$$\Delta p(x, t) = \Delta p_1(x, t) + \Delta p_2(x, t) \tag{3}$$

is a solution if Δp_1 and Δp_2 are each solutions. For example, a violinist can play an "A" and later a bassoonist can play a "C" ; it is also possible for both of them to play at the same time so that the combination is a sound wave which is just the sum of the two individual waves.*

F.2 Standing Waves of Sound

An example of the principle of superposition is the addition of two sound waves traveling in opposite directions. As already discussed in Chapter D, if a wave traveling to the left on a string is added to a wave of the same frequency traveling to the right, the composite wave can be a standing wave. The same phenomenon takes place with sound.

Consider for example a cylindrical tube open at one end and closed at the other end. Sound from a source such as a vibrating tuning fork can enter the tube at the open end. The sound wave propagates down the tube and is reflected at the closed end. The sum of the original and reflected waves can form a standing wave if the frequency is just right. At the open end of the tube, the air inside is in contact with the air outside, so there is an approximate *pressure node* at this end, with constant pressure equal to the amibient outside air pressure. The air can slosh back at forth as much as it likes at the open end, so there is also an approximate *displacement antinode* at that location. At the closed end of the tube the flow of air is necessarily zero, since the air cannot move in and out of the tube. Therefore the closed end must correspond to a *displacement node*, analogous to the fixed end of a rope where there is no displacement. There is nothing to prevent the pressure from increasing and decreasing as much as it wants, so the closed end also corresponds to a *pressure antinode*.

* It is interesting that if both of the wave amplitudes are *not* fairly small, the linearity property is lost, and the principle of superposition is lost as well. The wave equations are then not as simple as those displayed in Eqs. 1 and 2, and the sum of two solutions of the new equations is not itself a solution. Physically, nonlinearity means the following: If two sound-waves are individually possible, one the low-amplitude sound of a clarinet and the other the high-amplitude crack of nearby thunder, if both happen at once the sound will not be exactly the sum of the sounds of each individually. The sound of the clarinet may be badly distorted, for example.

Now suppose the tube is open at *both* ends. Sound from a vibrating source at one end propagates into the tube. In spite of the fact that the far end of the pipe is also open, a portion of the wave is nevertheless reflected at the far end back into the tube, so that the waves moving in both directions can form a standing wave if the frequency is just right. In this case, the pressure at each end is approximately the constant ambient pressure of the outside air, so there are approximate *pressure nodes* at both ends. These also are also regions where the flow of air is maximal, because there is nothing to stand in the way of air motion, so both ends also correspond to *displacement antinodes*. The need to maintain these conditions at the far open end of the tube is the reason why some of the wave has to be reflected back into the tube. If there were no reflected wave, there would not always be a pressure node and displacement antinode located there.

Pipe organs and other wind instruments use these principles to make music. Air is blown into or past an open end of an organ pipe, in a manner that excites sound waves. The waves travel down the pipe and are reflected at the other end, almost perfectly if the other end is closed, and quite well even if the other end is open. This sets up standing waves in the pipe. Some of the sound leaks out, and is heard by listeners. What are the characteristic frequencies and wavelengths of the standing waves in an organ pipe?

First suppose one end of the pipe is open and the other end is closed. As we have seen, the closed end is a pressure antinode; pressure can achieve its highest and lowest values at that end. Modes in a pipe with one end open and the other end closed are shown in Fig. 1. The lowest mode, shown in Fig. 1(a), has a pressure node at the open end and an antinode at the closed end, so there is only one-quarter of a wavelength inside. The next mode has another antinode within the pipe as well, as shown in the first picture of Fig. 1(b); there are now three-quarters of a wavelength inside. In general

$$L = \frac{2n+1}{4}\lambda_n \qquad n = 0, 1, 2, \ldots \tag{4a}$$

so that the allowed wavelengths in the pipe are

$$\lambda_n = \frac{4L}{2n+1} \qquad n = 0, 1, 2, \ldots, \tag{4b}$$

and the corresponding allowed frequencies are

$$\nu_n = \frac{v_s}{\lambda_n} = (2n+1)\frac{v_s}{4L} \qquad n = 0, 1, 2, 3, \ldots \tag{5}$$

Note that the lowest overtone in an open-closed pipe has three times the frequency of the fundamental. In fact, the overtones in an open-closed pipe are all odd multiples of the fundamental frequency.

Fig. 1. Modes in a pipe open at one end and closed at the other end. (a) The fundamental, with a pressure node (and also a density) node at the open end, and a pressure antinode at the closed end. The upper pictures show the air density at five different times during a full cycle. The lower pictures show the air flow velocities at the same times. The velocity is zero when the density contrast is large, and large when the density is uniform. The largest airflow is at the open end of the pipe, and the air is stationary at the closed end. (b) A different, symbolic diagram of the fundamental, showing the pressure (and density) node and antinode, and a symbolic diagram of the node and antinode of the velocity flow. A node is where the two lines come together. Antinodes are where the pressure reaches its maximum and minimum values. (c) Symbolic pictures for the first overtone and two additional modes, with density and pressure shown at the top and velocity at the bottom.

Now consider a pipe that is open at both ends. In this case there has to be a pressure node at each end. The lowest wave mode corresponds to a wavelength which is twice the length of the pipe; that is, there is room for one-half wavelength inside the pipe, with a pressure node at each end and a pressure antinode in the middle of the pipe, as shown in Fig. 2(a). This is the largest wavelength and the smallest frequency allowed in the pipe; it is ordinarily the most prominent frequency heard when the pipe is played, the "fundamental" frequency. Figure 2(b) is a schematic

Fig. 2. Modes in an open-open organ pipe. (a) The lowest-frequency "fundamental" mode, with pressure nodes (and density nodes) at each end. The upper pictures show the fundamental at the instant when the pressure (and density) antinode in the middle is at its maximum, when the pressure and density are uniform, when the pressure and density are at a minimum in the center, and so on. The pressure and density at each end stay constant. The lower pictures show the air velocities at the same times. The velocity is zero when the density contrast is large, and large when the density is uniform. The largest air flow is at the ends of the pipe; the air is stationary at the center. (b) A symbolic diagram of the fundamental, showing pressure (and density) nodes at each end and an antinode in the center in the upper picture. Beneath it is a symbolic diagram of the air velocity. (c) Similar pictures for the first overtone, with density and pressure shown at the top and velocity at the bottom. (d) Symbolic representation of two additional modes.

representation of the same mode; nodes are where the two lines come together, and antinodes are where the lines are farthest apart. The next mode has a node at each end. one in the middle, and two antinodes spaced evenly between them, as shown in Fig. 2(c). There is now one full wavelength inside, so the frequency is twice that of the fundamental, representing a note one octave higher. Two additional modes are shown in Fig. 2(d), each with an additional half wavelength more than the one before it. That is, there have to be an integer number of half-wavelengths inside the pipe:

$$L = n\,\lambda_n/2 \qquad n = 1, 2, 3, \ldots \tag{6a}$$

so the allowed wavelengths are

$$\lambda_n = \frac{2L}{n} \qquad n = 1, 2, 3, \ldots \tag{6b}$$

The corresponding allowed frequencies are

$$\nu_n = \frac{v_s}{\lambda_n} = n\,\frac{v_s}{2L} \qquad n = 1, 2, 3, \ldots \tag{7}$$

where v_s is the speed of sound. The antinodes in the pipe are locations where the pressure reaches its maximum value and its minimum value. The pressure becomes a maximum when air has rushed in from both sides, that is from the neighboring nodes, to this point. And the pressure becomes a minimum when air has rushed away from this point in both directions toward the neighboring nodes. That is, air does not move at an antinode; an antinode is rather a place toward which air flows to form high pressure, and from which air flows to form low pressure.

Note that the mode frequencies in an open-open pipe are all integer multiples of the fundamental frequency. So if a pipe has a fundamental corresponding to a "C" (256 Hz, say), then the first overtone has twice the frequency, which is one octave higher (a "C" at 512 Hz). The next overtone has a frequency 768 Hz, and so on. The loudest sound typically comes from the fundamental, and the overtones become successively softer. The differences in overtones between open-closed and open-open organ pipes makes them sound somewhat different.

It is seldom worthwhile trying to memorize the various wavelength and frequency formulas for open-closed and open-open pipes. The important thing is to understand the physics of where they come from, namely, the requirement that there be a pressure node at an open end and a pressure antinode at a closed end. This determines the wavelengths (and therefore the frequencies) that can exist inside the pipe.

Music for the pipe organ includes some of the greatest music ever composed. A prime example is *Passacaglia and Fugue in C minor* by J. S. Bach (BWV 582). Starting with the eight-measure theme shown first below, there follows twenty variations (such as the one beginning below the theme) followed with a fugue on the same theme. All sounds are achieved with a combination of open-open and open-closed pipes by an organist using, in some of the variations, both hands and both feet.

A flute is somewhat similar to an open-open pipe. The flautist blows across an open mouthpiece, establishing air vibrations; the position of the other end of the flute can be changed by depressing keys and opening holes along the flute, thus changing the fundamental frequency. A similar sound can be generated by blowing across the mouth of an open soda bottle, except that now the pipe is closed at the far end. There are no keys on the bottle, but the fundamental frequency can be changed by filling the bottle with varying amounts of water. Clarinets, oboes, and brass instruments are also closer to pipes with one end open and one end closed; in these cases the mouthpiece is essentially closed (by a vibrating reed or vibrating lips) and the other end of the instrument is open.

Sample Problem 1. Someone blows across the open end of a cylindrical pipe of length $L = 30$ cm whose other end is closed. (a) What fundamental frequency is heard? (b) What is the frequency of the first overtone?

Solution: The open end of the pipe develops a pressure node, and the closed end a pressure antinode. (a) The largest wavelength consistent with these conditions is $\lambda = 4L$, with a quarter-wavelength of sound inside the pipe. Therefore $\lambda = 1.2$ m, with corresponding frequency $\nu = v/\lambda = 330$ m/s / 1.2 m = 275 Hz. This is the frequency of the fundamental. (b) The next shortest wavelength consistent with the end conditions corresponds to $3\lambda/4 = L$, with three-quarters of a wavelength inside the pipe. The wavelength is 1/3 as large as the fundamental, so the frequency is three times as large, $3 \cdot 275$ Hz = 825 Hz.

F.3 Beats

An interesting example of the principle of superposition, sometimes used by musicians to tune their instruments, is the phenomenon of *beats*. Suppose two sound waves of slightly different frequency are generated, by two violins for example. What will this sound like to a listener?

Figure 3(a) shows two waves of different frequency, both moving toward the right. At some locations the waves add constructively, with large amplitude, and at some locations they add destructively, with small amplitude, as in Fig. 3(b). The intensity of the sound, proportional to the square of the amplitude, is sketched in Fig. 3(c). As the waves reach our ears, we hear a regular succession of high-intensity pulses, called beats.

Fig. 3. (a) Two waves of different frequency (b) The total wave (c) the wave intensity

We can work out the beat phenomenon analytically as well as graphically. At the location of the listener, the excess pressure is Δp_1 for one wave and Δp_2 for the other, where (say)

$$\Delta p_1 = \Delta p_0 \sin\omega_1 t \qquad \text{and} \qquad \Delta p_2 = \Delta p_0 \sin\omega_2 t \ . \tag{8}$$

That is, we assume the two waves have the same amplitude Δp_0 but different angular frequencies ω_1 and ω_2. The superposed sound wave is

$$\Delta p = \Delta p_0 \, [\, \sin\omega_1 t \ + \ \sin\omega_2 t] \, , \tag{9}$$

for which we can use the trig identity*

$$\sin\alpha + \sin\beta = 2\cos\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right) \tag{10}$$

to get

$$\Delta p = [\,2\Delta p_0 \cos(\Delta\omega t)]\,\sin\omega_0 t\,. \tag{11}$$

Here the *average* angular frequency of the two component waves is

$$\omega_0 = \frac{\omega_1 + \omega_2}{2} \tag{12a}$$

and *half the difference* angular frequency is

$$\Delta\omega = \frac{\omega_1 - \omega_2}{2}\,. \tag{12b}$$

If the frequency difference $\Delta\omega$ is small, then the factor $2\Delta p_0 \cos(\Delta\omega t)$ in Eq. 11 oscillates slowly, while the factor $\sin\omega_0 t$ oscillates rapidly. That is, the sound wave is basically an oscillation $\sin\omega_0 t$ at a frequency which is the average of the frequencies generated by the two violins, and whose amplitude is modulated by the factor $2\Delta p_0 \cos(\Delta\omega t)$, as illustrated in Fig. 3.

It is the modulation factor which is heard as the beats. The time interval between successive beats is the time between successive maxima or minima of $\cos(\Delta\omega t)$, because the sound is loudest whenever $\cos(\Delta\omega t) = +1$ or -1, as shown in Fig. 3. For example, the sound is loudest when $\Delta\omega t = 0$ and when $\Delta\omega t = \pi$, so the time interval between beats is

$$T_{\text{beat}} = \frac{\pi}{\Delta\omega} = \frac{2\pi}{|\omega_1 - \omega_2|}\,. \tag{13a}$$

The corresponding beat frequency is

$$\nu_{\text{beat}} = \frac{1}{T_{\text{beat}}} = \frac{|\omega_1 - \omega_2|}{2\pi} = |\nu_1 - \nu_2|\,. \tag{13b}$$

That is, *the beat frequency is simply the absolute difference between the frequencies of the two waves.*

*The identity can be proven using the identities for $\cos(A + B)$ and $\sin(A + B)$. See problem FP18.

Beats are observed in sound waves and in other kinds of waves at well. At the beach one sometimes observes that every third incoming wave (or every fourth, say) has a larger amplitude than those in between. This can be caused by a beating of two waves of different frequency, created independently in different parts of the ocean, arriving at the shore at the same time.

Sample Problem 2. A tuning fork sounds an "A" at 440 Hz. When another tuning fork of unknown frequency is sounded with the first, a beat maximum is heard every half-second. What is the frequency of the second tuning fork?

Solution: The beat frequency is $\nu_{\text{beat}} = |\nu_1 - \nu_2| = \dfrac{1}{T_{\text{beat}}} = \dfrac{1}{0.5\text{ s}} = 2\text{ s}^{-1}$. Taking ν_1 to be 440 Hz, ν_2 must be either 438 Hz or 442 Hz. From the beats alone, we can't tell which answer is correct.

F.4 Sound Waves in Three Dimensions; Intensity and Interference

Suppose that two stereo speakers emit sound waves of the same wavelength, as shown in Fig. 4.

Fig. 4. Two speakers emit sound waves of the same wavelength toward an ear at the right.

At the far right a human ear hears the sound. According to the principle of superposition, the two sound waves add up so that the excess pressure Δp is the sum of the excess pressures due to the individual sources:

$$\Delta p = \Delta p_1 + \Delta p_2 . \tag{14}$$

Each source emits a sinusoidal wave. If the ear is at a position where the sound waves are in phase (i.e., each has crests and troughs at the same times) the waves add *constructively*, as illustrated in Fig. 5(a), giving a net Δp twice that of one of the waves by itself. And if Δp is twice as much, the intensity heard is *four* times as much as that of a single wave, since $I \propto \Delta p^2$.

If the ear is at a position where the waves are 180° out of phase, so that it receives a crest of one wave as it receives a trough of the other wave, the waves add *destructively* and the waves cancel out, as illustrated in Fig. 5(b). In this case the ear hears nothing. This phenomenon of either adding or cancelling is called *interference*.

Fig. 5. Constructive interference (a) and destructive interference (b), depending upon whether two waves are in phase or 180° out of phase.

Where should one stand to hear constructive interference or destructive interference of the sound from two speakers? Let the phases (i. e., the arguments of the sine functions) of the sinusoidal waves from two speakers be

$$\Phi_1 = kr_1 - \omega t + \phi_1 \qquad \text{and} \qquad \Phi_2 = kr_2 - \omega t + \phi_2 \tag{15}$$

at some point in space that is at a distance r_1 from the first speaker and r_2 from the second speaker. The two waves will add constructively if the sine functions are in phase with one another, i. e. if

$$\Phi_2 - \Phi_1 = (kr_2 - \omega t + \phi_2) - (kr_1 - \omega t + \phi_1) = 2\,n\pi, \tag{16}$$

where n is a positive (or negative) integer or zero. Note that the two phase angles ϕ_1 and ϕ_2 need not necessarily be the same, although they are likely to be the same if the two speakers are driven by the same frequency source. If they *are* the same, the constructive interference condition of Eq. 16 becomes simply

$$k\Delta r = k(r_2 - r_1) = 2\,n\pi, \tag{17a}$$

or (since $k = 2\pi/\lambda$)

$$r_2 - r_1 = n\lambda. \tag{17b}$$

This condition is easy to understand: Assuming the waves are emitted in phase with one another at the speakers, the waves will still be in phase with one another at the listener if the distances from the two speakers to the listener are either exactly the same, as shown in Fig. 6, or if they differ by an integer number of wavelengths. In either case if one wave has a crest at the listener at some time, the other will have a crest at that same time.

Fig. 6. Waves from two speakers reaching a listener far to the right.

Suppose for simplicity the listener is quite far from the two speakers; that is, if d is the separation of the speakers, the distance of the listener from the speakers is $r >> d$. In that case the waves reaching the listener travel by nearly parallel paths. Let the angle of the two parallel beams be θ with respect to a line normal to the plane of the two speakers, as shown in Fig. 7. Then the path length from the upper speaker to the listener is longer than the shorter path by an amount $d\sin\theta$, as shown. If this extra path length is an integer number of wavelengths, then the two waves will be in phase with one another, so the listener will hear a loud sound. That is,

$$n\lambda = d\sin\theta \qquad n = 0, 1, 2, 3, \ldots \text{ (constructive interference)} \tag{18}$$

corresponds to *constructive* interference at the listener. If on the other hand the extra distance is equal to a half wavelength, or any odd number of half wavelengths, the two waves will interfere *destructively*, and the listener will hear nothing. This condition is

$$n\frac{\lambda}{2} = d\sin\theta \qquad n = 1, 3, 5, \ldots \quad \text{(destructive interference)} \tag{19}$$

Fig. 7. There is an extra distance $d \sin\theta$ along the longer path. If $d \sin\theta = n\,\lambda$ $(n = 0, 1, 2, \ldots)$, the two beams interfere constructively; if $d \sin\theta = n\dfrac{\lambda}{2}$, $(n = 1, 3, 5, \ldots)$, the two beams interfere destructively. The condition assumes that the distance between the two sources is much smaller than the distance from the sources to the listener.

Sound detectors placed in a row to the right of the speakers show alternating maxima and minima of sound. The intensity pattern observed is sketched in Fig. 8. The sound could be coming either from two speakers in phase, or from a single speaker at the far left, with a screen placed to the right of it containing two slits, as shown. Each slit can be thought of as a source of sound.

Fig. 8. The intensity of sound observed some distance to the right of a screen containing two slits. Sound penetrating one slit is in phase with sound penetrating the other slit, because the slits are equidistant from the speaker producing the sound.

Sample Problem 3. Two speakers a distance $d = 5.0$ m apart emit equal intensities of 660 Hz sound waves, in phase with one another. A listener $D = 30$ m away, initially equidistant from the two speakers, hears intensity I_0. The listener then walks sideways a distance x, as shown.

(a) What is the smallest value of x so the listener will hear a minimum in sound intensity?

(b) One of the speakers is then unplugged. What intensity will the listener hear?

Solution: (a) The listener walks to a point x such that the distance from the farther speaker is one-half wavelength larger than the distance from the nearer speaker; the sound waves will be out of phase and cancel one another. Therefore $d\sin\theta = \lambda/2$, as shown. This gives

$$\sin\theta = \frac{\lambda}{2d} = \frac{v}{2dv} = \frac{330 \text{ m/s}}{2(5.0 \text{ m})(660 \text{ Hz})} = 0.05 .$$

The distances x and D are related by $\frac{x}{D} = \tan\theta$. Because $\sin\theta$ is so small, $\tan\theta \cong \sin\theta = .05$, so $x = .05D = 1.5$ m .

(b) The listener was initially equidistant from the two speakers, so heard an interference maximum, with twice the amplitude and four times the intensity of a single speaker by itself. So when at x, essentially the same distance as before from each speaker, the listener will hear an intensity $I = I_0/4$ when one of the speakers is turned off.

F.5 Examples of Interference

Sound-wave interference, as well as the interference of waves on a rope, water waves, etc., is governed by the relative phases of the waves. If two waves are in phase, they add constructively; if they are 180° out of phase, they add destructively, as was already illustrated in Fig. 5.

What if two waves have an arbitrary phase difference? What is the sum of the two waves in that case? We will work out the answer to that question for the case of two sinusoidal sound waves of equal amplitude, both traveling to the right. The two waves are

$$\Delta p_1 = A \sin\Phi_1 \qquad \text{and} \qquad \Delta p_2 = A \sin\Phi_2 \tag{20a}$$

with phases

$$\Phi_1 = k_1 r_1 - \omega_1 t_1 - \phi_1 \qquad \text{and} \qquad \Phi_2 = k_2 r_2 - \omega_2 t_2 - \phi_2 . \tag{20b}$$

The two waves in general have different wave numbers k, angular frequencies ω, phase angles ϕ, distances r from the source, and can even be sampled at different times t.

By the principle of superposition, the total wave is

$$\Delta p = \Delta p_1 + \Delta p_2 = A[\sin\Phi_1 + \sin\Phi_2]. \tag{21}$$

Suppose that the two waves are out of phase with one another by an amount $\Delta\Phi$, so that $\Phi_2 - \Phi_1 = \Delta\Phi$. Using the trig identity of Eq. 10, the total wave is therefore

$$\Delta p = 2A \cos\left[\frac{\Phi_2 - \Phi_1}{2}\right]\sin\left[\frac{\Phi_2 + \Phi_1}{2}\right]$$

$$= 2A \cos\left[\frac{\Delta\Phi}{2}\right]\sin\left[\Phi_1 + \frac{\Delta\Phi}{2}\right]. \tag{22}$$

The *sum* of two waves with phases Φ_1 and Φ_2 has been converted into the *product* of two sinusoidal functions, one with phase $\frac{\Delta\Phi}{2}$ and the other with phase $\Phi_1 + \frac{\Delta\Phi}{2}$.

Example 1: Suppose the phase difference $\Delta\Phi = \pi$ (or in general $\Delta\Phi = \pm n\pi$ where n is an *odd integer*.) Then the total wave is $\Delta p = 0$, corresponding to complete destructive interference, since $\cos\left[\frac{\Delta\Phi}{2}\right] = \cos(\pm n\pi/2) = 0$. A nice application of destructive interference is the noise-cancelling headphones now available commercially. The headphones have small microphones built into each earpiece. The microphones pick up the surrounding background sound and internal electronics then phase-shifts the collected sound by 180°. The phase shifted signal is then added to the signal coming from the earpieces, such as music from a compact disk player. The phase-shifted signal interferes destructively with the background sound, effectively eliminating it. Such noise-cancelling headphones were originally designed for airline pilots so they could easily communicate with each other and with the control tower over the high background noise in the cockpit. Now anyone who can afford them can block out noise anywhere.

Noise-canceling headphones

Example 2: If the two waves are *in* phase, with $\Delta\Phi = 2\pi\, n$ (n = 0, 1, 2, ...), then $\cos\left[\frac{\Delta\Phi}{2}\right] = \cos \pi n = \pm 1$ depending upon whether n is even or odd. Equation 22 then gives

$$\Delta p = \pm\ 2A \sin(\Phi_1 + \pi n) = \pm\ 2A \sin \Phi_1 \cos \pi n = 2A \sin \Phi_1,$$

showing that the combined wave is the same as the first wave Δp_1 by itself, except that the amplitude has been doubled. This is constructive interference, as was illustrated in Fig. 5a. Constructive interference can be heard by an observer standing at equal distances from two speakers broadcasting identical single-frequency waves.

Example 3: If the two waves differ only in their phase angles ϕ, then Eq. 22 becomes

$$\Delta p = 2A \cos\left(\frac{\Delta\phi}{2}\right) \sin\left(kr - \omega t - \frac{\phi_1 + \phi_2}{2}\right),$$

a sinusoidal traveling wave with the same wavelength and frequency as both constituents but with amplitude $2A \cos\left(\frac{\Delta\phi}{2}\right)$, which can vary from zero to 2A in magnitude depending upon the phase-angle difference $\Delta\phi$. If $\Delta\phi = \pi/2$, for example, the amplitude is $\sqrt{2}A$, as shown.

Example 4: If two speakers both emit the same frequency sound, the phase difference arises not from different phase angles, but because the distance r from one speaker to a listener may be different than the distance from the other slit to the same listener. The phase difference is then

$$\Delta\Phi = k\Delta r = kd\sin\theta = \left(\frac{2\pi}{\lambda}\right)d\sin\theta, \tag{23}$$

where d is the separation between the speakers and θ is the angle to the listener, as shown in Fig. 9. If this phase difference is $2\pi\, n$, an integer times 2π, the two waves should add contructively. This is exactly the condition $n\,\lambda = d\sin\theta$ found in Section F.4.

Example 5: If the phase difference is due only to a difference in frequency, then

$$\Delta\Phi = (\omega_1 - \omega_2)\, t$$

so according to Eq. 22

$$\Delta p = 2A\cos\left[\frac{(\omega_1 - \omega_2)}{2}t\right]\sin\left[kr - \frac{(\omega_1 + \omega_2)}{2}t - \phi\right]$$

which is a traveling wave with an amplitude $2A\cos\left[\frac{(\omega_1 - \omega_2)}{2}t\right]$ that oscillates with angular frequency $\frac{(\omega_1 - \omega_2)}{2}$. This is the *beat* phenomenon discussed in Section F.3. An observer hears the sound wave become alternately louder and softer with a beat frequency. [For waves like sound in which different frequencies travel with the same velocity (i.e., they are *nondispersive*, as discussed in Section), the ratio $\omega_1/k_1 = \omega_2/k_2 = v$ is the same for both constituent waves, so if the frequencies ω differ the wave numbers k must differ as well. The reader can verify that the total wave still exhibits the beat phenomenon for an observer at some fixed location.]

The interference of sound waves, water waves, and seismic waves is a feature of other waves as well, including light waves. The interference of light waves, whether in the visible or other regions of the electromagnetic spectrum, is both interesting and useful in practical applications. In quantum mechanics particles themselves have wavelike properties, so that interference phenomena occur for individual particles like electrons, an observable and increasingly useful phenomenon.

State of the Search

1. The principle of superposition, valid for low-amplitude waves on a string, of sound, or water waves, states that for any two solutions of the wave equation, the sum of these solutions is also a solution.

2. Standing waves of sound can be set up in organ pipes and other wind instruments. They are superpositions of waves traveling in one direction and waves traveling in the opposite direction. Pipes with both ends open have pressure nodes at each end; pipes with one end open and the other end closed have a pressure node at the open end and a pressure antinode at the closed end. (A pressure node is a location at which the pressure stays constant. A pressure antinode is a location at which the pressure varies maximally, from its largest to its smallest value.) Therefore there are an integer number of half-wavelengths within an open-open pipe, so the fundamental frequency of an open-open pipe corresponds to a half-wavelength of sound within the pipe. For an open-closed pipe, the lowest frequency (the fundamental) corresponds to a quarter-wavelength within the pipe, and the higher (overtone) frequencies add an integer number of half-wavelengths to the fundamental.

3. Two waves moving parallel to one another, and which have slightly different frequencies, interfere with one another to produce alternating maxima and minima of intensity, called *beats*. The beat frequency is related to the individual frequencies ν_1 and ν_2 by

$$\nu_{\text{beat}} = |\nu_1 - \nu_2| .$$

4. Sound waves from two speakers a distance d apart exhibit constructive and destructive interference when heard by a distant listener. If the sound waves are in phase at the sources, and if the distance of the listener from the speakers is much larger than d, then the listener will hear maximum sound when $d\sin\theta = n\lambda$, where θ is the listener's angle from the normal to the plane of the two speakers, λ is the wavelength, and n is an integer.

5. The interference between two waves $\Delta p_1 = A \sin\Phi_1$ and $\Delta p_2 = A \sin\Phi_2$ depends upon the difference in their phases $\Delta\Phi = \Phi_2 - \Phi_1$. Phase differences can arise from differences in wavelength, frequency, phase angle, or distance from the source. In particular if $\Delta\Phi = 0$ (or any integer multiple of 2π) the two waves are in phase, and add constructively; If $\Delta\Phi = \pi$ (or any odd-integer multiple of π) the two waves are 180° out of phase, and add destructively.

PICTURE QUIZ F

1. An organ contains many pipes, some of them open-open and some open-closed.

(a) When playing the "A" pipe with fundamental frequency 440 Hz, it is found that there is also an overtone at 1320 Hz.

(A) The pipe is open-open (B) The pipe is open-closed (C) Either is possible.

(b) In addition, there is an overtone at 1760 Hz

(A) The pipe is open-open (B) The pipe is open-closed (C) Either is possible.

2. The speed of sound in a room is 360 m/s. When two flutes try to tune to one another, beats are heard every tenth of a second. If the sound from one flute has wavelength $\lambda = 0.500$ m, which of the following frequencies are possible for the sound from the other flute?

(A) 690 Hz (B) 710 Hz (C) 730 Hz (D) 750 Hz (E) none of these

3. A sound wave of wavelength 2.00 m passes through two slits a distance 4.00 m apart, as shown. Far from the slits, intensity maxima will occur at which of the following angles, measured from the forward direction?

(A) 15° (B) 30° (C) 45° (D) 60° (E) 75°

PROBLEM QUIZ F

1. An oboe plays an "A" at 440 Hz. A violin is trying to tune to the oboe. While playing an "A" on the violin, the violinist hears 3 beats/second with the oboe. What fractional tension change $\Delta F/F$ must the violinist make in the violin string to get into tune?

2. An organ builder designs an open-open organ pipe whose fundamental frequency is 330 Hz. (a) How long is the pipe? (b) What are the frequencies of the first and second overtones? (c) If the pipe is later closed off at one end, what would be the new fundamental frequency? The sound velocity is 330 m/s.

3. Two stereo speakers, 2.0 m apart, each emits a pure musical tone, a "C" at 512 Hz. They emit the tone in phase with one another. A listener stands 10.0 m away from the speakers, on the midline between them, as shown. How far sideways (the distance x shown) must the listener move to hear an interference minimum in the sound? How much farther must the listener move to hear the next interference maximum?

Special Topic: Fourier Analysis

A particularly useful example of the principle of superposition is the matching of an arbitrary periodic waveform with a summation of many sinusoidal waves. Consider an arbitrary wave moving to the right on a string, with the restriction that the wave is periodic in x at some fixed time, as illustrated at the right. The wave is not sinusoidal, but its form does repeat itself in a periodic way. The French mathematician Joseph Fourier, a friend of Napoleon, proved a remarkable theorem which has turned out to be enormously useful, and which may have a greater long-term impact on civilization than the conquests of Napoleon. What Fourier showed was that any smooth, periodic wave form of the type we are describing can be decomposed into the sum of sine and cosine waves. (In fact, even more general waves with some discontinuities can also be "Fourier analyzed"). A proof would be a long distraction here, so we simply restrict the discussion to a few examples.

Consider the triangular wave shown in Fig. S1(a); the wave consists of alternating lines of positive and negative slope. Because of the slope discontinuities it cannot be an actual sound wave or wave on a string, although it could be if the sharp edges were slightly rounded off. This wave is not sinusoidal, but it *is* the *sum* of sine waves according to Fourier. Actually, one must add up an infinite number of sine waves to replicate the function, but only a few are needed to match the wave pretty well, well enough so that it is hard to tell the difference except near the slope discontinuities. The best single sine wave is shown in Fig. S1(b). It has the same period as the triangular wave. It is too low in the middle and too high on the sides, however; we have to do better by adding in another sine wave which oscillates more rapidly. The next sine wave is shown in Fig. S1(c); it oscillates more rapidly, and when added to the first sine wave succeeds in building

Fig. S1. (a) A periodic triangular wave (b) The best single sine wave to replicate the triangular shape. (c) The next sine wave, which when added to the first does a better job, as shown in (d); the third sine wave shown in (e), which when added to the others replicates the triangular wave still more accurately, as shown in (f).

up the center and tearing down the sides, to more closely match the triangular shape. The sum of the two sine waves is shown in Fig. S1(d), superimposed upon the triangular wave; the third sine wave is shown in Fig. S1(e); it oscillates still more rapidly. When added to the first two sine waves it replicates the triangular shape even more closely, as shown in Fig. S1(f). Fourier methods tell us not only what oscillation frequencies are required for the various sine waves; it provides the amplitudes as well. So if the height of the triangular peak is "one", the amplitude of the first sine wave (the one shown in Fig. S1(b) turns out to be $\frac{8}{\pi^2}$, somewhat less that one, as expected. Relative to the first sine wave, the amplitude coefficient of the second is $-\frac{1}{9}$ (upside down and one-ninth as large), the amplitude coefficient of the third is $\frac{1}{25}$, etc. These values are all given precisely by Fourier analysis; they do not have to be guessed at by drawing pictures.

Fourier Series for Sound Waves

As for waves on a string, an arbitrary periodic sound wave can be decomposed by Fourier analysis into a series of sinusoidal waves. The distinctive sounds of musical instruments, for example, are due to their different wave forms. All have different combinations of "overtones", sinusoidal waves of higher frequency than the "fundamental" note being played. If a saxophone, trombone, and 'cello all play an "A", for example, they all sound different, not because the sinusoidal wave corresponding to the fundamental frequency of 440 Hz is different, but because of the distinctive combinations of overtones. Fig. S2 shows the wave forms for several instruments. Superimposed on the lowest-frequency note are waves of higher frequency which produce the complicated sound wave emitted by the instrument. Designers of electronic organs and synthesizers are quite successful at mimicking the sounds of various instruments by synthesizing the proper combinations of sinusoidal waves. Fourier analysis can tell musicians and organ designers which combinations of frequencies make up a given wave form.

Fig. S2. Characteristic wave forms for (a) a violin (b) an oboe (c) a French horn

Questions and Problems from Oscillation and Waves, Chapter F

FQ1. What is the purpose of the slide on a trombone?

FQ2. An organ pipe is tuned to exactly 326 Hz when the room temperature is 20° C. If the room temperature later increases to 22° C, does the pipe's frequency increase, decrease, or stay the same?

FQ3. Why do you sound like Mickey Mouse (with higher-pitched sounds than usual) when you take a deep breath of helium and then speak? (*Hint*: See Appendix E 1 in Chapter E.)

FQ4. For some of the very largest (i. e., lowest-frequency) pipes in a pipe organ, organ designers often choose open-closed pipes rather than open-open pipes. What is the advantage?

FQ5. Some restaurants are so noisy you can hardly carry on a conversation. What can be done to damp but background noise? Think of the room as a sort of organ pipe, with a wall corresponding to the closed end of a pipe, since a wall would have an antinode in density (and pressure.) About how far out from the wall might a tapestry be hung to most effectively absorb typical sound from human voices?

FQ6. The figure below shows the displacement of a standing sound wave in a 32-cm-long horizontal tube of air open at both ends. (a) Are the molecules moving horizontally or vertically? (b) At what distances from the left end of the tube do the molecules oscillate with maximum amplitude? (c) At what distances from the left end does the air pressure oscillate with maximum amplitude?

FQ7. A french horn player hears 3 beats per second when she plays a note and simultaneously sounds a 440 Hz tuning fork. After pulling her tuning valve out to slightly increase the length of her trumpet, she hears 5 beats per second against the tuning fork. Was her initial frequency 437 Hz or 443 Hz? Explain.

FQ8. A flute filled with helium will, until the helium escapes, play notes at a much higher pitch than normal. Why?

FQ9. You hear three beats per second when two sound tones are produced. The frequency of one tone is 550 Hz. The frequency of the other is (a) 544 Hz (b) 547 Hz (c) 553 Hz (d) 556 Hz (e) either a or d (f) either b or c.

FQ10. The two loudspeakers shown below are in phase. They emit equal-amplitude sound waves of wavelength 1.0 m. At the point shown, is the interference maximum constructive, perfect destructive, or something in between?

Section F.I The Principle of Superposition

FP1. Using the linear wave equation for a rope given by Eq. 1, show that (a) if $y_1(x,t)$ is a solution, then $Ay_I(x,t)$ is a solution, for any constant A. (b) if $y_1(x,t)$ and $y_2(x,t)$ are solutions, then $Ay_1(x,t) + By_2(x,t)$ is a solution, for any constants A and B.

Section F.2 Standing Waves of Sound

FP2. We want to design an open-open organ pipe whose fundamental frequency is 256 Hz, which is "middle C" on the piano. (a) How long should the pipe be? (b) What are the frequencies of the first and second overtones? (c) If the pipe were later closed off at one end, what would be its new fundamental frequency? Use the speed of sound $v_s = 330$ m/s.

FP3.The sound from an organ pipe is analyzed into its constituent frequencies; they are found to be 440 Hz, 1320 Hz, 2200 Hz, . . . (a) Is the pipe open-open or open-closed? (b) How long

is the pipe? Use the speed of sound $v_s = 330$ m/s.

FP4. (a) Find the speed of sound in air at - 40 C. (See *Sample Problem A2* in Appendix E I of Chapter E.) (b) A wooden organ pipe with both ends open is 40 cm long. How much will the fundamental frequency of the pipe be "out of tune" if the pipe is in tune at 20 C ? Assume the pipe does not change its length appreciably.

FP5. (a) Find the speed of sound in a low-pressure helium gas at 20 C. (See Sample Problem A2 in Appendix. E I of Chapter E. Helium gas is monatomic.) (b) An organ pipe with one end open and one end closed is 30 cm long. If the pipe is filled with helium, what is the frequency of the fundamental oscillation?

FP6. Shower tiles typically absorb only about 2% of the sound energy striking them, so if you continually sing a note that generates a standing wave within the shower stall, the sound intensity can get quite high. What are the lowest such "resonance" frequencies in a shower stall 8 ft high and 3 ft on each side due to the standing wave between (a) floor and ceiling (b) opposite walls? (Note that air cannot move perpendicular to a wall at the wall itself.) Assume the speed of sound in the shower is 350 m/s.

FP7. A small speaker is installed in a long thin tube capped at both ends. Find a formula for the resonant frequencies within the tube in terms of its length L. That is, find the frequencies of standing sound waves within the tube with both ends closed.

FP8. Use arrows to illustrate the air flow velocities at five equally spaced times for one complete cycle of the first overtone of an open-closed pipe. (Similar velocity flow diagrams are shown in Figs. 1 and 2.)

FP9. An organ designer needs a pipe to sound a fundamental "A" of frequency 440 Hz. How long should the pipe be if it is (a) open-open (b) open-closed? Assume $v_{\text{sound}} = 340$ m/s.

FP10. An organ has an open-open pipe of length d that sounds a first overtone of frequency ν_0. If the pipe were replaced by an open-closed pipe which has the same first overtone, how long would it be, as a multiple of d?

FP11. At air temperature 20 C, with certain keys depressed, a flute sounds a frequency 512 Hz. (a) If the air temperature suddenly drops to 0 C, what frequency will be heard? (b) If instead the air is suddenly replaced by helium gas at 20 C, what frequency will be heard? (See Appendix E1.)

FP12. When blowing across a large soda bottle, we hear the fundamental frequency 150 Hz. What frequency would we hear if we were to blow across the same bottle filled with argon gas at the same temperature? (See Appendix E1.)

FP13. A vertical glass tube is partially filled with water. The water level can be changed by means of a rubber hose connected to the tube through a plug at the lower end of the tube, as shown below. A tuning fork of frequency 256 Hz is excited and placed near the upper opening of the tube. At what distances d of the water surface beneath the upper opening will one hear loud resonances of the sound? Assume the speed of sound in the room is 350 m/s.

Section F.3 Beats

FP14. (a) What beat frequency is heard when one viola "C" string vibrates at 128 Hz while another vibrates at 132 Hz? (b) What fractional change in tension $\Delta F/F$ is needed to reduce the second viola's frequency to 128 Hz?

FP15. A tuning fork of frequency 440 Hz is used to tune a violin string. When both are sounded at once, 3.0 beats/second are heard. (a) What are the possible vibration frequencies of the violin string? (b) In each case what is the fractional change in tension $\Delta F/F$ needed to tune the string?

FP16. Two submarines each use sonar pulses whose fundamental frequency is 2250 Hz. Submarine A is at rest relative to the water, and submarine B is moving toward A at speed 12 m/s. In comparing the two signals, what beat frequency is heard by observers in submarine(a) A? (b) B? The speed of-sound in water is v_s = 1500 m/s.

FP17. Two train whistles, A and B, each have a frequency of 220 Hz. A is stationary, and B is moving toward the right (away from A) at a speed of 35.0 m/s. A listener is between the two trains and is moving toward the right with a speed of 15.0 m/s. No wind is blowing, and the speed of sound is 331 m/s. As heard by the listener, (a) What is the frequency from A? (b) What is the frequency from B? (c) What is the beat frequency?

FP18. Ultrasound has many medical applications, one of which is to monitor fetal heartbeats by reflecting ultrasound off a fetus in the womb. (a) Consider an object moving at speed v_0 toward a sound at rest that is emitting sound waves of frequency ν_0. Show that the reflected wave that returns to the source has a Doppler-shifted frequency

$$\nu_{\text{echo}} = \frac{v + v_0}{v - v_0}\nu_0$$

where v is the speed of sound in the medium. (b) Suppose the object's speed is much less than the wave speed. Then $\nu_{\text{echo}} \cong \nu_0$, and a microphone that is sensitive to these frequencies will detect a beat frequency if it listens to the two frequencies simultaneously. Use the binomial approximation and other approximations to show that the beat frequency is $\nu_{\text{beat}} \cong (2v_0/v)\nu_0$. (c) The reflection of 2.40 MHz ultrasound waves from the surface of a fetus's beating heart is combined with the 2.40 MHz wave to produce a beat frequency that reaches a maximum of 65 Hz. What is the maximum speed of the surface of the heart? The speed of ultrasound waves within the body is 1540 m/s. (d) Suppose the surface of the heart moves in and out sinusoidally at 90 beats/min. What is the amplitude in mm of the heartbeat?

FP19. Piano tuners tune pianos by listening to the beats between the harmonics of two different strings. When properly tuned, the note A should have a frequency of 440 Hz and the note E should be at 659 Hz. (a) What is the frequency difference between the third harmonic of the A and the second harmonic of the E? (b) A tuner first tunes the A string very precisely by matching it to a 440 Hz tuning fork. She then strikes the A and E strings simultaneously and listens for beats between the harmonics. What beat frequency indicates that the E string is properly tuned? (c) The tuner starts with the tension in the E string a little low, then tightens it. What is the frequency of the E string when she hears four beats per second?

Section FA Sound Waves in Three Dimensions: Intensity and Interference

FP20. Two loudspeakers a distance d apart emit sound waves of the same wavelength λ, which are in phase with one another. A listener is a distance D ($D >> d$) away, equidistant from the two speakers, as shown below. The listener walks a distance x perpendicular to the line between herself and the speakers. Show that if $\lambda << d$, the distances x for which she will hear intensity maxima are given by $x = n\lambda D/d$, where n is an integer.

FP21. Two loudspeakers are separated by a distance d = 6.60 m , and emit sound of the same pitch ν = 44.0 Hz, but are 180° out of phase with one another. At what distances from one of the speakers, along a line between them, will the sound be most intense?

FP22. Two small balls, a distance d = 8.0 m apart, sit near the center of a still circular pool of water of radius R = 120 m and depth h = 16

m. The center of the pool is halfway between the two balls. Each ball oscillates up and down at frequency $\nu = 1.00$ Hz, generating surface water waves of this frequency. (a) Are the generated waves shallow water waves or deep water waves? (*Hint:* Try each, using the velocity formulas given in Chapter E, and see which of them is a good approximation.) (b) What is the typical distance an observer on shore will have to walk to go from one interference maximum to the next?

FP23. A tsunami of wavelength 0.30 km is known to be approaching the shore. A concrete break-water 2.0 km offshore has two openings in it, a distance 0.50 km apart, as shown below. To minimize the wave amplitude experienced by her boat, a sailboat owner should tie up the boat at an onshore dock at what distance x from an onshore breakwater point halfway between the two break-water openings?

FP24. The three identical loudspeakers shown below play a 170 Hz tone in a room where the speed of sound is 340 m/s. You are standing 4.0 m in front of the middle speaker. At this point, the amplitude of the wave from each speaker is a. (a) What is the total amplitude at this point? (b) How far must speaker 2 be moved to the left to produce a maximum amplitude at the point where you are standing? (c) When the amplitude is maximum, by what factor is the sound intensity greater than the sound intensity from a single speaker?

FP25. Two loudspeakers in a plane, 5.0 m apart, are playing the same frequency. If you stand 12.0 m in front of the plane of the speakers, centered between them, you hear a sound of maximum intensity. As you walk parallel to the plane of the speakers, staying 12.0 m in front of them, you first hear a minimum of sound intensity when you are directly in front of one of the speakers. (a) What is the frequency of the sound? Assume a sound speed of 340 m/s. (b) If you stay 12.0 m directly in front of one of the speakers, for what other frequencies between 100 Hz and 1000 Hz is there a minimum sound intensity at this point?

FP26. Prove the trig identity of Eq. 10. *Hint*: Use the identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B,$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B,$$

and

$$\sin^2 A + \cos^2 A = 1.$$

Special Topic: Fourier Analysis

FP27. A "square" wave has the profile shown below: (a) Redraw the square wave and beneath it sketch the single sine wave (the "fundamental") that most closely matches the profile.(b) Sketch beneath that graph the next two higher-frequency sine waves, which when added together and to the fundamental would give the closest three-frequency approximation to the profile. (c) Add up the three waves graphically and show where the sum overestimates and where it underestimates the actual profile.

Oscillation and Waves: Review of Definitions and Relationships

Simple Harmonic Oscillator Equation ($F = ma$): $\ddot{x} + \omega^2 x = 0$

Solution: $x(t) = A\cos(\omega t + \phi_0)$

Period: $T = \dfrac{1}{\nu} = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{m}{k}}$.

Effective force constant about a potential energy minimum: $k = \left.\dfrac{d^2U}{dx^2}\right|_{x_0} = U''(x)|_{x_0}$

Pendulum equation: $\ddot{\theta} = -\dfrac{g}{\ell}\sin\theta$ Small-amplitude angular frequency $\omega = \sqrt{\dfrac{g}{\ell}}$

Damped harmonic oscillator equation: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$ $\omega_0 = \sqrt{k/m}$

Underdamped solution $(\beta < \omega_0)$ $x(t) = A\, e^{-\beta t}\cos(\omega_1 t + \phi_0)$ where $\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$

Damped oscillator driven by the force $F = F_0\sin\omega t$: position $x(t) = x_c(t) + x_p(t)$ where $x_c(t)$ is the solution of the unforced damped oscillator and

$x_p(t) = C\sin(\omega t - \delta)$, where $\delta = \tan^{-1}\left(\dfrac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$

and $C = \dfrac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$

Waves on a rope: speed $v = \sqrt{\dfrac{F}{\mu}}$

Wave equation $\dfrac{\partial^2 y}{\partial x^2} = \left(\dfrac{\mu}{F}\right)\dfrac{\partial^2 y}{\partial t^2}$ Solution $y(x, t) = f(x - vt) + g(x + vt)$

Sinusoidal wave moving to the right: $y(x, t) = A\sin(kx - \omega t - \phi)$

$k = 2\pi/\lambda$ is the wave number (λ the wavelength), $\omega = 2\pi\nu = 2\pi/T$ is the angular frequency (ν is the frequency and T is the period), and ϕ is a phase angle.

(Note that the same symbol (k) is used for wave number and for the force-constant of a spring. One can tell which is which from the context or by checking dimensions.

Standing waves : $y(x, t) = 2A\cos\omega t \sin kx$

Sound waves: speed $v = \sqrt{\dfrac{\Delta p}{\Delta \rho}} = \sqrt{\dfrac{B}{\rho}}$ (where p is the pressure and ρ is the density of the medium, and B is the bulk modulus)

Surface water waves:

Shallow water waves (with $\lambda >>$ depth h) have velocity $v = \sqrt{g\,h}$

Deep water waves (with $\lambda << h$) have velocity $v = \sqrt{\dfrac{g\lambda}{2\pi}}$

The time-average power carried by a sinusoidal wave: $< P > = \dfrac{1}{2} A^2 \mu\, v\, \omega^2$

(A is the amplitude, μ is the string's mass/length, v is the wave velocity, and ω is the angular frequency)

The intensity of a sound wave (the power/area) spreading into three dimensions falls off as the square of the distance from the source. The *sound level* SL is related to the intensity I by

$$\mathrm{SL} = 10 \log\left(\frac{I}{I_o}\right) \quad \text{(in decibels, or } db\text{)}.$$

Doppler shift: $\nu_{ob} = \left(\dfrac{1 \pm v_{ob}/v_s}{1 \mp v_{em}/v_s}\right)\nu_{em}$ (v_s is the speed of sound, v_{ob} and v_{em} are the velocities of the observer and emitter relative to the medium. Upper signs are used if the observer and emitter are moving toward one another, and lower signs are used if they are moving apart.

Beat frequency: $\nu_{beat} = |\nu_1 - \nu_2|$

Two slit (or two speaker) interference, originally in phase: Maxima when $d\sin\theta = n\lambda$

Solutions to Self-Quiz Problems

PICTURE QUIZ 15

1. (A) Forward. The effective gravity due to the car's forward acceleration is backward, toward the rear of the car. The balloon floats in a direction opposite to the total effective gravity, so tips forward. Another way to think about it is to realize that as the car accelerates forward the air tends to flow toward the rear, and the balloon floats on the denser air.

2. Both (B) and (D) are correct. If the elevator accelerates downward with $a = 12.1$ m/s^2, then there is an effective gravity $(12.1 - 9.8)$ m/s^2 = 2.3 m/s^2 *upward* within the elevator. If the elevator accelerates downward with $a =$ 7.5 m/s^2, there is an effective gravity (9.8 – 7.5) m/s^2 = 2.3 m/s^2 *downward* within the elevator.

3. The answer is (D). The rotation period T can depend upon the radius R and the effective gravity g. The dimensions of these quantities, in terms of time T and length L, are $[T] = \mathrm{T}$, $[R] = \mathrm{L}$, and $[g] = L/T^2$. Therefore the only way to combine R and g to give a time is to let $T = k\sqrt{\dfrac{R}{g}}$, where k is some dimensionless constant. Therefore if g stays the same while R is cut in half, we find that $T = T_0/\sqrt{2}$.

PROBLEM QUIZ 15

1. (a) The isolation diagrams are

The important $\mathbf{F} = m\mathbf{a}$ equations are $T = m_1 a$ and $m_2 g - T = 0$ (no vertical acceleration) Therefore $m_2 g = T = m_1 a$, so the acceleration of the system is $a = (m_2/m_1)g$. The net force F_0 on the entire system must therefore be

$$F_0 = (M + m_1 + m_2)a = (M + m_1 + m_2)(m_2/m_1)\, g\,.$$

(b) In the accelerating frame, the isolation diagrams are

Nothing accelerates in this frame, so the forces balance. All results (including T and F_0) are the same as in part (a), setting $g_{\mathrm{eff}} = a$.

2. When riding in the *same* direction as the rotation direction, her speed is $v = R\omega + R\omega/2 = 3R\omega/2$ in an inertial frame, since the rim speed itself is $R\omega$. Her centripetal acceleration is therefore $v^2/R = (9/4)R\omega^2$. The normal force provided by the seat must given by $N = ma = (9/4)mR\omega^2$, which is equal in magnitude to her weight. If she is riding in the direction *opposite* to the rotation direction, her speed in the inertial frame is only $v = R\omega - R\omega/2 = R\omega/2$, so her weight will be only $(1/4)mR\omega^2$.

3. The colonist should throw the ball parallel to the rim and opposite to the rotation direction, with velocity $R\omega$. That cancels out the rim velocity as seen by an outside inertial observer, so the ball will remain at rest to the inertial observer, while the rim (and colonists) rotate around. The colonists will see the ball skim around the rim without touching anything, once each period of rotation.

To the inertial observer

To colonists

PICTURE QUIZ 16

1. (a) The answer is A. The initial angular momentum of the system is the same in A and B. The moment of inertia of the upper ring in B is greater than that of the upper disk in A, so the ring is more effective in slowing the rotating rate when the upper object comes in contact with the lower disk. (b) The answer is B. The initial angular momentum of the system is the same in A and B. The moment of inertia of the upper disk in A is $\frac{1}{2}MR^2$, and the moment of inertia of the upper disk in B is smaller, only $\frac{1}{2}(2M)(R/2)^2 = \frac{1}{4}MR^2$. The larger moment of the inertia of the upper disk in A is more effective in slowing the rotation rate when the upper disk comes in contact with the lower disk. (c) The answer is B. The initial angular momentum of the ring in B is greater than that of the disk in A, because the moment of inertia of the ring is twice that of the disk. The slowing effect of adding the disk (with $I = \frac{1}{2}MR^2$) to the ring (with $I = MR^2$) in B is less than the slowing effect of adding the ring to the disk in A.

2. The answer is D. The ring must have mass $m = 4M$ to keep the total angular momentum the same.

3. The answer is A. Even though he has zero angular momentum throughout the stunt, he can still land on his back. One way to do this to "windmill" one's arms forward about the shoulders, which makes the entire body tilt backward to conserve angular momentum. Because the moment of inertia of the arms is less than that of the body, a single 360° rotation of the arms produces about a 20° rotation of the body.

4. The answer is A. You can twist your body around even though your angular momentum remains zero at all times. For example, you can extend your arms away from your body and move them both clockwise, so your body turns counterclockwise to compensate. Then pull your arms back to your side, which makes your body turn back toward its original position, but not as far. Repeat several times, and you have turned your entire body around.

5. The answer is B. Station A has a smaller moment of inertia, since some of its mass is in the form of disks at the endcaps. The lengths of the stations don't matter as long as the total masses are the same.

PROBLEM QUIZ 16

1. Conserving angular momentum, $I_1\omega_1 + I_2\omega_2 = \text{constant}$. That is,

$$\frac{1}{2}MR^2\omega_o + \frac{1}{2}(2M)\left(\frac{R}{2}\right)^2\left(-2\omega_o\right) = \frac{1}{2}MR^2\left(\frac{\omega_o}{2}\right) + \frac{1}{2}(2M)\left(\frac{R}{2}\right)^2\omega_f .$$

Solving for ω_f, we find that $\omega_f = -\omega_o$.

2. Picture the washer as a solid uniform disk of radius R_2, with an inner disk of radius R_1 removed from it. The area of the washer is $\pi R_2{}^2 - \pi R_1{}^2$, the area of the outer disk minus the area of the inner disk. If the disk in

the center had not been removed, the mass of the solid outer disk would be $M_2 = \dfrac{R_2^2}{R_2^2 - R_1^2} M$, larger than the mass M of the washer by the factor $\dfrac{\pi R_2^2}{\pi R_2^2 - \pi R_1^2} = \dfrac{R_2^2}{R_2^2 - R_1^2}$, the ratio of their areas. Similarly, the mass of the inner disk would be $M_1 = \dfrac{R_1^2}{R_2^2 - R_1^2} M$, the fraction $\dfrac{\pi R_1^2}{\pi R_2^2} = \dfrac{R_1^2}{R_2^2}$ of the mass of the outer disk.

Now to find the moment of inertia of the washer, we subtract the moment of inertia of the small disk from that of the large disk:

$$I_{\text{washer}} = \frac{1}{2} M_2 R_2{}^2 - \frac{1}{2} M_1 R_1{}^2 = \frac{1}{2}\left(\frac{R_2^2}{R_2^2 - R_1^2} M\right) R_2^2 - \frac{1}{2}\left(\frac{R_1^2}{R_2^2 - R_1^2} M\right) R_1^2$$

$$= \frac{M}{2}\left(\frac{R_2^4 - R_1^4}{R_2^2 - R_1^2}\right) = \frac{1}{2} M\left(R_1^2 + R_2^2\right).$$

In the final step we used the identity $R_2^4 - R_1^4 = \left(R_2^2 - R_1^2\right)\left(R_2^2 + R_1^2\right)$.

3. (a) The moment of inertia of the combination about its central axis is $I = 4md^2$, so $m = I/4d^2$.
(b) $L = I\omega$, where $\omega = v/d$. So $L = Iv/d$.

4. The answer is (D) $\dfrac{1}{24} MD^2$. If the stick were uniform, its moment of inertia would be $\dfrac{1}{12} MD^2$. In fact, the mass density decreases with distance from the rotation axis, so the moment of inertia has to be smaller than $\dfrac{1}{12} MD^2$. (That is because the r^2 factor used in in calculating moments of inertia amplifies the importance of mass far from the axis.) The only answer smaller than $\dfrac{1}{12} MD^2$ is $\dfrac{1}{24} MD^2$.

To calculate I directly, we must first relate the mass M to the constants λ_0 and D. The right half of the stick (with $x \geq 0$) has half the mass, so

$$\frac{M}{2} = \int_0^{D/2} dm = \int_0^{D/2} \lambda(x)\, dx = \lambda_0 \int_0^{D/2} (1 - 2x/D) dx = \lambda_0 \left(x - x^2/D\right)\Big|_0^{D/2} = \lambda_0 \frac{D}{4}.$$

Therefore $\lambda_0 = \dfrac{2M}{D}$. The right half of the stick also has half the moment of inertia, so

$$\frac{I}{2} = \int_0^{D/2} \lambda(x)\, x^2\, dx = \lambda_0 \int_0^{D/2} (1 - 2x/D)\, x^2 dx = \lambda_0 \left(\frac{x^3}{3} - \frac{2}{D}\frac{x^4}{4}\right)\Bigg|_0^{D/2} = \frac{1}{48} MD^2, \text{ and } I = \frac{1}{24} MD^2.$$

5. The moment of inertia about the cylinder's symmetry axis, which runs through the CM of the cylinder, is $I_{CM} = \frac{1}{2}MR^2$. Using the parallel-axis theorem, the moment of inertia about an axis parallel to the symmetry axis, a distance $R/2$ from it, is $I = I_{CM} + M\left(\frac{R}{2}\right)^2 = \frac{3}{4}MR^2$.

PICTURE QUIZ 17

1. (a) is not in equilibrium, because both torques are clockwise. The wheel will turn clockwise.

(b) is not in equilibrium, because the counterclockwise torque due to the force $2F_0$ is four times as large as the clockwise torque due to the force F_0. The wheel will turn counterclockwise.

(c) The force $F_0/4$ exerts a counterclockwise torque $\tau_{CCW} = RF_0/4$. The force F_0 exerts a clockwise torque $\tau_{CW} = (R/2)F_0 \sin 30^0 = RF_0/4$. The two torques are equal but opposite, so the wheel stays in equilibrium.

(d) The force F_0 exerts no torque about the center. The force $F_0\sqrt{2}$ exerts the counterclockwise torque $\tau_{CCW} = (R/2)F_0\sqrt{2}\sin 45^0 = (R/2)F_0$, and the force $F_0/\sqrt{3}$ exerts the clockwise torque $\tau_{CW} = R(F_0/\sqrt{3})\sin 60^0 = (RF_0/\sqrt{3})(\sqrt{3}/2) = RF_0/2$. The torques are equal but opposite, so the wheel stays in equilibrium.

2. The answer is A. The torque on the stick due to gravity is the same as though all the mass of the stick were located at its center of mass, a distance 1/6 of the stick's length to the right of the support point. Since the distance of the CM from the support is twice that of the 1-kg weight, the stick must have a mass of 2 kg.

3. The answer is (D). The counterclockwise torque due to the 10-kg weight is $10\text{ kg}\cdot g\cdot 2R\cos 30^0 = 10\text{ kg}\cdot g\cdot\sqrt{3}R$, and the clockwise torque due to M is MgR. These balance if $M = 10\sqrt{3}$ kg.

PROBLEM QUIZ 17

1. An isolation diagram for the weight M shows that in equilibrium the tension of the hanging cable is $T_a = Mg$.

An isolation diagram for the strut reveals three forces that exert torques about an origin located at the hinge. The clockwise torques are

$$\tau_{CW} = \ell T_a \sin(90^\circ - \theta) + \frac{\ell}{2}mg\sin(90^\circ - \theta) = \ell(T_a + mg/2)\cos\theta = \ell(M + m/2)g\cos\theta,$$

where is the length of the strut. The counterclockwise torque is $\tau_{CCW} = \ell T_b \sin\theta$. Balancing torques, $(M + m/2)g\cos\theta = T_b\sin\theta$, so $T_b = (M + m/2)g\cot\theta$. The net force on the strut must also be zero.

$F_{\text{horizontal}} = (F_x)_{\text{hinge}} - T_b = 0$, so $(F_x)_{\text{hinge}} = T_b = (M + m/2)g\cot\theta$

$F_{\text{vertical}} = (F_y)_{\text{hinge}} - T_a - mg = 0$, so $(F_y)_{\text{hinge}} = (M + m)g$

2. Compute the torque on the wheel about the point where the wheel touches the curb. The clockwise torque is $\tau_{CW} = F_o r_\perp = F_o(R - h)$, since $(R - h)$ is the perpendicular part of the radius vector between the curb and the line of force. The counterclockwise torque is $\tau_{CCW} = (Mg - N)\, r_\perp = (Mg - N)\sqrt{R^2 - (R-h)^2} = (Mg - N)\sqrt{2Rh - h^2}$, where N is the normal force up on the wheel from the street. (We found $r_\perp$ in this case from the law of Pythagoras on the triangle shown.)

In equilibrium the two torques are equal, so $F_o(R - h) = (Mg - N)\sqrt{2Rh - h^2}$. As F_o increases, N decreases. The right-hand side is maximized when $N \to 0$. Therefore $(F_o)_{\max} = Mg\dfrac{\sqrt{2Rh - h^2}}{R - h}$. Note that $(F_o)_{\max} = 0$ if $h = 0$, i.e., if there is no curb at all. As h increases to $h = R$, $(F_o)_{\max} \to \infty$. In this case it is not possible to raise the wheel at all, because the curb gives no upward force at all. If $F_o > (F_o)_{\max}$, the wheel rolls up over the curb.

3. An isolation diagram of the ladder is shown. To balance vertical forces, set $N_1 = mg$. To balance horizontal forces, $F_f = N_2$. Balancing torques about the point of contact with the floor, we have $mg(d/2)\cos 60^o = N_2(d)\sin 60^o$ where d is the length of the ladder. (Note that $(d/2)\cos 60^o$ is the component of the radius vector perpendicular to $m\mathbf{g}$, and $d\sin 60^o$ is the component of $\mathbf{r}$ perpendicular to $\mathbf{N}_2$.) Solving the third equation for N_2 gives $N_2 = (mg/2)\cot 60^o = mg/2\sqrt{3}$, so $F_f = N_2 = mg/2\sqrt{3}$ using the second equation. However, $F_f \le \mu_s N_1 = \mu_s mg$ from the first equation, so $mg/2\sqrt{3} \le \mu_s mg$, giving $\mu_s \ge 1/2\sqrt{3}$. Therefore $\mu_{s\,\min} = 1/2\sqrt{3}$.

PICTURE QUIZ 18.

1. (C) is correct. The acceleration might depend upon M, g, and the pulley's radius R and shape-factor f. Reasoning dimensionally, the acceleration can depend only upon g and the dimensionless shape factor f. Therefore if M doubles the acceleration is unaffected.

2. (B) is correct. If M is doubled without changing m, the gravitational torque is doubled and the inertia of the system is increased, but not doubled. That is, the inertia of the hanging weight increases, but not the inertia of the pulley. Therefore the acceleration will increase, but not double.

3. (B) is correct. The torque τ exerted on the larger wheel is twice as large, because the radius of the wheel is twice as large and the forces exerted are the same on each wheel. The angular momentum increase is $\Delta L = \tau \cdot t$ for constant torque, so ΔL is greater for the larger wheel.

PROBLEM QUIZ 18.

1. Isolation diagrams for the weight and for the wheel are shown. $F = ma$ applied to the weight gives (1) $mg - T = ma$. Equation $\tau = I\alpha$ applied to the wheel gives (2) $RT = MR^2\alpha = MRa$, since a and α are related by $a = R\alpha$, assuming the string does not slip on the wheel.
(a) Multiply Eq.(1) by R and add it to Eq.(2), to eliminate the tension T. Then solve for a to find $a = mg/(M + m)$. (b) The angular acceleration is then $\alpha = a/R = mg/(M + m)R$. (c) The momentum of m is $p = mv = mat$, since a is constant, and the weight starts at rest. Therefore $p = \left(\dfrac{m^2 g}{M + m}\right)t$. (d) The angular momentum of the wheel is $L = I\omega = I\alpha t = MR^2\,\alpha t$, since α is constant, and the wheel starts at rest. Therefore $L = \left(\dfrac{mMgR}{M + m}\right)t$.

2. The star has *constant* (negative) angular acceleration $\alpha = -\tau_0/I$. (a) Therefore we can use the formula $\omega = \omega_0 + \alpha t$ (with $\omega = 0$) to get $t = -\omega_0/\alpha = \omega_0 I/\tau_0$ as the time to come to rest. (b) We can also use the constant-angular-acceleration formula $\theta = \theta_0 + \omega_0 t + (1/2)\alpha t^2$ to find that $\theta - \theta_0 = \omega_0(\omega_0 I/\tau_0) + (1/2)(-\tau_0/I)(\omega_0 I/\tau_0)^2 = \omega_0{}^2 I/2\tau_0$, substituting in the time found in part (a). The number of revolutions before coming to rest is therefore $N = (\theta - \theta_0)/2\pi = \omega_0{}^2 I/4\pi\tau_0$.

3. (a) The net torque on the saucer is $2F_0R = I\alpha$, so the angular acceleration is $\alpha = 2F_0R/I$. Therefore $\omega_0 = \alpha t_0 = (2F_0R/I)t_0$. (b) It keeps spinning with the same angular velocity ω_0.

4. Isolation diagrams for the wheel and for the weight are shown. The torque on the wheel about its axis of symmetry is $\tau = (R/2)T$, where T is the tension in the cord. Therefore $\tau = I\ddot{\theta}$ becomes

$$(R/2)T = I\ddot{\theta}.$$

The $F = ma$ equation for the weight is

$$mg - T = ma = m(R/2)\ddot{\theta}\ ,$$

since the weight's acceleration a is related to the angular acceleration $\ddot{\theta}$ of the wheel by $a = (R/2)\ddot{\theta}$. Elimination of $\ddot{\theta}$ between the two equations and solving for the tension gives

$$T = \frac{mg}{\left[1 + \dfrac{mR^2}{4I}\right]}\ ,$$

independent of time (for $t > 0$). Note that $T < Mg$, the value it would have if the weight were not accelerating.

5. (a) From $\tau = I\ddot{\theta}$ we have $A\sin\beta t = I\ddot{\theta}$, so $\ddot{\theta} = (A\sin\beta t)/I$. (b) The angular velocity is

$$\dot{\theta} = \dot{\theta}_0 + \int_0^t \ddot{\theta}\,dt = \dot{\theta}_0 + (A/I)\int_0^t \sin\beta t\,dt = \dot{\theta}_0 + (A/I)\left.\frac{-\cos\beta t}{\beta}\right|_0^t = \omega_0 + (A/\beta I)(1-\cos\beta t).$$

(c) The rotation angle is therefore

$$\theta = \theta_0 + \int_0^t \dot{\theta}\,dt = 0 + \int_0^t \left[\omega_0 + (A/\beta I)(1-\cos\beta t)\right]dt = \omega_0 t + (A/\beta I)\left[t - \frac{\sin\beta t}{\beta}\right]$$

$$= \omega_0 t + \left(A/\beta^2 I\right)[\beta t - \sin\beta t].$$

PICTURE QUIZ 19

1. (C) is correct. The rotational kinetic energy of a hoop is proportional to the square of its radius and to the square of its angular speed. So their rotational kinetic energies are the same.

2. (C) is correct. The torque about the center of the pulley is $\tau = RT$, so the equation $\tau = I\alpha$ is $RT = (1/2)MR^2\alpha$, and so the angular acceleration is $\alpha = 2T/MR$. The angular acceleration is constant, so $\omega^2 = 2\alpha\theta = 2\alpha(2\pi)$ after one complete revolution. That is, $\omega^2 = 4\pi\,(2T/MR)$, so $T = MR\omega^2/8\pi$. You can check using dimensions; (C) is the only answer with the correct dimensions of force. We expect that T will need to increase if M is increased; this eliminates two answers. It should also increase as ω is increased, which eliminates one other answer. The problem can also be solved by the work-energy theorem; the work done by the cord tension shows up in rotational kinetic energy of the pulley.

3. (E) is correct. When rolling without slipping, the contact point is instantaneously at rest, the center is moving with speed v, and the uppermost point is moving with speed $2\,v$. (See Fig. 4)

4. (C) is correct. The translational K.E. is $(1/2)mv^2 = (1/2)mR^2\omega^2$, since $v = R\omega$ if the wheel rolls without slipping. This is the same as the rotational kinetic energy $(1/2)I\omega^2$ of the idealized bicycle wheel, since $I = mR^2$.

5. (C) is correct. They each lose the same gravitational potential energy, so they each have the same total kinetic energy at the bottom. The smaller disk has translational kinetic energy $\text{K.E.}_T = (1/2)mv^2$ and rotational kinetic energy $\text{K.E.}_R = (1/2)I\omega^2 = (1/2)[(1/2)mR^2]\omega^2 = (1/2)\text{K.E.}_T$ using the rolling-without-slipping condition $v = R\omega$. The rotational kinetic energy of the larger disk is also one-half of its translational kinetic energy. The translational kinetic energy of either disk is therefore two-thirds of its total kinetic energy. It follows that since both have the same total kinetic energy at the bottom, each has the same translational kinetic energy there as well.

PROBLEM QUIZ 19

1. The fan is equivalent to half a disk. If M_0 is the mass of the whole disk from which the fan-wheel is cut, the moment of inertia I of the half disk is only half that of the full disk, namely $\frac{1}{2}\left(\frac{1}{2}M_0R^2\right)$. However, the full disk also has twice the mass of the half disk, so $M_0 = 2M$. Therefore $I = \frac{1}{2}MR^2$. The angular momentum of the wheel is $L = I\omega_0 = \frac{1}{2}MR^2\omega_0$, and its kinetic energy is $K = \frac{1}{2}I\omega_0^2 = \frac{1}{4}MR^2\omega_0^2$.

2. The translational kinetic energy of the moving car is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}(1200\text{ kg})(11\text{ m/s})^2 = 72{,}600\text{ J}.$$

If 75% of this energy goes into rotational kinetic energy of the wheel, then

$$0.75 \cdot 72{,}600\text{ J} = 54{,}450\text{ J} = \frac{1}{2}I\omega^2 = \frac{1}{4}MR^2\omega^2 = \frac{1}{4}(12\text{ kg})(0.21\text{ m})^2\omega^2 = 0.132\text{ kg m}^2\,\omega^2.$$

Therefore the angular speed of the wheel is $\omega = \sqrt{\frac{54{,}450\text{ J}}{0.132\text{ kg m}^2}} = 642\text{ s}^{-1}$. The rotational frequency is $\nu = \frac{\omega}{2\pi} = 102\text{ rev/s} = 60(102)\text{ rev/min} = 6.1\cdot 10^3\text{ rpm}.$

3. The disk arrives first. The reason is that at any given altitude the total kinetic energy of each must be the same, because each loses the same gravitational potential energy. However, more of this kinetic energy must go into rotational kinetic energy for the hoop, so there is less left over for translational kinetic energy. The rotational and translational kinetic energies of the hoop are always equal to one another, but the rotational kinetic energy of the disk is only half that of its translational kinetic energy. This follows from the rolling-without-slipping condition $v = R\omega$ and the shape-factors 1 for the hoop and (1/2) for the disk. The fact that at any altitude the disk has more translational kinetic energy than the hoop means its speed will always be greater, so it will reach the bottom first.

4. An isolation diagram of either object is shown at the right.

There is no acceleration perpendicular to the plane, so the normal force $N = Mg\cos\theta$. The force of static friction therefore obeys the condition $F_{fric} \le \mu_s N = \mu_s Mg\cos\theta$.

The $F = ma$ equation down the plane gives $Mg\sin\theta - F_{fric} = Ma$, and the $\tau = I\alpha$ equation about the center of the object is $R\,F_{fric} = I\alpha = Ia/R$, since it rolls without slipping. Solve the $\tau = I\alpha$ equation for F_{fric}, and eliminate a using the $F = ma$ equation. The result is

$$F_{fric} = \left(\frac{I}{R^2}\right)a = \left(\frac{I}{R^2}\right)\left(\frac{Mg\sin\theta - F_{fric}}{M}\right),$$ which can be solved for F_{fric} to give

$F_{\text{fric}} = \dfrac{(Ig/R^2)\sin\theta}{1+(I/MR^2)} = \left(\dfrac{k}{1+k}\right)Mg\sin\theta \le \mu_s Mg\cos\theta$ using the shape factor k ($I = kMR^2$), and the inequality for F_{fric}. Dividing by $\cos\theta$ and solving for $\tan\theta$, the object will roll without slipping as long as $\tan\theta \le \mu_s\left(\dfrac{1+k}{k}\right)$. For a solid cylinder, $\left(\dfrac{1+k}{k}\right) = \left(\dfrac{1+1/2}{1/2}\right) = 3$, and for a sphere, $\left(\dfrac{1+k}{k}\right) = \left(\dfrac{1+2/5}{2/5}\right) = 3.5$. The cylinder will slip first, because its maximum angle is less.

PICTURE QUIZ 20

1. (D) west The direction follows from the right-hand rule.

2. (A) $+x$. Using unit vectors, $\hat{\mathbf{z}} \times (-\hat{\mathbf{y}}) = -\hat{\mathbf{z}} \times \hat{\mathbf{y}} = +\hat{\mathbf{x}}$.

3. (C). The $\mathbf{r}$ vector points from the origin to the place where the force is applied; the resulting torque is counterclockwise, out of the page by the right-hand rule.

4. (D) $\dfrac{2}{3}$. The orbital angular momentum about a point on the field is $\mathbf{L}_{\text{orb}} = \mathbf{R} \times M\mathbf{v}$, whose magnitude is $L_{\text{orb}} = R_\perp Mv$, where $R_\perp$ is the component of the $\mathbf{R}$ vector extending from the point on the field to the center of the ball, which is perpendicular to the ball's velocity v. The ball is moving horizontally, so $R_\perp$ is just the ball's radius R. Also $v = R\omega$, since the ball rolls without slipping with angular speed ω, so $L_{\text{orb}} = RM(R\omega) = MR^2\omega$. A soccer ball is nearly a hollow sphere, with moment of inertia $I = \dfrac{2}{3}MR^2$ about an axis through its center, so the magnitude of the spin angular momentum is therefore $L_{\text{spin}} = I\omega = \dfrac{2}{3}MR^2\omega$. The spin angular momentum is in the same direction as the orbital angular momentum, so their ratio is 2/3.

5. (b), out of the page. By the right-hand rule, the initial angular momentum of the wheel is to the *left*. The torque τ on the axle-wheel combination about the point of support (where the chain is attached to the axle) is *into* the page, also by the right-hand rule. That must also be the direction of $\Delta\mathbf{L}$, the initial change of angular momentum in the short time Δt, since $\Delta\mathbf{L} = \tau\Delta t$. The angular momentum after time Δt is the sum, $\mathbf{L}(t+\Delta t) = \mathbf{L}(t) + \Delta\mathbf{L}$. The vectors are viewed from above in the diagram shown. The new angular momentum is still along the direction of the axle, so the wheel must have precessed out of the page.

PROBLEM QUIZ 20

1. (b) $(2\,\hat{\mathbf{y}} + \hat{\mathbf{z}})$ m/s. If $\mathbf{v}$ had any x component, it would give a y or z component of force. Answer (b) is the only one without an x component.

2. Both wires are 3 m from the x axis, and 4 m from either the positive or negative y axis. Therefore each is 5 m from the bottom of the pole, by the law of Pythagoras. Since the upper end of each wire is 5 m above the ground, that means that the force on the pole exerted by each wire has equal vertical and horizontal components, each $\cos 45^\circ = 1/\sqrt{2}$ times the wire tension. From the given geometry, the force of each wire in the x direction is 4/5

times the horizontal component, and the force of each wire in the y direction (+ or −) is 3/5 times the horizontal component. In terms of unit vectors,

$$\mathbf{F}_o = -T_o\,\hat{\mathbf{x}} = -25\,\hat{\mathbf{x}}\ \text{N}. \qquad \mathbf{F}_1 = F_1\left[-\frac{1}{\sqrt{2}}\hat{\mathbf{z}} + \frac{1}{\sqrt{2}}\left(\frac{4}{5}\hat{\mathbf{x}} + \frac{3}{5}\hat{\mathbf{y}}\right)\right]$$

$$\mathbf{F}_2 = F_2\left[-\frac{1}{\sqrt{2}}\hat{\mathbf{z}} + \frac{1}{\sqrt{2}}\left(\frac{4}{5}\hat{\mathbf{x}} - \frac{3}{5}\hat{\mathbf{y}}\right)\right]$$

where F_1 and F_2 are the magnitudes of these two forces, the tension in the wires. There is also the gravitational force downward on the pole, and the force of the socket on the bottom of the pole, with both vertical and horizontal components. However, these forces exert no torque on the pole about the bottom of the pole as origin, because $\mathbf{r} = 0$ for each. The nonzero torques are

$$\tau_o = \mathbf{r}_o \times \mathbf{F}_o = (10\,\hat{\mathbf{z}}) \times (-25\,\hat{\mathbf{x}})\ \text{N m} = -250\ \hat{\mathbf{y}}\ \text{N m}$$

$$\tau_1 = \mathbf{r}_1 \times \mathbf{F}_1 = (5\,\hat{\mathbf{z}}) \times F_1\left[-\frac{1}{\sqrt{2}}\hat{\mathbf{z}} + \frac{1}{\sqrt{2}}\left(\frac{4}{5}\hat{\mathbf{x}} + \frac{3}{5}\hat{\mathbf{y}}\right)\right] = \frac{1}{\sqrt{2}}F_1\,(4\hat{\mathbf{y}} - 3\hat{\mathbf{x}})$$

$$\tau_2 = \mathbf{r}_2 \times \mathbf{F}_2 = (5\,\hat{\mathbf{z}}) \times F_2\left[-\frac{1}{\sqrt{2}}\hat{\mathbf{z}} + \frac{1}{\sqrt{2}}\left(\frac{4}{5}\hat{\mathbf{x}} - \frac{3}{5}\hat{\mathbf{y}}\right)\right] = \frac{1}{\sqrt{2}}F_2\,(4\hat{\mathbf{y}} + 3\hat{\mathbf{x}})$$

Note that the x components of torque sum to zero only if $F_1 = F_2$. We could also have realized this equality from the symmetry of the situtation. The y components of torque sum to zero if

$$-250\ \text{N} + 2\frac{4}{\sqrt{2}}F_1 = 0, \text{ so the tension in each wire is } F_1 = \frac{250\sqrt{2}}{8}\ \text{N} = 44\ \text{N}.$$

3. An isolation diagram of the ball is shown. The only horizontal force is sliding friction, $F_f = -\mu_k N = \mu_k Mg$, where the positive direction is to the right. The acceleration of the ball is therefore $a = -\mu_k g$, which is constant, so the velocity of the ball is $v = v_o + at = v_o - \mu_k g t$. The orbital angular about a point on the alley is $\mathbf{L}_{orb} = \mathbf{R} \times M\mathbf{v}$, with magnitude $L_{orb} = RMv$, as explained in the solution to Picture Quiz Problem 4. Therefore the orbital angular momentum as a function of time has the magnitude $L_{orb} = RM(v_o - \mu_k g t)$. The torque on the ball due to friction is $\tau = F_f R = \mu_k MgR$ about its center. The torque is constant, so the spin angular momentum increases at the constant rate $L_{spin} = \tau t = \mu_k MgRt$, the integral of $\tau = dL/dt$. The total angular momentum is constant, since the sum of L_{orb} and L_{spin} is always RMv_o, the same as it was at the beginning.

4. The onboard gyro might be mounted with a vertical axis, for example. Suppose it is spinning so its angular momentum is vertically upward (spinning counterclockwise as seen from above). Then if the ship starts to roll to the right this causes a torque on the gyro directed toward the front of the ship. The gyro therefore starts to precess with the upper axis tilting toward the front of the ship. Sensors can detect this precession, and then direct motors to tilt the vanes on the ship sides to raise the right side of the ship and lower the left side.

5. Suppose for example the motorcycle is moving eastward on a level road. The angular momentum vector of the front wheel is therefore pointing north, using the right-hand rule. If the rider pushes the handlebars to the left, the angular momentum vector of the front wheel gains a small component toward the west (although most of its angular momentum is still directed toward the north). In reaction, the bike leans additionally to the right, which gives the bike a small additional angular momentum to the east, tending to cancel the angular momentum component toward the west of the front wheel as the bike "tries" to conserve angular momentum. That is, the gyroscopic effect *helps* the bike lean to the right.

PICTURE QUIZ A

1. (D) is correct. Each satellite obeys $\frac{GMm}{r^2} = ma = \frac{mv^2}{r}$ where M is the mass of the asteroid and m is the mass of the satellite. Therefore $v = \sqrt{\frac{GM}{r}}$. The speeds of satellites vary as $r^{-1/2}$, so if r is quadrupled, $v = v_0/2$.

2. (C) is correct. The angular momentum of an object in circular orbit is $L = mvr$, so the outer satellite has angular momentum $L = (M/2)(v_0/2)(4R) = M v_0 R = L_0$ (Here M is the mass of the inner satellite).

3. (C) is correct. The semimajor axis of the orbit is 9 A. U. The period of orbits goes like the semimajor axis to the 3/2 power, according to Kepler's third law. Therefore the period will be increased over the period of the Earth by the factor $9 \cdot 3 = 27$.

4. (A) is correct. By conservation of angular momentum the product $r\,v$ is the same at perihelion and aphelion. So since the radius at aphelion is 17 times larger, the velocity there must be $v_0/17$.

PROBLEM QUIZ A

1. As it orbits the Sun, Jupiter (treated as a point particle) has angular momentum $L_J = m_J r_J^2 \omega_J = m_J r_J^2 \left(\frac{2\pi}{T_J}\right)$. Treating the Sun as a sphere of uniform density, its angular momentum is $L_S = I_S \omega_S = \frac{2}{5} m_S R_S^2 \left(\frac{2\pi}{T_S}\right)$. The ratio is

$$\frac{L_J}{L_S} = \frac{m_J R_J^2 / T_J}{\frac{2}{5} m_S R_S^2 / T_S} = \frac{5}{2}\left(\frac{m_J}{m_S}\right)\left(\frac{R_J}{R_S}\right)^2\left(\frac{T_S}{T_J}\right) \sim \frac{5}{2}(.001)(1000)^2\left(\frac{25 \text{ days}}{12 \cdot 365 \text{ days}}\right) \sim 14,$$

so in this model Jupiter has much more angular momentum than the Sun. If the Sun were treated more accurately as a sphere with larger density in the center than near the rim, its moment of inertia would be smaller (i.e. the shape factor would be less than 2/5), so $\frac{L_J}{L_S}$ would be even larger.

2. (a) The orbital shape (an ellipse) is sketched at the right.

(b) The semimajor axis of this elllipse is

$$a = \frac{R_o + 3R_o}{2} = 2R_o.$$

Therefore by Kepler's third law the period of the satellite's orbit is $T = T_o\left(\frac{a}{R_o}\right)^{3/2} = T_o(2)^{3/2} = 2\sqrt{2}\,T_o.$

(c) The satellite's velocity at perigee is Earth's velocity plus the boost velocity, i.e. $v_p = \frac{2\pi R_o}{T_o} + \Delta v$. According to conservation of angular momentum,

$$L = mv_p r_p = mv_a r_a, \quad \text{so} \quad v_a = (r_p/r_a)v_p = 3v_p = 3(\frac{2\pi R_o}{T_o} + \Delta v).$$

3. (a) Note that $\dot{r} = v$, so the radial velocity is $v_r = \dot{r} = v$ until the elevator stops at the rim. (b) The tangential velocity $v_\theta = r\dot{\theta} = r\omega = vt\omega = (v\omega)\,t$, increasing linearly with time. (c) The force on the elevator in the radial direction is

$$F_r = Ma_r = M(\ddot{r} - r\dot{\theta}^2) = M(0 - vt\omega^2) = -Mv\omega^2 t$$

(minus means toward the center). This is the force exerted on the elevator by the cables, linearly increasing with t.

(d) The force on the elevator in the tangential direction, which is due to the tube, is

$$F_\theta = M(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = M(0 + 2v\omega) = 2Mv\omega,$$

independent of time until the elevator starts slowing down.

4. The speed v of the car is increasing at constant rate as it moves in a circle, so its angular speed ω also increases at constant rate. Therefore we can use Eq. 25(d) of Day 18,

$$\omega^2 = \omega_o^2 + 2\alpha_o(\theta - \theta_o).$$

Multiplying this equation by R^2, the square of the track radius, we have (using $v = R\omega$) $v^2 = v_o^2 + 2(R\alpha_o)s$, where $s = R(\theta - \theta_o)$ is the distance traveled. Therefore $R\alpha_o = (v^2 - v_o^2)/2s$. The tangential component of acceleration is $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = R\alpha_o + 0 = R\alpha_o = (v^2 - v_o^2)/2s = [(41.66\text{ m/s})^2 - (6.94\text{ m/s})^2]/240\text{ m} = 7.0\text{ m/s}^2$.

The radial component of acceleration is $a_r = \ddot{r} - r\dot{\theta}^2 = 0 - r\omega^2 = -v^2/r = -(41.66\text{ m/s})^2/250\text{ m} = 6.9\text{ m/s}^2$.

The radial and tangential accelerations are perpendicular to one another, so the magnitude of the total acceleration is $a = \sqrt{a_r^2 + a_\theta^2} = 9.8\text{ m/s}^2$.

PICTURE QUIZ B

1. (A) is correct. If the position of the block is $x = A\cos\omega t$, then its velocity is $v = \dot{x} = -A\omega\sin\omega t$, which corresponds to the graph with $A\omega = 2.0$ m/s. The period of the motion is $T = 2\pi/\omega = 4.0$ s, as read from the graph, so the angular frequency is $\omega = 2\pi/T = 1.6\ \text{s}^{-1}$, and the amplitude is $A = 2.0\ \text{m/s}/\omega = 1.3$ m.

2. (D) is correct. The force constant of the half-spring is twice that of the original spring. The reason is that a half-spring stretches half as much as the full spring under the same force, so the half-spring is twice as stiff, which means it has twice the force constant. (Think of the full spring as two half-springs end to end; if it stretches a certain distance under a given force, each half of it will stretch half that much.) The oscillation frequency is $\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$, so if k doubles, ν rises by the factor $\sqrt{2}$.

3. (a) (A) is correct; the period is $T = 2.0$ s, so $\nu = 1/T = 0.50$ Hz. (b) (D) is correct; the angular frequency is $\omega = 2\pi\nu = 3.1\ \text{s}^{-1}$ (c) (C) is correct; the amplitude is $A = 3.0$ Amperes. (d) (B) is correct; to make $i = 0$ at time $t = 0$, we need $\phi_0 = \pm\frac{\pi}{2}$, so that the current is $i = A\cos(\omega t \pm \frac{\pi}{2}) = A[\cos\omega t\cos\frac{\pi}{2} \mp \sin\omega t\sin\frac{\pi}{2}] = \mp A\sin\omega t$. The graph shows a *positive* sine function, so we need the lower sign. The phase angle is therefore $\phi_0 = -\frac{\pi}{2}$. (e) (B) is correct; at time $t = 1.0$ s, the phase is $\omega t + \phi_0 = (3.1\ \text{s}^{-1}\cdot 1.0\ \text{s}) - \frac{\pi}{2} = 1.6$.

PROBLEM QUIZ B

1. (a) The angular frequency is $\omega = \sqrt{k/m} = 2\pi/T_0$, so the mass is $m = kT_0^2/4\pi^2$. (b) The amplitude is x_0, since the block is released from rest with that compression. (c) Since the position of the block is $x = A\cos\omega t$, its velocity is $v = \dot{x} = -A\omega\sin\omega t$, so its maximum speed is $v = A\omega$, which is $2\pi x_0/T_0$.

2. Suppose the car is supported by one spring above each wheel (the answer at the end is the same if there is *more* than one spring above each wheel.) Then the extra weight mg on each spring causes a sag d, so (balancing forces) the force constant of each spring is $k = mg/d$. When the people step out, the car is not in its equilbrium position; it then oscillates with frequency $\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k_{\text{eff}}}{M}} = \frac{1}{2\pi}\sqrt{\frac{4k}{M}}$. (The four springs together have an effective force constant $k_{\text{eff}} = 4k$.)

3. The position is $x(t) = A\cos(\omega t + \phi_0)$, and its velocity is $v(t) = -A\omega\sin(\omega t + \phi_0)$. At time $t = 0$ these reduce to $x = A\cos(\phi_0) = 49$ cm, and $v = -A\omega\sin(\phi_0) = -32$ cm/s. The angular frequency is $\omega = 2\pi/T = 1.047\ \text{s}^{-1}$, so ϕ_0 can be eliminated in the x and v equations to give (a) the amplitude $A = 58$ cm.

(b) The oscillator energy is $E = \frac{1}{2}kA^2 = \frac{1}{2}(m\omega^2)A^2 = 1.3 \cdot 10^{-2}$ J. The maximum speed corresponds to $\frac{1}{2}mv^2 = \frac{1}{2}kA^2$ when all the energy is kinetic energy. The result is $v = \sqrt{\frac{2}{m}\left(\frac{1}{2}kA^2\right)} = 61$ cm/s.

4. (a) Conserve momentum during the brief time that the bullet comes to rest relative to the block, so the upward velocity of the block-bullet combination is $V = mv/(M+m)$ just after the collision. The kinetic energy at this time is $(1/2)(M+m)V^2 = (1/2)m^2v^2/(M+m)$. Before impact, with the block hanging on the spring, the spring is stretched a distance $x_o = Mg/k$ (x_o measured positive downward), found by balancing gravity with the spring force. Just after capture, the energy of the system is $(1/2)kx_o^2 - (M + m)gx_o + (1/2)m^2v^2/(M+m)$, taking the zero of gravitational potential energy to be the bottom of the unstretched spring. At the highest position x_f of the block + bullet, when they have instantaneously come to rest, the energy of the system is purely potential energy, $(1/2)kx_f^2 - (M + m)gx_f$. Conserving energy, we have $(1/2)kx_f^2 - (M + m)gx_f = (1/2)kx_o^2 - (M + m)gx_o + (1/2)m^2v^2/(M+m)$. This is a quadratic eqt. in x_f, with solution (using the quadratic formula) $x_f = (Mg/k) - (mg/k)\left[\sqrt{1 + kv^2/(M+m)g^2} - 1\right]$ using $x_o = Mg/k$. The height achieved by the system above its starting point is therefore $h = x_o - x_f = (mg/k)\left[\sqrt{1 + kv^2/(M+m)g^2} - 1\right]$. (b) The equilibrium position of the system is $x_{eq} = (M + m)g/k$, so the amplitude of oscillation is $A = x_{eq} - x_f = (mg/k)\sqrt{1 + kv^2/(M+m)g^2}$. (c) The angular frequency is $\omega = \sqrt{k/(M+m)}$.

PICTURE QUIZ C

1. The period of the spring-mass system does not depend upon gravity, while the period of the pendulum is inversely proportional to the square root of g. Therefore as g is reduced as a consequence of moving the experiment to the Moon, the period of the pendulum is increased while the period of the SHO is unchanged. Answer (B) is correct: on the Moon, $T_{\text{Pendulum}}/T_{\text{SHO}} > 1$.

2. For a simple-harmonic oscillator type equation, the angular oscillation frequency is equal to the square root of the coefficient of the undotted variable, which is $\sqrt{g/\ell}$ for the small-amplitude pendulum equation. Therefore answers (A) and (D) are excluded. Answer (B) is also wrong, because $A\tan(\omega t)$ does not satisfy the equation. Answers (C) and (E) are both solutions.

PROBLEM QUIZ C

1. To be off by 1%, the first correction term $(\theta_o^2/16)$ in Eq. 10 must be about 1% = 0.01 (The second correction term would then be much smaller still.) This gives $\theta_o = 0.4$ rad = 23°.

2. According to *Sample Problem 1*, the angular frequency of a small-amplitude physical pendulum is $\omega = \sqrt{\frac{Mgd}{I}}$, where I is the moment of inertia of the object about the chosen axis of rotation, and d is the distance between that axis and the center of mass of the object. In this problem, the distance $d = \ell$. The moment of inertia of the rod about its center is $\frac{1}{12}ML^2$, so by the parallel-axis theorem, $I = \frac{1}{12}ML^2 + M\ell^2$.

The period of small-amplitude oscillations is therefore

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{Mgd}} = 2\pi\sqrt{\frac{\frac{1}{12}ML^2 + M\ell^2}{Mg\ell}} = 2\pi\sqrt{\frac{\frac{1}{12}L^2 + \ell^2}{g\ell}}.$$

The period is a minimum if the quantity inside the square root is a minimum; that is, if

$$\frac{d}{d\ell}\left(\frac{\frac{1}{12}L^2 + \ell^2}{g\ell}\right) = \frac{1}{g}\left[\frac{2\ell}{\ell} - \frac{L^2/12 + \ell^2}{\ell^2}\right] = \frac{1}{g}\left[1 - \frac{L^2}{12\ell^2}\right] = 0.$$

The period is a minimum if $\ell = \frac{1}{\sqrt{12}}L = \frac{1}{2\sqrt{3}}L$.

3. Suppose the hoop rolls to the right through a small angle θ along the surface. Then the center of the hoop moves a distance $s = R\theta$ to the right, and a point at the top of the hoop moves a distance $2s = 2R\theta$ to the right. (That is, a point at the top moves the distance s shown at the bottom of the hoop, *plus* the distance s shown at the top of the hoop. See also Fig. 3 in Chapter 4 on Rotation.)

The spring at the right is compressed a distance $2s = 2R\theta$, and the spring at the left is *stretched* the same distance $2R\theta$. Both springs therefore exert forces to the left, with total force $k(4R\theta)$. These forces exert torques about a point on floor beneath the hoop; the total torque has magnitude $k(4R\theta)(2R) = 8kR^2\theta$ and direction *counter*clockwise (i.e., out of the page.) The only other horizontal force on the hoop is due to friction with the surface; this force, however, exerts no torque relative to a point on the surface. We can therefore use $\tau = I\ddot{\theta}$ where $\ddot{\theta}$ is positive for increasing θ, i.e. for clockwise rotation. Taking this clockwise direction as positive, we have

$$-8kR^2\theta = I\ddot{\theta} = (2MR^2)\ddot{\theta}, \qquad \text{so} \qquad \ddot{\theta} + (4k/M)\theta = 0,$$

since $I = 2MR^2$ about the point on the surface, using the parallel-axis theorem. The angular frequency of oscillation is therefore $\omega = 2\sqrt{k/M}$, so the period is $T = \frac{2\pi}{\omega} = \pi\sqrt{\frac{M}{k}}$.

4. (a) The effective potential energy is a minimum if

$$\frac{d}{dr}(U_{\text{eff}}) = \frac{d}{dr}\left(\frac{L^2}{2mr^2} - \frac{GMm}{r}\right) = \frac{-L^2}{mr^3} + \frac{GMm}{r^2} = 0.$$

Solving for r, the radius of the circular orbit for given m and L is $r_0 = \frac{(L/m)^2}{GM}$.

(b) To find the oscillation period about the circular orbit, we can use $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k_{\text{eff}}}}$, where $k_{\text{eff}} = U''(r_o)$, the second derivative of the potential energy, evaluated at the minimum. The second derivative is

$$\frac{d^2}{dr^2}(U_{\text{eff}}) = \frac{d}{dr}\left(\frac{-L^2}{mr^3} + \frac{GMm}{r^2}\right) = \frac{3L^2}{mr^4} - \frac{2GMm}{r^3}.$$

Substituting $r = r_o = \frac{(L/m)^2}{GM}$, we get $U''(r_o) = k_{\text{eff}} = \frac{(GM)^4 m}{(L/m)^6}$.

The period of these oscillations is

$$T = 2\pi\sqrt{\frac{m}{k_{\text{eff}}}} = 2\pi\sqrt{\frac{(L/m)^6}{(GM)^4}} = 2\pi\frac{(L/m)^3}{(GM)^2}.$$

How does this compare with the time to orbit the Sun? For a circular orbit, $F = ma$ becomes

$$\frac{GMm}{r^2} = ma_{\text{centripetal}} = mr\omega^2 = mr\left(\frac{2\pi}{T}\right)^2$$

Solving for T, $T = 2\pi\sqrt{\frac{r^3}{GM}} = 2\pi\sqrt{\frac{((L/m)^2/GM)^3}{GM}} = 2\pi\frac{(L/m)^3}{(GM)^2}$, which is exactly the same as the oscillation period! What does that mean?

It means that the planet oscillates inside and outside of the orbit once each revolution, as sketched below at the left. The path is closed, meaning that the orbit repeats over and over again, going from the closest point to the Sun (called perihelion) to the farthest point (called aphelion). Because the orbit is closed, these points are always at the same location in space. An example of a hypothetical nonclosed orbit is shown at the right for comparison.

5. (a) The amplitude is cut by a factor of two as time goes from zero to 4.0 s. Therefore the damping factor satisfies $e^{-\beta t} = e^{-\beta(4.0\text{ s})} = 0.5$. Solving for β (using a calculator or logarithm tables), the constant $\beta \cong 0.17\text{ s}^{-1}$. (Note that $\beta \equiv b/2m$, so the factor $b = 2m\beta = 2(0.10\text{ kg})(0.17\text{ s}^{-1}) = .034$ kg/s.)

(b) The oscillation period is $T = 2.0$ s, so the angular frequency is $\omega_1 = 2\pi/T = 3.14\text{ s}^{-1}$. The natural frequency of the undamped spring is

$$\omega_o = \sqrt{\omega_1^2 + \beta^2} = \sqrt{3.14^2 + (0.17)^2}\text{ s}^{-1} = 3.14\text{ s}^{-1}$$

using Eq. 6, so the force-constant of the spring is

$$k = m\omega_0^2 = 0.1 \text{ kg } (3.14 \text{ s}^{-1})^2 = 0.99 \text{ kg/s}^2 = 0.99 \text{ N/m}.$$

(c) *Sample Problem 1* shows that $\phi_0 = -\tan^{-1}(\beta/\omega_1)$, so here $\phi_0 = -\tan^{-1}(0.17/3.14) = -.054$ radians.

PICTURE QUIZ D

1. (a) From the time plot, the period is $T = 12$ s, so (D) is correct. (b) the angular frequency is $\omega = 2\pi/T = 0.52 \text{ s}^{-1}$, so (B) is correct. (c) From the position plot, the wavelength is $\lambda = 4$ m, so (D) is correct. (d) the wave number is $k = 2\pi/\lambda = 1.6 \text{ m}^{-1}$, so (E) is correct. (e) The speed is $v = \omega/k = 0.33$ m/s, so (A) is correct. (f) Assuming the wave has the form $y = A\sin(kx \pm \omega t - \phi)$, the phase angle $\phi = \frac{\pi}{2}$, since that gives the correct phase for both plots. For example, the upper plot $y(x, 0) = A\sin(kx - \frac{\pi}{2}) = -A\cos kx$, which is the function drawn, if the amplitude A taken to be positive, which is customary. Therefore (C) is correct.

2. (a) (D) is correct, since one sees 6 antinodes (maxima or minima) on the standing wave. (b) The standing wave has the form $y = 2A\cos\omega t \sin kx$, where $k = 2\pi/\lambda$ is the same for the standing wave as for the two traveling waves from which it is made. Therefore (B) is correct; the wavelength is 0.20 m. (c) (D) is correct; $\nu = v/\lambda = 30$ m/s. (d) (E) is correct; $F = \mu v^2 = 3.6$ N.

PROBLEM QUIZ D

1. Solutions must be a function of either $x + vt$ or $x - vt$. Only answers (b) and (d) qualify. If t increases by Δt in answer (b), x must *decrease* by $(4/2)\Delta t$ to keep the phase constant. Therefore this wave travels to the *left* at speed (4/2) m/s = 2 m/s. If t increases by Δt in answer (d), x must *increase* by $(7/3)\Delta t$ to keep the phase constant. Therefore this wave travels to the *right* at speed (7/3) m/s.

2. The tension in the string is 10 N. (*Not* 20 N! - An isolation diagram of the left-hand weight, for example, shows that the tension must be 10 N to support the 10 N weight against gravity.) The traveling wave velocity is therefore $v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{10 \text{ N}}{0.004 \text{ kg/m}}} = 50$ m/s. The longest standing-wave wavelength that can be set up on the string is 1.0 m, so the corresponding frequency is $\nu = v/\lambda = 50 \text{ m/s}/1.0 \text{ m} = 50$ Hz. (b) Doubling the frequency keeping λ the same requires that the tension be quadrupled, so one would have to add a 30 N weight to each side.

3. (a) The viola string would have to be 1.5 times as long as the violin string and the cello string would have to be $1.5 \cdot 2 = 3$ times as long as the violin string. That is because the wavelength of the fundamental tone scales as the length of the string, and the frequency is inversely proportional to the wavelength if the wave velocity is fixed, as is true if both μ and F remain the same. (b) The viola and cello are not as large as the results found in part (a). The viola cannot be that large because it would be very difficult to tuck under the chin and hold up, and the cello would be awkwardly large to play while seated. Instead, some combination of increased μ and decreased F are

needed to reduce the wave velocity, so the frequencies are lowered more than one would expect by increasing the wavelength by only a factor of 1.17 or 2.13. Typically μ is increased by using heavier strings. The lowest two viola strings are wrapped in fine copper wire, for example, to increase their densities. (c) The fundamental of the viola C string has only 2/3 the frequency of the violin's G string. Therefore if there is one full wavelength on the violin string and one and one-half wavelengths on the viola string, they will both have the same frequency. This frequency is the G one octave above the fundamental of the violin string, i.e., a frequency twice as high as the fundamental G.

PICTURE QUIZ E

1. (a) (D) is correct, since if the wave spreads out in three dimensions its intensity goes like $1/r^2$. (b) (C) is correct. The sound level is $SL = 10 \log(I/I_0)$ decibels initially, rising to

$$SL = 10 \log(100I/I_0) = 10 [\log 100 + \log I/I_0] = 10 [2 + \log I/I_0]$$

when the intensity has increased by a factor of 100. Therefore the sound level has increased by 20 decibels.

2. (a) (E) is correct; the average power carried by the wave is proportional to its amplitude squared, so the power increases by a factor of $3^2 = 9$. (b) (E) is correct; the average power is also proportional to the frequency squared, so if the wavelength is cut in half the frequency is doubled, so the power increases by a factor of four.

3. (C) is correct. The Doppler effect has increased the frequency by 22 Hz, which is 5% of the original frequency. Therefore the truck must be moving about 5% the speed of sound, or 18 m/s.

PROBLEM QUIZ E

1. (a) The phase velocity of shallow surface-water waves is $v_p = \omega/k = \sqrt{gh}$, so $\omega = \sqrt{gh}\ k$. The group velocity is then $v_g = \dfrac{\partial\omega}{\partial k} = \sqrt{gh}$. The group velocity is the same as the phase velocity in this case. (b) The phase velocity of deep surface-water waves is $v_p = \omega/k = \sqrt{\dfrac{g\lambda}{2\pi}} = \sqrt{\dfrac{g}{k}}$, so $\omega = \sqrt{gk}$. The group velocity of these waves is therefore $v_g = \dfrac{\partial\omega}{\partial k} = \dfrac{1}{2}\sqrt{\dfrac{g}{k}} = \dfrac{1}{2}v_p$. The group velocity is half the phase velocity in this case.

2. We can use dimensional reasoning. In terms of the fundamental dimensions of mass M, length L, and time T, the dimensions of the relevant quantities are $[I] = \left[\dfrac{P}{A}\right] = \dfrac{ML^2/T^3}{L^2} = M/T^3$, $[\Delta p] = M/LT^2$, $[\rho] = M/L^3$, and $[v] = L/T$. We know that I is proportional to $(\Delta p)^2$, with dimensions M^2/L^2T^4. So we have to divide $(\Delta p)^2$ by ρ to eliminate one of the M's, with overall dimensions $[\ (\Delta p)^2 / \rho\ ] = ML/T^4$. Then if we divide by v, we get the correct dimensions for I. That is, $I = C\ (\Delta p)^2 / \rho v$, where C is an unknown dimensionless constant. Therefore if the two fluids have different densities, with $\rho_B = 2\rho_A$, but the other variables are the same for particular sound waves, the $I_B/I_A = \rho_A/\rho_B = 1/2$.

3. (a) The observer is moving toward the emitter, and will hear a higher pitch, with

$$\nu_{ob} = (1 + v_{ob}/v_s)\nu_{em} = (1 + 15/331)\ 950\ \text{Hz} = 993\ \text{Hz}.$$

(b) Relative to the rest-frame of the air, the observer is moving toward the emitter at 15 m/s, and the emitter is moving away from the observer at 15 m/s. Therefore

$$\nu_{ob} = \left[\frac{1 + v_{ob}/v_s}{1 + v_{em}/v_s}\right]\nu_{em} = \left[\frac{1 + 15/331}{1 + 15/331}\right]\nu_{em} = \nu_{em} = 950\ \text{Hz}.$$

4. From *Sample Problem* 2, the speed of sound is $v = \sqrt{\gamma \frac{RT}{M}}$, where γ is the ratio of specific heats, R is the gas constant, T is the absolute temperature, and M is the molecular weight. For helium gas, which is monatomic, $\gamma = 5/3$ (See the discussion in *Sample Problem* 2.) Substituting in the values, we find that the speed of sound in helium is 972 m/s, nearly three times as fast as the speed of sound in air for the same temperature. The main reason for the difference is the much smaller molecular weight in helium, although there is also a somewhat larger value of γ for helium, which also serves to raise its sound velocity. The higher sound speed in helium leads to interesting effects if you talk after breathing in helium from a balloon.

5. The bulk modulus of ethanol is $B = \rho v^2 = (790\ \text{kg/m}^3)\ (1207\ \text{m/s})^2 = 1.15 \cdot 10^9\ \text{N/m}^2$. The fractional volume change in compressing a one meter-long cylinder by 1 mm is $\Delta V/V = 10^{-3}$. The extra pressure required therefore has the magnitude $\Delta p = B(\Delta V/V) = 1.15 \cdot 10^6\ \text{N/m}^2$. The area of the piston is $A = \pi r^2 = \pi(.030\ \text{m})^2 = 2.83 \cdot 10^{-3}\ \text{m}^2$, so the force one must exert on the piston is $F = (\Delta p)A = 3.3 \cdot 10^3$ N. For comparison, to support one cubic meter of ethanol against the force of gravity on the Earth would require more than twice the force, $mg = 790\ \text{kg} \cdot 9.8\ \text{m/s}^2 = 7.7 \cdot 10^3$ N.

PICTURE QUIZ F

1. (a) (C) is correct; the overtone has three times the frequency of the fundamental, so the pipe could be open-open (with n = 3 in Eq. 5) or open-closed (with n = 1 in Eq. 7). (b) (A) is correct; the additional overtone has a frequency four times that of the fundamental, so the pipe must be open-open. Only odd multiples of the fundamental can exist in an open-closed pipe.

2. (B) and (C) are both correct. The frequency of the first flute is $v/\lambda = 360\ \text{m/s} / 0.500\ \text{m} = 720$ Hz. The beat frequency is 10 Hz, so the second flute could have a frequency of 710 Hz or 730 Hz.

3. (B) is correct. There must be an integer number of extra wavelengths in the lower beam, to interfere constructively with the upper beam. Therefore $\sin\theta = \lambda/d = 1/2$, so $\theta = 30^\circ$. One cannot fit *two* extra wavelengths unless $\theta = 90^\circ$.

PROBLEM QUIZ F

1. The beat frequency is $3\text{ Hz} = |\nu_1 - \nu_2|$, so the violin is playing either 437 Hz or 443 Hz. The frequency of the string vibration is related to string tension by $\nu = v/\lambda = \frac{1}{\lambda}\sqrt{\frac{F}{\mu}}$. For small frequency changes we can take the differential $\Delta\nu = \frac{1}{\lambda\sqrt{\mu}}\frac{1}{2}F^{-1/2}\Delta F$, and then divide by ν to find $\frac{\Delta\nu}{\nu} = \frac{1}{2}\frac{\Delta F}{F}$. The fractional change in tension required is therefore

$$\frac{\Delta F}{F} = 2\frac{\Delta\nu}{\nu} = \pm\, 2\,\frac{3\text{ Hz}}{440\text{ Hz}} = \pm\, 0.014\,.$$

2. (a) The fundamental has a wavelength equal to twice the length of an open-open pipe, i.e. $\lambda = v_S/\nu = 2L$. Therefore the length is $L = v_S/2\nu = 330\text{ m/s}/660\text{ Hz} = 0.50\text{ m}$. (b) The frequencies of the first and second overtones are twice and three times the fundamental, 660 Hz and 990 Hz (See Fig. 1). (c) The fundamental has only one-quarter of a wavelength within an open-closed pipe, so the new fundamental wavelength is twice as long and the frequency is half as much, 165 Hz.

3. There must be an extra half wavelength in the longer path, so (as explained in the text) $\lambda/2 = d\sin\theta$, so

$$\sin\theta = \frac{\lambda}{2d} = \frac{v/\nu}{2d} = \frac{331\text{ m/s}}{2\cdot 2.0\text{ m}\cdot 512\text{ Hz}} = 0.16.$$

Therefore $\theta = 0.16$ rad, and $\tan\theta = 0.16$ as well. The distance x is related to D, the observer's distance from the slits, by $x/D = \tan\theta = 0.16$, so $x = 0.16D = 1.6$ m.

Answers to Some of the Odd-Numbered Problems

The Station

15P1. (a) $y'(t) = y_0 + v_{0y}t - (1/2)a_{ship}t^2$ (b) $t = 2v_{0y}/a_{ship}$

15P3. (a), (b) $t = \sqrt{3h/2g}$

15P5. (a) $T = \dfrac{3m_1m_2}{m_1+m_2}g$ (b) $\dfrac{3}{2}\left(\dfrac{m_1-m_2}{m_1+m_2}\right)g$ (c) $\dfrac{6m_1m_2}{m_1+m_2}g$

15P7. (a) $\mu_K = 0.14$ (b) no

15P11. (a) $\dfrac{2}{\sin\theta}\sqrt{\dfrac{h}{g}}$ (b) $\sqrt{gh}$

15P13. 0.91 s

15P15. $F_0 = \dfrac{(M+2m)2g\sin\theta}{1+\cos\theta}$

15P17 (a) $4\pi^2Rm/T^2$ (b) $2\pi^2Rm/T^2$, 0 (c) $4\pi^2R/T^2$, $2\pi^2R/T^2$, 0

15P19. (a) $4\pi^2R/T^2$ (b) $(2\pi R/T + v)^2/R$ (c) $(2\pi R/T - v)^2/R$

15P21. (b) $\theta = \sqrt{3} - \dfrac{\pi}{3}$ = 0.685 radians

15P23. (b) $2R\omega/\pi$ (c) $v = R\omega\sqrt{1+(4/\pi^2)}$ $\theta = \tan^{-1}(\pi/2)$

15P25. (a) $\sqrt{3}\pi R/T$ (b) 30° (c) 1 m/s^2 $\approx g_{Earth}/10$ (d) 17 s (e) 8.7 m/s (f) 3.8 m

15P27. (a) The Coriolis force acts. (b) The waterfall bends to the right as seen by those inside watching it.

15P29. (a) The Coriolis force is perpendicular to the velocity vector. (b) It doesn't change.

16P1. (a) 0.26 rad/s (b) 24 s (c) 3.1 cm/s

16P3. (a) $7.3 \cdot 10^{-5}$ s^{-1} (b) 470 m/s, 330 m/s

16P5. (a) $(4/5)\,\omega_0$ (b) $(1/5)\omega_0$

16P7. (a) $\omega_0/2$ (b) $\omega_0/2$

16P9. $\omega_{2f} = -(1/3)\,\omega_{1o}$

16P11. (a) $\left(\dfrac{M}{M+m/2}\right)\omega_0$ (b) $\left(\dfrac{m/2}{M+m/2}\right)\omega_0$

16P13. $(M/2 + m)R^2$

16P17. $\left(\dfrac{\ell + R/2}{\ell + R}\right)MR^2$

16P19. (a) $\rho = R$ (b) $\rho = R/\sqrt{2}$

16P23. (a) $(\pi\,\ell n2)\rho_0R^2t$ (b) $(1-\ell n2)\pi t\rho_0R^4$ (c) $\left[\dfrac{1-\ell n2}{\ell n2}\right]MR^2$

16P25. (a) $MR^2/4$ (b) $2MR^2/3$

16P27. (a) (x, y) = (6, 4) cm (b) 2 x 10^{-3} kg m^2 (c) 1.28 x 10^{-3} kg m^2

16P29. $57MR^2/160$ (M = original mass)

16P31. (a) $(M + 6m)\, L^2/12$ (b) $(M + 3m)\, L^2/3$

16P33. (a) $(1/3)\, ML^2$ (b) $(1/18)\, ML^2$

16P35. (a) $m\ell^2/6$ (b) $ML^2/6$

16P37. $(13/32)MR^2$ (M = original mass)

16P39. $\omega_o/2$

16P41. (a) $\dfrac{\omega_o}{1 + 2mr^2/MR^2}$ (b) $\dfrac{\omega_o}{1 + 2m/M}$

16P43. $(1 + 2m/M)\omega_o$

16P45. $\left[\dfrac{1+500mR^2/I}{1+250mR^2/I}\right]^2 g$

16P47. (a) in the center (b) decreases by $2.8 \cdot 10^{-12}$ s

16P49. (a) 1.8 times/s (b) $L_o = L_f = 2.2 \cdot 10^{39}$ kg m^2/s

17P1. $F_2 = F_1$ in magnitude, acting in the clockwise direction.

17P3. (a) $F\sqrt{3}/2$ (b) $r\sqrt{3}/2$ (c) all $rF\sqrt{3}/2$

17P5. $F_2 = F_1/3\sqrt{2}$

17P7. 40°

17P9. 21 1/2°, CW

17P11. $\theta = \cos^{-1}(2/3)$

17P13. $(3/4)\,\ell$

17P15 $\dfrac{2}{3}\ell\cos 15^o$

17P17. (a) $T = Mg$ (b) $F_H = Mg\sqrt{3}/2$ $F_V = Mg/2$

17P19 (a) dF clockwise (b) dF clockwise

17P21. 29°

17P23. (b) $F_{Av} = mg$, $F_{Ah} = mgd/2\sqrt{\ell^2 - d^2}$, $F_{Bv} = 0$, $F_{Bh} = -mgd/2\sqrt{\ell^2 - d^2}$

17P25. $\dfrac{\ell}{d_o}\left(1 - \sqrt{1 - d_o^2/\ell^2}\right)$

17P27. (a) $mg/2\cos\phi$

17P29. (a) $T = \alpha F_o/\sin\phi$ (b) $\mu_s = \left(\dfrac{1-\alpha}{\alpha}\right)\tan\phi$

17P31. It slides before toppling over.

18P1. (a) $[L] = ML^2/T$ (b) $\tau = c - 2dt$

18P3. (a) $\omega_2 = 2\omega_1$ (b) 8 (c) 8

18P5. (a) $MR^2[3\beta t^2 - \delta\gamma\cos(\delta t)]$ (b) $MR^2[6\beta t + \delta^2\gamma\sin(\delta t)]$

18P7. (a) $\left(\dfrac{2m^2 g}{M + 2m}\right)t$ (b) $\left(\dfrac{mMgR}{M + 2m}\right)t$

18P9. (a) $(1/2)MR^2(2\beta t + \gamma\delta e^{\delta t})$

18P11. (a) $\left(\dfrac{m_2 - m_1/2}{m_1R^2/4 + m_2R^2 + I}\right)Rg$

18P13. $\dfrac{(M_B - M_A)g}{M_A + M_B + 3I/R^2}$

18P15. (b) $\dfrac{Mg(I + mr(R + r))}{I + mr^2 + MR^2}$

18P17. (a) $(F/MR)\,t$ (b) $(F/2MR)\,t^2$

18P19. (a) $M \sim FT^2/R$ (b) $M = FT^2/24\pi R$

18P21. (a) $(2/3)g$ (b) $MgR/3$ (c) $(2/3)gt$ (d) $gt^2/6\pi R$

18P23. (a) $4.3 \cdot 10^3$ kg m^2/s^2 (b) 250 mi/hr

18P25. $\sqrt{2}\,t_o$

18P27. (b) $0.445\sqrt{gR}$

18P29. (a) $(5/6)\,MR^2$ (b) $(5M/12F_o)\sqrt{g_{eff}R}$

18P31. (a) $(2/3)\mu_k MgR$

18P33. $\left(\dfrac{36\pi^2R^2A}{M}\right)^{1/3}$

18P35. $[\,144\pi^2A/MR^2\,]^{1/3}$

19P1. (a) The ring has the greatest kinetic energy, and the sphere the least.

(b) $\omega_r = \omega_d/\sqrt{2}$, $\omega_s = \sqrt{5/4}\,\omega_d$

(c) The ring has the largest angular momentum, and the sphere has the smallest.

19P3. $\sim 10^{-4}$ s

19P5. (a) $\{2mgh/[(m + M/2)r^2 + I]\}^{1/2}$ (b) $mgr\omega_f$

19P7. (a) $F_o^2t^2/M$ (b) F_oRt

19P9. $3v^2/g$

19P11. 15.8 J

19P13. (a) $\omega_f = (1 + 2m/M)\omega_0$ (b) $(1/2)mR^2\omega_0^2(1 + 2m/M)$

19P15. (a) $(5/9)\omega_o$ (b) $(5/9)\omega_o L$

19P17. 4 $\cdot 10^{29}$ N m

19P19. (b) $K\,(2\pi)^{n-1}\,p^{2-r}$

19P21. (a) $2\,\tau_o\,t^3/3MR^2$ (b) $\tau_o\,t^4/6MR^2$

19P23. $-(7/16)\,MgR$

19P25. $-(3/4)\,mgR$

19P27. $8 \cdot 10^{-5}$

19P29. $-(3/8)\,MgR\sin\theta$

19P31. $\sqrt{(20\pi/27)NgR}$

19P33. (a) 1/3 (b) 1/2

19P35. The sphere wins

19P37. (a) v_o^2/g (b) $3v_o^2/4g$

19P39. $h_{min} = 27R/10$

19P41. $\theta_{pig} = \cos^{-1}(2/3)$ $\theta_{ball} = \cos^{-1}(10/17)$

19P43. (a) $\dfrac{1}{r\sin\theta}\sqrt{\dfrac{h}{g}\left(\dfrac{R^4 + 2r^2R^2 + 3r^4/2}{R^2 + r^2/2}\right)}$

20P1. (a) $\omega = 15\,\hat{\mathbf{y}}\ \mathrm{s}^{-1}$ (b) $L = 52\,\hat{\mathbf{y}}$ kg m^2/s

20P3. (a) $\boldsymbol{\omega} = (-0.55\hat{\mathbf{x}} + 0.15\hat{\mathbf{y}})\ \mathrm{s}^{-1}$ (b) $\mathbf{L} = (-5.2\hat{\mathbf{x}} + 1.4\hat{\mathbf{y}}) \times 10^3$ kg m^2/s

20P5. (a) $m = M$ (b) Out of the page (counterclockwise) (c) Into the page (clockwise)

20P11. (a) 42,000 km (b) 1.5×10^{14} kg m^2/s *north*

20P13. $L = 8.0$ kg m^2/s south in each case

20P15. (a) $-\hat{\mathbf{x}} + 14\hat{\mathbf{y}} - 15\hat{\mathbf{z}}$ (b) $-19\hat{\mathbf{x}} + 23\hat{\mathbf{y}} + 9\hat{\mathbf{z}}$

20P25. (a) $-0.13\,\hat{\mathbf{z}}$ N m (b) $0.18\ \mathrm{s}^{-1}$ (c) 0.90 kg m^2/s

20P27. (a) $48\,t\,\hat{\mathbf{z}}$ kg m^2/s

20P29. $\mathbf{A} = (-813\hat{\mathbf{x}} - 2170\hat{\mathbf{y}} + 15{,}900\hat{\mathbf{z}})$ lbs $\mathbf{F}_1 = (3260\hat{\mathbf{x}} - 4350\hat{\mathbf{y}} - 8160\hat{\mathbf{z}})$ lbs $\mathbf{A} = 6520(\hat{\mathbf{y}} - \hat{\mathbf{z}})$ lbs

20P31. $\mathbf{A} = (-11\hat{\mathbf{x}} + 0.5\hat{\mathbf{y}} + 25\hat{\mathbf{z}})$ N $F_1 = 10$ N $F_2 = 4.7$ N

20P33. (a) MRv in magnitude (b) $(1/2)\,MRv$ in magnitude (c) 2/3

20P35. 8.3×10^{-6}

20P37. $Mv\,r/[(1/2)MR^2 + mr^2]$

20P39. (c) $V_{CM} = v_o/3$, $\omega = 4v_o/5L$ (d) $(2/15)\,Mv_o^2$

20P41. (a) $v_o/4$ (b) zero (c) $-v_o/2$

20P43. $5v_o/7$

20P45. (a) h (b) $2R/5$

20P47. (a) $\frac{3}{2}RT/\left(I + MR^2\right)$ (b) $\frac{3}{2}R^2T/\left(I + MR^2\right)$

20P49. $\frac{2}{3}\left[\dfrac{1 + \alpha + \alpha^2}{1 + \alpha}\right]d$

20P51. (a) 2.9 s (b) 1.1 s

20P55. (a), (b) no effect (c) If wheel spins with $\mathbf{L}$ *up*, car would tilt to the right.

20P57. (a) *iii* (b) *ii*

Chapters A - F

AP1. (a) 240 min (b) 5.6 km/s (c) 0.026 rad/min

AP3. (a) 6.6 R_E (b) 3.1 km/s (c) 0.23 m/s^2

AP5. (a) $1/\sqrt{2}$ (b) $2\sqrt{2}$ (c) 1/2 (d) 1/2 (e) 1/2

AP7. (a) $M = r_s v_s^2 T_e^2 M_o / 4\pi^2 r_e^3$ (b) $1.0 \cdot 10^{11} M_o$

AP9. (a) $n = 1$ (b) $T = 2\pi/\sqrt{k}$ (c) zero

AP11. (a) $2.2 \cdot 10^8$ km (b) 0.565 (c) 1.81 yrs

AP13. $r_{IO} = 422 \cdot 10^3$ km $r_{CAL} = 1880 \cdot 10^3$ km $T_{GANY} = 7.15$ d

AP17. (a) $v_p = 6.2$ km/s (b) $1.2 \cdot 10^{21}$ m^2

AP19. (c) $r_{min} = L^2 / GMm^2$ $E_{min} = -G^2 M^2 m^3 / 2L^2$

AP21. (a) $T = 2\pi\sqrt{mR/Mg}$ (b) $L = \sqrt{mMgR^3}$

AP23. (a) energy, and angular momentum about O (b) $v = v_o/\sqrt{2}$, $\theta = 45^o$

AP25. (a) 1.0 in/s (b) 10 in/s (c) – 37 in/s^2 (d) 7.0 in/s^2

AP27. $t = \left[\frac{R^2}{\alpha^2}\left(\left(\frac{\mu_s g}{\alpha}\right)^2 - 1\right)\right]^{1/4}$

AP29. (a) $v_r = \beta v/\sqrt{\beta^2 + r^2}$ $v_\theta = rv/\sqrt{\beta^2 + r^2}$

AP31. (a) – 39 m/s^2 (b) 150 m/s^2

AP33. (b) $v_x \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$ $v_y = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$ (c) $v = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$

AP37. 0.40 yrs

AP39. (a) $4.2 \cdot 10^4$ km (c) 5.3 hrs

BP1. $T = 3.0$ s, $\nu = 0.33$ Hz

BP3. (a) $T_o/\sqrt{2}$ (b) $\sqrt{2}/T_o$

BP5. $y = 4y_o$

BP7. (a) 2.0 s^{-1} (b) 0.32 Hz (c) 3.1 s (d) – 0.13 m (e) 0.16 m/s

BP9. (a) 6 m/s (b) 1 m

BP11. (a) 1.4 s (b) 0.096 m (plus or minus) (c) 0.21

BP13. (a) $\pi^2 M/T_o^2$ (b) $\sqrt{\frac{M+4m}{M}}\,T_o$

BP15. $v = \sqrt{g/\pi^3 R}$

BP17. 0.67 m

BP19. 0.5 m

BP23. (a) 2.0 cm (b) 31 s^{-1} (c) $\pi/2$ (d) increased by π

BP25. (a) 2.2 s^{-1} (b) 0.35 Hz (c) 2.9 s (d) 28 cm (e) – 18 cm/s

BP27. (a) $\sqrt{g/\ell}$ (b) $x(0) = 0.20$ m $v(0) = -1.8$ m/s (c) $A = 22$ cm $\phi_o = 27^o$

BP29. (a) 2.5 m (b) $1.74 \cdot 10^{-3}$ s^{-1} (c) 0 min or 60 min

BP31. (a) $x = \pm A/\sqrt{2}$ (b) all x

BP37. (a) quadrupled (b) doubled (c) doubled (d) unchanged

BP39. 42 minutes

BP41. $t = \frac{1}{2}\sqrt{\frac{3\pi}{G\rho}}$ for both

BP43. (a) $U = -\frac{kmM}{2Rr}\ell n\left(\frac{r+R}{r-R}\right)$ (b) it isn't

BP45. (a) $t = \frac{\pi}{\sqrt{\frac{GM}{R^3} - \Omega^2}}$

CP1. (a) 56° (b) 4.2 m/s

CP3. (b) $mg(3 - 2\cos\theta_0)$

CP5. (a) 200.91 s (b) 200.40 s (c) yes; stopwatch with multiple trials. There are also many more precise methods. (d) 9.77 m/s^2

CP11. $\ell = R$

CP15. $\frac{1}{2\pi}\sqrt{\frac{gr}{R^2/2 + r^2}}$

CP17. $2\pi\sqrt{2R/g}$

CP19. (a) $2\pi\sqrt{I/\kappa}$

CP21. (a) $\sqrt{2k/3m}$ (b) $v_0/\omega r$ (c) $\sqrt{\frac{km\,v_0^2}{6}}$

CP25. (a) $r_0 = (b/a)^{1/6}$ $U(r_0) = -a^2/b$ (b) $\omega = \sqrt{\frac{72}{m}\left(\frac{a^7}{b^4}\right)^{1/3}}$

CP27. (a) $r = r_0$ $U(r_0) = 0$ (b) $2a^2D_e$ (c) $2\pi\sqrt{\frac{m}{2a^2D_e}}$

CP29. (a) $a = 16.1$ eV Å^4 $b = 17.7$ eV Å^3 (b) $\nu \sim 2.5 \cdot 10^{13}$ Hz (c) infrared

CP31. (a) $1/\sqrt{LC}$ (b) $q = q_0\cos\omega t$

CP33. (a) 26.0 s (b) 8.2

CP35. (a) $E = A\cos\phi_0$, $F = -A\sin\phi_0$ (b) $\phi_0 = -\tan^{-1}(F/E)$, $A = \sqrt{E^2 + F^2}$

CP37. (b) ≈ 630 (c) 50

CP39. (a) $\gamma = \beta$ (b) $A = x_0$, $B = \beta x_0$

CP41. (a) 2320 (b) 4640 (c) 1160

CP43. (a) α, the same as the input amplitude (b) $\omega = \sqrt{2}\,\omega_0$

DP1. $\omega = 3.1$ s^{-1} $k = 1.8$ cm^{-1} $v = 1.7$ cm/s

DP3. (a) $A\sin 2\pi\left[\frac{x}{\lambda} - vt\right]$

DP5. (a) 0.02 m (b) $3 \cdot 10^2$ m^{-1} (c) 2.5 Hz (d) 15.7 s^{-1} (e) 0.05 m/s (f) $\phi = 0$

DP7. (a) 0.10 s (b) $f(x - 2.30\text{ m})$

DP9. (b) $y(x, t) = a^3/[a^2 + (x - vt)^2]$

DP11. (a) $\lambda = 21.0$ m, $\nu = 0.19$ Hz (b) $\frac{\partial y}{\partial x} = 0.045\cos(0.3x + 1.2t)$ (c) 4.0 m/s

DP13. 150 m/s

DP15. (a) $\sqrt{\frac{g\ell}{2\cos\theta}}$, $\sqrt{\frac{g\ell\tan\theta}{2}}$

DP17. (b) $t = 2\sqrt{L/g}$

DP21. (a) $A = 0.2$ cm, $k = 0.63$ cm^{-1}, $\lambda = 10$ cm, $\omega = 31$ s^{-1}, $\nu = 5.0$ Hz, $T = 0.20$ s, $v = 0.50$ m/s, moving to the right. (b) $v_{max} = 6.3$ cm/s.

DP23. (a) $y = 0.20$ cm $\cdot \sin(0.40\,x + 17.9\,t + \pi/6)$ (x in meters and t in seconds)

(b) $v_y = 3.6$(cm/s) $\cos(0.40\,x + 17.9\,t + \pi/6)$

DP25. 1.67 s

DP27. (a) 42 N (b) 590 Hz

DP29. (a) 48 Hz (b) A factor 4 higher, 20 N.

DP31. (a) 350 Hz (b) 400 kg

DP33. (a) 6 (b) $2\nu_0$

EP1. (a) 343 m/s (b) it decreases

EP3. (a) m = – 8/3 n = 5/3 (b) $\sqrt{\frac{5k}{3}\frac{\hbar\rho^{1/3}}{M^{4/3}}}$

EP5. 1100 m/s

EP7. (a) $v \sim \Delta x\sqrt{\kappa/I}$

EP9. $v = g/2\pi\nu_0$

EP11. (b) $3.5 \cdot 10^{-6}$

EP13. $v \sim \sqrt{\sigma k/\rho}$

EP15. (a) 16 m/s to the right (b) 7.7 N (c) $\lambda = 8.4$ m $\nu = 1.9$ Hz (d) $\langle P\rangle = 0.35$ W

EP19. (a) $y = (A/r^n)\sin k(r - vt)$ where n = 1/2.

EP21. 50 dB

EP23. (a) $I_0/10$ (b) $I_0/100$

EP25. $\nu_{ob} = \nu_{em}(1 + v_{em}/v_s)^{-1}$

EP27. $\Delta\nu = \nu_{em}(v/v_s)^2$

EP29. 43 kHz

EP33. 618 Hz, 582 Hz

EP35. (a) $5.1 \cdot 10^3$ m/s (b) $8.4 \cdot 10^2$ m/s

FP3. (a) open-closed (b) 19 cm

FP5. (a) 1000 m/s (b) 830 Hz

FP7. $\nu = (n/2L)v_{sound}$ n = 1, 2, 3, ...

FP9. (a) 38.6 cm (b) 19.3 cm

FP11. (a) 494 Hz (b) $1.50 \cdot 10^3$ Hz

FP13. $d = (2n + 1)(0.342$ m) n = 1, 2, 3, ...

FP15. (a) 443 Hz or 437 Hz (b) $\pm$ 0.014

FP17. (a) 210 Hz (b) 208 Hz (c) 2 Hz

FP19. (a) 2 Hz (b) 2 Hz (c) 658 Hz

FP21. $x = 1.43$ m or $x = 5.18$ m

FP23. 0.63 km

FP25. (a) 170 Hz (b) 510 Hz, 850 Hz

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Greek Alphabet

α,	Α	Alpha	ν,	Ν	Nu
β,	Β	Beta	ξ,	Ξ	Xi
γ,	Γ	Gamma	ο,	Ο	Omicron
δ,	Δ	Delta	π,	Π	Pi
ε,	Ε	Epsilon	ρ,	Ρ	Rho
ζ,	Ζ	Zeta	σ,	Σ	Sigma
η,	Η	Eta	τ,	Τ	Tau
θ,	Θ	Theta	υ,	Υ	Upsilon
ι,	Ι	Iota	φ,	Φ	Phi
κ,	Κ	Kappa	χ,	Χ	Chi
λ,	Λ	Lambda	ψ,	Ψ	Psi
μ,	Μ	Mu	ω,	Ω	Omega

Some Constants and Physical Properties

speed of light	$c = 3.00 \times 10^{8}$ m/s
gravitational constant	$G = 6.67 \times 10^{-11}$ N m^2/kg^2
Planck's constant	$h = 6.63 \times 10^{-34}$ J s
proton charge	$e = 1.60 \times 10^{-19}$ Coulomb
Boltzmann's constant	$k = 1.38 \times 10^{-23}$ J/molecule K
electron mass	$m_e = 9.11 \times 10^{-31}$ kg
proton mass	$m_p = 1.67 \times 10^{-27}$ kg
dry air at 1 atm and 20° C	density $\rho = 1.21$ kg/m^3;
	viscosity $\eta = 1.8 \times 10^{-5}$ N·s/m^2
water density	$\rho = 1$ gm/cm^3 $= 1000$ kg/m^3
Earth: mass	$m = 5.98 \times 10^{24}$ kg
mean (i.e., average) radius	$r = 6.37 \times 10^{3}$ km
mean Earth-Sun distance	$d = 1.49 \times 10^{8}$ km
Sun: mass	$M = 1.99 \times 10^{30}$ kg
radius	$R = 6.96 \times 10^{8}$ m

Some Conversion Factors

mass:	1 kg = 1000 g 1 tonne = 1000 kg 1 ton = 2000 lbs
	1 u = 1.66×10^{-27} kg (atomic mass unit)
length:	1 m = 100 cm = 39.4 in = 3.28 ft
	1 in = 2.54 cm
	1 mi = 5280 ft = 1.61 km
time:	1 y $= 3.16 \times 10^{7}$ s
speed:	1 mi/hr = 1.47 ft/s = 1.61 km/hr = 0.447 m/s
force:	1 N = 10^5 dyne = 0.225 lb
energy:	1 J = 10^7 erg = 0.738 ft-lb
	1 eV = 1.60×10^{-19} J
	1 kiloton explosion of TNT = $4.2 \cdot 10^9$ J
power:	1 horsepower = 746 W = 550 ft lb/s